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THE MINIMAL CONTINUOUS SEMANTICS OF THE LAMBDA-CALCULUS

by

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0:PROLOGUE:

0.0:Contents:-

0.1:Abstract:-

A semantics of the λ-calculus is presented which is different from any of the lattice models, so far analysed, of Scott and the term models of Morris and Barendregt. The original motivation was to do for λ-expressions what Scott had done for flow-diagrams, namely construct a 'syntactic' inverse limit lattice, E_∞, in which to represent them. A further motivation was to abstract out the essential notion ("continuous semantics") behind the theorem that Wadsworth proved concerning some of Scott's models, namely that the meaning of a λ-expression can be found as the (continuous) limit of its approximate reductions. That this idea is relevant to E_∞ can be seen since the coordinates of the image of a λ-expression in E_∞ form a subset of its approximate reductions.

To establish the basic fact of β-modelship about E_∞, it has to be shown that Wadsworth's theorem does indeed apply - i.e. that the

E_∞-coordinates provide a sufficiently complete subset of all the possible approximate reductions. Translating this back to the λ-calculus gives an algorithm ("i'th reductions") for producing β-reductions which must be proven 'correct' - i.e. that it goes sufficiently far in all cases : this notion is christened "weak completeness". I'th reductions are generalised to a non-deterministic evaluation mechanism called "inside-out reductions" which behaves in almost the opposite manner to Church's "standard reductions". This generalisation is not too drastic since it is easy to show that a weak completeness result for one implies the same for the other. The weak completeness of inside-out reductions is established.

The E_∞-semantics is a 'pure' β-model in that the only η-reductions modelled are when there are equivalent β-reductions - otherwise they are not even comparable. Further, λ-expressions with a normal form are maximal and isolated in E_∞, unsolveable expressions are $\perp$, the fixed-point combinators $\{Y_i | i \geq 0\}$ are equivalenced and the model itself is substitutive, normal, solvable and implies Morris' "extensional equivalence". Finally, it is the minimal continuous semantics in the sense that Wadsworth's theorem is true in another semantics if and only if it is continuously derivable from E_∞.

0.2:Acknowledgements:-

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0.3:The λ-Calculus:-

0.3.0:INTRODUCTION:-

The λ-calculus, invented in the 1930's by Church, provided for the first time a systematic notation for functions. That there was a need for some form of standardisation was argued by Curry and Feys - [00] - where they draw attention to the dangers and absurdities of ad hoc techniques like those habitually used for differential and integral operators.

The λ-calculus gives us more than the strict notion of a function as a set of argument-value pairs in that it also encodes a set of rules whereby we can evaluate the function on any argument. This, of course, restricts the sort of functions that are λ-definable to those for which rules of evaluation exist - i.e. the computable ones. This is just as well since, in the pure λ-calculus, the λ-expressions themselves provide the notation for the objects in the domain and range of the functions. So, if all functions were λ-definable, we would have the mathematically impossible situation of a domain being the same as its own function space!

This near paradox, together with the fact that nobody really bothered, is the reason why it took so long - nearly thirty five years - for the first successful models of the λ-calculus to be discovered. Indeed, it was not clear that the evaluation - or "reduction" - rules gave consistent results independent of the order in which they were applied. In fact, an early formulation - [01] - turned out to be inconsistent! The original proof of consistency, the Church-Rosser theorem, was amazingly long and complicated when we consider the simplicity of the definition of λ-calculus. The situation has been gradually improved over the years and possibly the shortest proof so far is that found by

Tait and Martin-Lof - see Barendregt [O2] - and, independently, by Park (see O.4.O). Even so, it has taken a very long time to reach this elegance!

Clearly, we must be very careful in dealing with λ-calculus. We have an intuition about how it behaves, but this is notoriously faulty. This is why we have included so many of the details of the technical results in this thesis. It is only too easy to say,"Well, of course, that's true!", and then find, when we finally get around to proving it, either that the induction will not go through, no matter how hard we squeeze the hypothesis (e.g. 3.4.11) or, worse still, a counter-example (e.g. 5.4.2(iii)). We quote Curry and Feys from the preface to their Combinatory Logic, volume 1, - [O3] - : "Some half dozen persons have written technically on combinatory logic and most of these, including ourselves, have published something erroneous. Since some of our fellow sinners are among the most careful and competent logicians on the contemporary scene, we regard this as evidence that the subject is refractory. Thus fullness of exposition is necessary for accuracy; and excessive condensation would be false economy here, even more than it is ordinarily."

O.3.1:DEF:-

Let I be a countably infinite set. We call I the set of VARIABLES. Then, the set of λ-EXPRESSIONS, EXP, is given by the context-free grammar :-

$$\mathrm{EXP} ::= \mathrm{I} \mid \lambda \mathrm{I}.\mathrm{EXP} \mid (\mathrm{EXP})(\mathrm{EXP})$$

O.3.2:NOTATION:-

Mostly, we will use small Roman letters to represent variables and small Greek letters, capital Greek letters and capital Roman letters to represent λ-expressions. Occasionally, these will be subscripted by natural numbers.

We adopt, also, the usual conventions about the "dot" and associating to the left so as to avoid too much bracketing.

0.3.3:DEF:-

Let $x, y \in I$ and $\varepsilon, \delta \in EXP$. Then,

x is NOT FREE IN y if $(x \neq y)$,

x is NOT FREE IN $\lambda y.\varepsilon$ if $(x = y)_{\vee}$(x is not free in ε)

and x is NOT FREE IN $\varepsilon(\delta)$ if (x is not free in ε)

$_{\wedge}$(x is not free in δ).

0.3.4:REMARK:-

We have not followed the usual procedure of defining "free" and "bound" variables since we believe that they are confusing (they are not complimentary) and unnecessary ("not freeness" is all we need) concepts.

0.3.5:DEF:- (Substitution Operators)

Let $\varepsilon \in EXP$ and $x \in I$. Then, we define a map, $[\varepsilon/x]:EXP \to EXP$, inductively, by the following three equations :-

$$[\varepsilon/x]y \quad := \left\{\begin{matrix} \varepsilon, \text{ if } x = y. \\ y, \text{ if } x \neq y. \end{matrix}\right\},$$

$$[\varepsilon/x]\lambda y.\delta := \left\{\begin{matrix} \lambda y.\delta, \text{ if } P. \\ \lambda y.[\varepsilon/x]\delta, \text{ if } Q. \\ \lambda z.[\varepsilon/x][z/y]\delta, \text{ if } R. \end{matrix}\right\},$$

where $P \equiv (x = y)$,

$Q \equiv (x \neq y)_{\wedge}$(y is not free in ε)

and $R \equiv (x \neq y)_{\wedge}$(y is free in $\varepsilon)_{\wedge}$(z is the first variable in some fixed ennumeration of I such that $z \neq x$ and z is not free in ε, δ),

and $[\varepsilon/x]\mu(\nu) := [\varepsilon/x]\mu([\varepsilon/x]\nu)$.

0.3.6:DEF:- (Reduction Rules)

(α) $\lambda x.\varepsilon \xrightarrow{1\alpha} \lambda y.[y/x]\varepsilon$, if y is not free in ε.

(β) $(\lambda x.\varepsilon)(\delta) \xrightarrow{1\beta} [\delta/x]\varepsilon$.

(η) $\lambda x.\varepsilon x \xrightarrow{1\eta} \varepsilon$, if x is not free in ε.

0.3.7:REMARK:-

Intuitively, the substitution operator, $[\varepsilon/x]$, means "replace all free occurrences of x with ε", α-conversion is "change of dummy variables in the function", β-reduction is "application of function to argument" and η-reduction reflects the principle of extensionality, which is "two functions are the same if they give the same answers for the same data".

Next, we introduce Wadsworth's idea of contexts - [04] - so as to generalise the reduction rules so that they need only apply to sub-expressions. Intuitively, a context is a λ-expression with a hole into which we may plug another λ-expression.

0.3.8:DEF:-

The set of CONTEXTS, $\mathcal{C}[\ ]$, for λ-expressions is that generated by the context-free grammar :-

$$\mathcal{C}[\ ] ::= [\ ] \mid \lambda I.\mathcal{C}[\ ] \mid (\mathcal{C}[\ ])(\mathrm{EXP}) \mid (\mathrm{EXP})(\mathcal{C}[\ ])$$

0.3.9:LEMMA:-

Any context, $C[\ ] \in \mathcal{C}[\ ]$, determines a map from λ-expressions to λ-expressions as follows :-

$$\begin{aligned} C[\ ] : \mathrm{EXP} &\longrightarrow \mathrm{EXP} \\ \varepsilon &\longmapsto C[\varepsilon], \end{aligned}$$

where $C[\varepsilon]$ is defined by,

$$(C[\ ] = [\ ]) \Rightarrow (C[\varepsilon] = \varepsilon),$$

$$(C[\ ] = \lambda x.C'[\ ]) \Rightarrow (C[\varepsilon] = \lambda x.C'[\varepsilon]),$$

$$(C[\ ] = (C'[\ ])(\delta)) \Rightarrow (C[\varepsilon] = (C'[\varepsilon])(\delta))$$

and $(C[\ ] = \delta(C'[\ ])) \Rightarrow (C[\varepsilon] = \delta(C'[\varepsilon])$.

Proof:-

-Clear.

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0.3.10:GENERALISATION:-

(i) If $\varepsilon = C[\varepsilon']$ and $\delta = C[\delta']$ and $\varepsilon' \xrightarrow{1\alpha} \delta'$, we will also say that $\varepsilon \xrightarrow{1\alpha} \delta$.

(ii) Similarly, extend $\xrightarrow{1\beta}$ and $\xrightarrow{1\eta}$.

(iii) $\xrightarrow{\alpha}$, $\xrightarrow{\beta}$, $\xrightarrow{\eta}$ are the transitive closures of $\xrightarrow{1\alpha}$, $\xrightarrow{1\beta}$, $\xrightarrow{1\eta}$ respectively.

(iv) $\xrightarrow{n\alpha}$ means a sequence of n $\xrightarrow{1\alpha}$'s.

(v) Similarly, for $\xrightarrow{n\beta}$ and $\xrightarrow{n\eta}$.

(vi) $\xrightarrow{\leq n\beta}$, $\xrightarrow{< n\alpha}$ etc... mean what they say.

(vii) We can mix the reductions together so that $\xrightarrow{\alpha,\beta,\eta}$, for example, means a sequence of $\xrightarrow{\alpha}$, $\xrightarrow{\beta}$ and $\xrightarrow{\eta}$'s in any order and with any number of repeats.

0.3.11:DEF:-

Let $\sim$ be a relation on EXP. Then, $\sim$ is SUBSTITUTIVE if $(\varepsilon \sim \delta) \Rightarrow (C[\varepsilon] \sim C[\delta])$, for all contexts C[ ].(*)

0.3.12:LEMMA:-

All the relations defined in 0.3.10 are substitutive.

Proof:-

-The relations defined in parts (i) and (ii) are the substitutive closures of the original definition, 0.3.6.

-Clearly, their transitive closures and the others remain substitutive.

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0.3.13:THEOREM:-

(i) $[x/x]\varepsilon \xrightarrow{\alpha} \varepsilon$.

(ii) (x is not free in ε') $\Rightarrow$ (x is not free in $[\varepsilon'/x]\varepsilon$).

(iii) (x is not free in ε,ε') $\Rightarrow$ (x is not free in $[\varepsilon'/y]\varepsilon$).

(iv) (x is not free in ε) $\Rightarrow$ ($[\varepsilon'/x]\varepsilon \xrightarrow{\alpha} \varepsilon$).

(*) This is sometimes called the "replacement property".

(v) $(x \text{ is not free in } \varepsilon) \Rightarrow ([\varepsilon'/x][x/y]\varepsilon \xrightarrow{\alpha} [\varepsilon'/y]\varepsilon)$.

(vi) $\xrightarrow{\alpha}$ is an equivalence relation on EXP.

(vii) $(\varepsilon \xrightarrow{\alpha} \delta) \Rightarrow ([\varepsilon'/x]\varepsilon \xrightarrow{\alpha} [\varepsilon'/x]\delta)$.

(viii) $(\varepsilon' \xrightarrow{\alpha} \delta') \Rightarrow ([\varepsilon'/x]\varepsilon \xrightarrow{\alpha} [\delta'/x]\varepsilon)$.

(ix) $(z \neq x) \wedge (z \text{ is not free in } \varepsilon, \varepsilon') \Rightarrow$

$$([\varepsilon'/x](\lambda y.\varepsilon) \xrightarrow{\alpha} \lambda z.[\varepsilon'/x][z/y]\varepsilon).$$

(x) $(x \text{ is not free in } \varepsilon) \wedge (\varepsilon \xrightarrow{\beta} \delta) \Rightarrow (x \text{ is not free in } \delta)$.

(xi) $(x \cdot \neq y,z) \wedge (z \text{ is not free in } \varepsilon') \Rightarrow$

$$([\varepsilon'/x][y/z]\varepsilon \xrightarrow{\alpha} [y/z][\varepsilon'/x]\varepsilon).$$

(xii) $(z \cdot \neq x) \wedge (z \text{ is not free in } \varepsilon') \Rightarrow$

$$([\varepsilon'/x][\delta/z]\varepsilon \xrightarrow{\alpha} [[\varepsilon'/x]\delta/z][\varepsilon'/x]\varepsilon).$$

(xiii) $(\varepsilon \xrightarrow{\alpha,1\beta} \delta) \Rightarrow ([\varepsilon'/x]\varepsilon \xrightarrow{\alpha,1\beta} [\varepsilon'/x]\delta)$.

(xiv) $(\varepsilon' \xrightarrow{\alpha,1\beta} \delta\ ) \Rightarrow ([\varepsilon'/x]\varepsilon \xrightarrow{\alpha,\beta} [\delta'/x]\varepsilon)$.

(xv) $(\varepsilon \xrightarrow{\alpha,1\beta} \delta) \wedge (\varepsilon \xrightarrow{\alpha,1\beta} \gamma) \Rightarrow$

$$(\delta \xrightarrow{\alpha,\beta} \omega) \wedge (\gamma \xrightarrow{\alpha,\beta} \omega), \text{ for some } \omega \in \text{EXP}$$

Proof:-

-See Curry and Feys - [05].

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0.3.14:REMARK:-

Above are a list of some of the elementary properties of α- and β-reductions. Nevertheless, their proofs are only simple when we compare them with some of the other proofs in λ-calculus.

We present parts (i) to (xi) since these properties are reflected later when we construct our lattice model (chapters 1,2 and 3). Because of part (vi), we will tend to consider the set of α-equivalence classes, EXP/α, instead of the pure λ-expressions. Hence, in future, we will write $\xrightarrow{\beta,\eta}$ for $\xrightarrow{\alpha,\beta,\eta}$ etc..., assuming that in any reduction sequence α-conversions can take place at any stage. Thus, we would write part (xv), for example :-

$$(\varepsilon \xrightarrow{1\beta} \delta) \wedge (\varepsilon \xrightarrow{1\beta} \gamma) \Rightarrow (\delta \xrightarrow{\beta} \omega) \wedge (\gamma \xrightarrow{\beta} \omega).$$

Note that part (ix) means that we can simplify the second equation of definition 0.3.5(ii) to just the third alternative (i.e. *R*) and in which any suitable z will do (i.e. not just the first one).

The hardest part to prove is (xii), from which parts (xi), (xiii) and, eventually, (xv) follow.

0.4:Some Classical Results:-

0.4.0:THEOREM:- (Church-Rosser)

$(\varepsilon \xrightarrow{\beta} \delta) \wedge (\varepsilon \xrightarrow{\beta} \gamma) \Rightarrow (\delta \xrightarrow{\beta} \omega) \wedge (\gamma \xrightarrow{\beta} \omega)$, for some $\omega \in$ EXP.

Proof:-

-We will outline the proof due to Tait/Martin-Lof/Park.

-Let $\xrightarrow{1\sigma}$ be a reduction rule with the following properties :-

(a) "Church-Rosser" property,

$(\varepsilon \xrightarrow{1\sigma} \delta) \wedge (\varepsilon \xrightarrow{1\sigma} \gamma) \Rightarrow (\delta \xrightarrow{1\sigma} \omega) \wedge (\gamma \xrightarrow{1\sigma} \omega)$

and (b) its transitive closure, $\xrightarrow{\sigma}$, is the same as $\xrightarrow{\beta}$.

-Then, we can get the Church-Rosser property for $\xrightarrow{\sigma}$ by filling in the diagram :

$$\begin{tikzcd}
 & \delta \\
\varepsilon \arrow[ur, "\sigma"] \arrow[dr, "\sigma"'] & \\
 & \gamma,
\end{tikzcd}$$

in the manner suggested by the diagram :-

$$\begin{tikzcd}
 & & \delta_1 = \delta \arrow[dr] & & & \\
 & \delta_0 \arrow[ur] \arrow[dr] & & \omega_1 \arrow[dr] & & \\
\varepsilon \arrow[ur] \arrow[dr] & & \omega_0 \arrow[ur] \arrow[dr] & & \omega_3 \arrow[dr] & \\
 & \gamma_0 \arrow[ur] \arrow[dr] & & \omega_2 \arrow[ur] \arrow[dr] & & \omega_5 = \omega, \\
 & & \gamma_2 \arrow[ur] \arrow[dr] & & \omega_4 \arrow[ur] & \\
 & & & \gamma_3 = \gamma \arrow[ur] & &
\end{tikzcd}$$

where each of the lines represents $\xrightarrow{1\sigma}$, by using property (a).

-By property (b), we have the Church-Rosser Property for $\xrightarrow{\beta}$.

-But, $\xrightarrow{1\beta}$ just fails as a suitable candidate for $\xrightarrow{1\sigma}$, since 0.3.11(xv) is not quite strong enough.

-However, we define "grand" reductions, written $\Rrightarrow$, as follows :-

(i) $x \Rrightarrow x$.

(ii) $\lambda x.\varepsilon \Rrightarrow \omega$, if $\omega \xrightarrow{\alpha} \lambda x.\varepsilon'$ and $\varepsilon \Rrightarrow \varepsilon'$.

(iii) $$\varepsilon(\delta) \Rrightarrow \left\{ \begin{array}{l} \underline{\text{either}}\ \varepsilon'(\delta'),\ \text{if}\ \varepsilon \Rrightarrow \varepsilon'\ \text{and}\ \delta \Rrightarrow \delta' \\ \underline{\text{or}}\ [\delta'/x]\varepsilon'',\ \text{if}\ \varepsilon \Rrightarrow \lambda x.\varepsilon''\ \text{and}\ \delta \Rrightarrow \delta' \end{array} \right\}.$$

-The fact that $\Rrightarrow$ is non-deterministic is irrelevant.

-Clearly, $\Rrightarrow$ has property (b) and it is not all that complicated to show that it has property (a) also.

-Hence, $\Rrightarrow$ will do for $\xrightarrow{1\sigma}$ and so we have the result.

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0.4.1:DEF:-

Let $\gamma = C[\omega]$. Then, if $\omega = \lambda x.\varepsilon$, for some $x \in I$ and $\varepsilon \in EXP$, ω is an α-REDEX of γ. If $\omega = (\lambda x.\varepsilon)(\delta)$, for some $x \in I$ and $\varepsilon,\delta \in EXP$, then ω is a β-REDEX of γ. If $\omega = \lambda x.\varepsilon x$, for some $x \in I$ and $\varepsilon \in EXP$ such that x is not free in ε, then ω is an η-REDEX of γ.

0.4.2:DEF:-

Let $\varepsilon \in EXP$. Then, ε is in α-, β- or η-NORMAL FORM, respectively, if ε contains no α-, β- or η-redexes. We write α-, β- or η-NF to denote the set of α-, β- or η-normal forms.

0.4.3:LEMMA:-

β-NF is given by NF in the context-free grammar :-

$$HD ::= I \mid (HD)(NF)$$
$$NF ::= HD \mid \lambda I.NF$$

Proof:-

-Let $A[\![\varepsilon]\!] \equiv (\varepsilon \in \beta\text{-NF}) \Rightarrow (\varepsilon \in NF)$.

-Clearly, $A[\![x]\!]$ and $(A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!])$.

-Claim: $A[\![\varepsilon]\!]_\wedge A[\![\delta]\!] \Rightarrow A[\![\varepsilon(\delta)]\!]$:-

-Let $\varepsilon(\delta) \in \beta$-NF. Then, $\varepsilon \notin \lambda I.EXP$ and $\varepsilon,\delta \in \beta$-NF.

-So, by $A[\![\varepsilon]\!]_\wedge A[\![\delta]\!]$, ε and $\delta \in NF$.

-Thus, $\varepsilon \in NF \backslash \lambda I.EXP$ - i.e. $\varepsilon \in HD$.

-$\therefore$, $\varepsilon(\delta) \in HD \subset NF$.

-Hence, by structural induction, $(\forall \varepsilon \in EXP) A[\![\varepsilon]\!]$.

-We need a "cross-product" induction to go the other way.

-Let $P[\![\mu]\!] \equiv (\mu \in \beta\text{-}NF)$.

-Let $Q[\![\nu]\!] \equiv (\nu \in \beta\text{-}NF) \wedge (\nu \notin \lambda I.EXP)$.

-Then, clearly, we have the following results :-

-$Q[\![x]\!]$.

-$Q[\![\mu]\!] \wedge P[\![\nu]\!] \Rightarrow Q[\![\mu(\nu)]\!]$.

-$Q[\![\mu]\!] \Rightarrow P[\![\mu]\!]$.

-$P[\![\mu]\!] \Rightarrow P[\![\lambda x.\mu]\!]$.

-$\therefore$, we conclude, $(\forall \mu \in HD) Q[\![\mu]\!]$ and, in particular, $(\forall \nu \in NF) P[\![\nu]\!]$.

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0.4.4:REMARK:-

We shall not be interested in α-normal forms at all, not very interested in η-normal forms but very interested in β-normal forms. Hence, and in view of the above characterisation, we shall drop the β- when refering to these objects and write NF for β-NF.

Further, this context free grammar of normal forms provides a guideline for the construction of the syntactic lattices (0.7.27)

0.4.5:COR:-

$$(\varepsilon \xrightarrow{\beta} \delta) \wedge (\varepsilon \xrightarrow{\beta} \gamma) \wedge (\delta, \gamma \in NF) \Rightarrow (\delta \xrightarrow{\alpha} \gamma).$$

Proof:-

-By 0.4.0, $(\delta \xrightarrow{\beta} \omega) \wedge (\gamma \xrightarrow{\beta} \omega)$, for some $\omega \in EXP$.

-But, since δ and $\gamma \in NF$, there can be no β-reductions from them.

-Hence, considering remark 0.3.14, $(\varepsilon \xrightarrow{\alpha} \omega) \wedge (\gamma \xrightarrow{\alpha} \omega)$.

-So, $\delta \xrightarrow{\alpha} \gamma$, by 0.3.13(vi).

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0.4.6:REMARK:-

The above corollary is what we mean by the "consistency" of

β-reduction. In a normal form, there are no β-redexes (= function-argument pairs) still unevaluated. The result shows that, in any "completed" computation, it does not matter in what order the β-redexes were taken.

0.4.7:DEF:-

β-$\underline{\text{cnv}}$ is the equivalence relation generated by $\xrightarrow{\beta}$. Again, we will write just $\underline{\text{cnv}}$ for β-$\underline{\text{cnv}}$.

Similarly, we define η-$\underline{\text{cnv}}$ and β-η-$\underline{\text{cnv}}$.

0.4.8:EXAMPLES:-

Not every computation terminates in a normal form, thus setting up the conditions for 0.4.5. For example, if $\Delta = \lambda x.xx$, we can only have :-

$$\Delta\Delta \xrightarrow{1\beta} \Delta\Delta \xrightarrow{1\beta} \Delta\Delta \xrightarrow{1\beta} \ldots\ldots\ldots\ldots$$

A more interesting case is the paradoxical, or fixed-point combinator, $Y = \lambda f.\Sigma\Sigma$ where $\Sigma = \lambda y.f(yy)$. Here, we have :-

$$Y = \lambda f.\Sigma\Sigma \xrightarrow{1\beta} \lambda f.f(\Sigma\Sigma) \xrightarrow{1\beta} \lambda f.f(f(\Sigma\Sigma)) \xrightarrow{1\beta}$$
$$\lambda f.f(f(f(\Sigma\Sigma))) \xrightarrow{1\beta} \lambda f.f(f(f(f(\Sigma\Sigma)))) \xrightarrow{1\beta} \ldots\ldots\ldots\ldots$$

It is called the "fixed-point" combinator because :-

$$\varepsilon(Y\varepsilon) \;\underline{\text{cnv}}\; Y\varepsilon.$$

There is a family of "fixed-point" combinators. Starting with $Y_0 := Y$ and letting $G = \lambda y.\lambda f.f(yf)$, we define $Y_{i+1} := Y_iG$. Suppose, as an induction hypothesis, that Y_i is a "fixed-point" combinator, i.e. :-

$$\varepsilon(Y_i\varepsilon) \;\underline{\text{cnv}}\; Y_i\varepsilon.$$

Then,

$$\begin{aligned} Y_{i+1}\varepsilon = (Y_iG)(\varepsilon) \;\underline{\text{cnv}}\; & G(Y_iG)(\varepsilon) \\ & \xrightarrow{\beta} \varepsilon((Y_iG)(\varepsilon)) = \varepsilon(Y_{i+1}\varepsilon). \end{aligned}$$

So, the induction goes through and all the Y_i's are fixed-point combinators. We feel that the Y_i's are doing essentially the same thing, yet it is not too hard to see that $Y_0 \;\underline{\text{c}\cancel{\text{n}}\text{v}}\; Y_1$. Further, it is probably true that Y_i $\underline{c\!\!/v}$ Y_j, for all $i \neq j$!

0.4.9:REMARK:-

Returning to the Church-Rosser theorem and its extension for $\xrightarrow{\eta}$, Curry - [06] - develops techniques and results which we shall find extremely useful later (chapter 6). We list them here.

0.4.10:DEF:-

Let $\varepsilon \xrightarrow{1\beta} \delta$ and let S be a redex in ε. Then, see Curry - [07] - for an exhaustive definition of the RESIDUALS OF S IN δ.

0.4.11:NOTATION:-

Let $\varepsilon \xrightarrow{1\beta} \delta$ by contraction of the redex R in ε. Then, we write :-

$$\varepsilon \xrightarrow{R} \delta.$$

If we have numbered the reduction, e.g. :-

$$\varepsilon \xrightarrow{R} \delta \dots\dots\dots\dots\dots\dots\dots\dots ①$$

or :-

$$\varepsilon \xrightarrow{R \quad ①} \delta,$$

and S is a redex in ε, then we write $\{S\}/①$, or $\{S\}/R$, to denote the set of residuals of S in δ.

If we have a set, $\mathcal{S}$, of redexes in ε, then we generalise the definition of residuals of redexes to residuals of sets of redexes by :-

$$\mathcal{S}/① := \cup\{\{S\}/① \mid S \in \mathcal{S}\}.$$

We further generalise as follows. Let,

$$② \dots\dots \varepsilon_0 \xrightarrow{R_1} \varepsilon_1 \xrightarrow{R_2} \varepsilon_2 \xrightarrow{R_3} \dots\dots \xrightarrow{R_n} \varepsilon_n,$$

and let $\mathcal{S}$ be a set of redexes in ε_0. Write $\mathcal{S}_0 := \mathcal{S}$. Then, $\mathcal{S}_{i+1} := \mathcal{S}_i/R_{i+1}$, the set of residuals of $\mathcal{S}_i$ in ε_{i+1}. Then, we define :-

$$\mathcal{S}/② := \mathcal{S}_n.$$

0.4.12:DEF:-

Let $\mathcal{S}$ be a set of redexes in ε. Then, $\varepsilon \xrightarrow{\beta \quad ③} \delta$ is a REDUCTION RELATIVE TO $\mathcal{S}$ IN ε if the only redexes contracted are residuals of redexes of S.

Further, it is a <u>COMPLETE RELATIVE REDUCTION OF S IN ε</u> if $S/③ = \emptyset$. In this case, we write :-

$$\varepsilon \xrightarrow{\quad S \quad} \delta.$$

<u>0.4.13:THEOREM:-</u> (Curry's Strong Property E)

Let S be a set of redexes in ε and let R also be a redex in ε. Then, there is a complete relative reduction of S in ε and all complete relative reductions end in the same δ(up to α-conversion).

Further, R has the same residuals in δ no matter which complete relative reduction is used.

<u>Proof:-</u>

-See Curry - [08].

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<u>0.4.14:COR:-</u> (Parallel Moves)

Let $\varepsilon \xrightarrow{\beta} \delta$ and let S be a set of redexes in ε. Then, we can construct the following diagram :-

$$\begin{tikzcd}
\varepsilon \arrow[r, "③"] \arrow[d, "S"'] & \delta \arrow[d, "S/③"] \\
\varepsilon' \arrow[r, "\beta"] & \delta'.
\end{tikzcd}$$

<u>Proof:-</u>

-Suppose the reduction sequence ③ is of the form ② in 0.4.11.

-Define $S_0, S_1, \ldots\ldots, S_n$ as in 0.4.11. Consider the diagram :-

$$\begin{tikzcd}
\varepsilon = \varepsilon_0 \arrow[r, "R_1"] \arrow[d, "S_0"'] \arrow[dr, phantom, "A_0"] & \varepsilon_1 \arrow[r, "R_2"] \arrow[d, "S_1"'] \arrow[dr, phantom, "A_1\ \cdots\cdots"] & \cdots\cdots \arrow[r, "R_i"] & \varepsilon_i \arrow[r, "R_{i+1}"] \arrow[d, "S_i"'] \arrow[dr, phantom, "A_i"] & \varepsilon_{i+1} \arrow[d, "S_{i+1}"] & \cdots\cdots & \varepsilon_n = \delta \arrow[d, "S_n"] \\
\varepsilon' = \varepsilon'_0 \arrow[r] & \varepsilon'_1 \arrow[r] & \cdots\cdots \arrow[r] & \varepsilon'_i \arrow[r] & \varepsilon'_{i+1} & \cdots\cdots & \varepsilon'_n = \delta'
\end{tikzcd}$$

-Consider square A_i. The sequence $\varepsilon_i \longrightarrow \varepsilon_{i+1} \longrightarrow \varepsilon'_{i+1}$ is a complete relative reduction of $\mathcal{S}_i \cup \{R_{i+1}\}$ in ε_i. Now, construct $\varepsilon'_i \longrightarrow \varepsilon''_{i+1}$ by a complete relative reduction of the residuals of R_{i+1} in ε_i. Then, the sequence $\varepsilon_i \longrightarrow \varepsilon'_i \longrightarrow \varepsilon''_{i+1}$ is another complete relative reduction of $\mathcal{S}_i \cup \{R_{i+1}\}$ in ε_i. Hence, by 0.4.13, $\varepsilon'_{i+1} \xrightarrow{\alpha} \varepsilon''_{i+1}$.

-This happens in each square and, so, we have the result.

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0.4.15:REMARK:-

The Church-Rosser theorem for $\xrightarrow{\beta}$ is now a simple corollary of 0.4.14. We just use it repeatedly with $\mathcal{S}$ a singleton set each time - see Curry [09].

Curry goes on to establish the Church-Rosser property for $\xrightarrow{\beta,\eta}$ and develops some useful results. We list them in the following lemma.

0.4.16:LEMMA:-

(i) $(\forall \varepsilon \in \mathrm{EXP})(\varepsilon \xrightarrow{\eta} \varepsilon^* \in \eta\text{-NF})$

(ii) $(\varepsilon \xrightarrow{\eta} \delta) \wedge (\varepsilon \xrightarrow{\eta} \gamma) \Rightarrow (\delta \xrightarrow{\eta} \omega) \wedge (\gamma \xrightarrow{\eta} \omega)$

(iii) $(\varepsilon \xrightarrow{\beta,\eta} \delta) \Rightarrow (\varepsilon \xrightarrow{\beta} \gamma \xrightarrow{\eta} \delta)$

(iv) $(\varepsilon \xrightarrow{\beta,\eta} \delta) \Rightarrow (\varepsilon^* \xrightarrow{\beta,\eta} \delta^*)$, where ε^* and δ^* are the η-normal forms of ε and δ respectively.

(v) $(\varepsilon \xrightarrow{\beta,\eta} \delta) \wedge (\varepsilon \xrightarrow{\eta} \gamma) \Rightarrow (\delta \xrightarrow{\eta} \delta^*) \wedge (\gamma \xrightarrow{\beta,\eta} \delta^*)$

(vi) $(\varepsilon \xrightarrow{\beta,\eta} \delta) \wedge (\varepsilon \xrightarrow{\beta,\eta} \gamma) \Rightarrow (\delta \xrightarrow{\beta,\eta} \omega) \wedge (\gamma \xrightarrow{\beta,\eta} \omega)$.

Proof:-

(i) -Clearly, an η-reduction strictly shortens the length of an expression and, so, we can only do a finite number from one.

(ii) -Straightforward - see Curry [10]. Hence, η-normal forms are unique.

(iii) -Non-trivial: although we can show quite easily that :-

$$(\varepsilon \xrightarrow{1\eta} \delta \xrightarrow{1\beta} \gamma) \Rightarrow (\varepsilon \xrightarrow{\beta} \omega \xrightarrow{\eta} \gamma),$$

we cannot guarantee that the length of either the $\xrightarrow{\beta}$ or $\xrightarrow{\eta}$ will be restricted to at most one.

-See Curry [11] for a correct proof.

(iv) -Simpler that part (iii) - see Curry [12]. In fact, the sequence from ε^* to δ^* is of a special form which Curry calls "θ-reductions" : in these, β-reductions are allowed only from expressions in η-normal form.

(v) -This is a useful corollary to part (iv).

-Let $\varepsilon \xrightarrow{\beta,\eta} \delta$ and $\varepsilon \xrightarrow{\eta} \gamma$.

-Now, $\varepsilon \xrightarrow{\eta} \varepsilon^* \in \eta$-NF, by part (i).

-By the Church-Rosser part (ii), this ε^* is unique and, so, $\gamma \xrightarrow{\eta} \varepsilon^*$.

-But, by part (iv), $\delta \xrightarrow{\eta} \delta^*$ and $\varepsilon^* \xrightarrow{\beta,\eta} \delta^*$.

-Thus, $\gamma \xrightarrow{\beta,\eta} \delta^*$.

(vi) -Trivial, using 0.4.0 and parts (iii) and (v) above - [12].

$\nmid$

0.4.17:REMARK:-

Frequently, when proving theorems in the λ-calculus, we perform inductions on the complexity of an expression ε. As far as possible, we shall try to do this with structural inductions over the context-free grammar given in 0.3.1. However, it is neccessary occasionally to use an ordinary natural number induction over the "length" or "rank" of an expression (e.g. 0.3.13 and 0.4.20). In section 0.5, we make use of a restricted notion of "rank" which should not be confused with the following one which applies to all λ-expressions. The following is taken from Curry [28].

0.4.18:DEF:-

$$\mathrm{rank}[\![x]\!] = 0$$

$$\mathrm{rank}[\![\lambda x.\varepsilon]\!] = 1 + \mathrm{rank}[\![\varepsilon]\!]$$

$$\mathrm{rank}[\![\varepsilon(\delta)]\!] = 1 + \mathrm{rank}[\![\varepsilon]\!] + \mathrm{rank}[\![\delta]\!].$$

0.4.19:LEMMA:-

(i) $\mathrm{rank}[\![[a/b]\varepsilon]\!] = \mathrm{rank}[\![\varepsilon]\!]$

(ii) $(\varepsilon \xrightarrow{\alpha} \delta) \Rightarrow (\mathrm{rank}[\![\varepsilon]\!] = \mathrm{rank}[\![\delta]\!])$

(iii) $\mathrm{rank}[\![[\delta/x]\varepsilon]\!] = \mathrm{rank}[\![\varepsilon]\!] + n{*}\mathrm{rank}[\![\delta]\!]$,

where n is the number of times x occurs free in ε.

(iv) $(\varepsilon \xrightarrow{1\eta} \delta) \Rightarrow (\mathrm{rank}[\![\varepsilon]\!] = 2 + \mathrm{rank}[\![\delta]\!])$.

Proof:-

(i),(ii) and (iii) -Trivial induction on the rank of ε.

(iv) -Trivial structural induction on ε.

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0.4.20:THEOREM:-

$$(\varepsilon \xrightarrow{\beta,\eta} \delta \in \mathrm{NF}) \Leftrightarrow (\varepsilon \xrightarrow{\beta} \varepsilon^* \in \mathrm{NF}).$$

Proof:-

($\Leftarrow$) -Immediate.

($\Rightarrow$) -This looks trivial, but, like so much else in this subject, lacks a trivial proof.

-First, just note that $\varepsilon \xrightarrow{\beta,\eta} \delta^* \in \beta\text{-}\eta\text{-NF}$, by 0.4.16(i).

-The result now follows from the non-trivial lemma of Curry, Hindley and Seldin - [76].

ǂ

0.4.21:REMARK:-

The above result says that, if we are interested in reaching normal form, we need not bother with η-reductions at all. Later (0.5.9), we show that η-reductions are similarly irrelevant to the question of reaching "head normal form" ("solvability"). This uses a proof strongly based on the one for 0.4.20.

There is one other classical result on β-reductions in which we are interested in this thesis. The β-reduction rule does not specify any order in which to take β-redexes. However, it is possible to give a deterministic rule (in fact : "always take the left-most outermost") which guarantees termination (i.e. reaches normal form) if termination is at all possible. This is not a trivial property since there are reduction rules (e.g. : "always take the innermost") which occasionally fail to terminate when this is clearly possible (e.g. $(\lambda x.y)(\Delta\Delta)$). Later (chapters 5 and 6), we shall give other reduction rules which also guarantee terminatior whenever possible.

0.4.22:DEF:-

Let R and S be different β-redexes in ε. Then, R is LEFT-OUTSIDE of S in ε is defined inductively by :-

(i) $\varepsilon = x$: not possible,

(ii) $\varepsilon = \lambda x.\varepsilon'$: if R is left-outside of S in ε'

and (iii) $\varepsilon = \omega(\delta)$: if either $R = \omega(\delta)$

or $\omega = C[R]$ and $\delta = C'[S]$

<u>or</u> R is left-outside of S in ω

<u>or</u> R is left-outside of S in δ.

Further, R is the <u>LEFTMOST-OUTERMOST</u> redex in ε if it is left-outside of all the others.

<u>0.4.23:LEMMA:-</u>

(i) Left-outside is a total ordering on the set of redexes in ε. In particular, if ε has any redexes at all, then it has a unique leftmost-outermost one.

(ii) Let R be left-outside of S in ε and $\varepsilon \xrightarrow{S} \delta$. Then, R has precisely one residual, R´, in δ. Further, if S is left-outside of T in ε, then R´ is left-outside of any residuals of T in δ. Also, if T´ is a new redex in δ (i.e. with no ancestor in ε), then R´ is left-outside of T´ in δ.

<u>Proof:-</u>

-By simple structural inductions on ε.

$\nmid$

<u>0.4.24:DEF:-</u>

Let $\varepsilon_0 \xrightarrow{R_1} \varepsilon_1 \xrightarrow{R_2} \varepsilon_2 \xrightarrow{R_3} \ldots\ldots \xrightarrow{R_i} \varepsilon_i \xrightarrow{R_{i+1}} \ldots\ldots \xrightarrow{R_n} \varepsilon_n$.

Then, this reduction sequence is <u>STANDARD</u> if R_{i+1} is not the residual of a redex that is left-outside of R_i in ε_{i-1}, for all i = 1,2,3,......,n-1. The sequence is <u>NORMAL</u> if R_i is the leftmost-outermost redex in ε_{i-1}, for all i = 1,2,3,......,n.

<u>0.4.25:LEMMA:-</u>

(i) Normal reductions are standard.

(ii) Take a standard reduction sequence with notation as above. Let S_{i-1} be a redex that is left-outside of R_i in ε_{i-1}, for some i $\in$ {1,......,n-1}. Then, S_{i-1} has precisely one residual, S_j, in ε_j and it is left-outside of R_{j+1}, for alll j $\in$ {i,......,n-1}. Finally, S_{i-1} has a unique residual, S_n, in ε_n.

<u>Proof:-</u>

(i) -Clear.

(ii) -By 0.4.23(ii), S_{i-1} has a unique residual, S_i, in ε_i.

-Suppose R_{i+1} is a residual of a redex, T, in ε_{i-1}. Then, R_i is left-outside of T, by the definition of standard and by 0.4.23(i). But then, by 0.4.23(ii), S_i is left-outside of R_{i+1} in ε_i.

-Otherwise, R_{i+1} is a new redex in ε_i and, so, again by 0.4.23(ii), S_i is left-outside of R_{i+1}.

-We repeat this argument at each reduction step to complete the lemma.

⫲

0.4.26:REMARK:-

Thus, if ever we miss out a redex on the left-outside during a standard reduction, it remains left out! We see that standard reductions take their redexes in an "outside-in" order. In this, they can be compared with the "call-by-name" mechanism for passing parameters to procedures and, so, have the same sort of inefficienc - the measure being the length of the reduction sequences. The reduction rules we shall be introducing in chapters 5 and 6 work in an "inside-out" manner. Their 'normal' form (i.e. not allowed to miss out any redexes) is analogous to the "call-by-value" mechanism and, so, they should be more efficient.

0.4.27:THEOREM:- (Standardisation)

Let $\varepsilon \xrightarrow{\beta} \delta$. Then, there is a standard reduction sequence from ε to δ.

Proof:-

-See Curry [13].

⫲

0.4.28:COR:- (Normalisation)

Let $\varepsilon \xrightarrow{\beta} \delta \in NF$. Then, there is a normal reduction sequence from ε to δ.

Proof:-

-By 0.4.27, there is a standard reduction sequence from ε to δ.

-If the redex contracted each time were not innermost-outermost, there would be a redex left in δ, by 0.4.25(ii).

-But, this is not possible since δ is in normal form - ✗.(*)

-Hence, the standard reduction sequence must be normal.

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0.4.29:COR:-

The predicate $P[\![\varepsilon]\!]$, $\equiv$ (there exists $\delta \in$ NF)($\varepsilon \xrightarrow{\beta} \delta$), is semi-decidable.

Proof:-

-Do a normal reduction from ε.

-If $P[\![\varepsilon]\!]$ is true, this will terminate after a finite number of reductions.

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0.4.30:REMARK:-

In fact, the predicate $P[\![\varepsilon]\!]$ - "ε has a normal form" - is not decidable - see Barendregt [14]. In computing terms, this can be thought of as the "halting problem" - although there are dangers. We might be tempted to consider all non-terminating programs as equivalent rubbish! While it makes sense to lump expressions like $\Delta\Delta$ or $\Delta\Delta\Delta$ together as "useless", Y, which does not have normal form, certainly has its uses. However, it would make sense to equivalence all the Y_i's. For these reasons, the concept of "head normal form" was introduced by Wadsworth - [15] - and, equivalently but independently, "solvability" by Barendregt - [16].

(*) We use the sign ✗ to indicate a contradiction.

0.5:Head Normal Form and Solvability:-

0.5.0:DEF:-

We define the set of λ-expressions in "head normal form", HNF, by the context-free grammar :-

$$\text{HEAD} ::= \text{I} \mid (\text{HEAD})(\text{EXP})$$

$$\text{HNF} ::= \text{HEAD} \mid \lambda\text{I}.\text{HNF}$$

We also define the set of λ-expressions "not in head normal form", NOH, by the context-free grammar :-

$$\text{NOH} ::= (\lambda\text{I}.\text{EXP})(\text{EXP}) \mid (\text{NOH})(\text{EXP}) \mid \lambda\text{I}.\text{NOH}$$

0.5.1:LEMMA:-

(i) $\text{NF} \subset \text{HNF}$.

(ii) $\text{HEAD} \cap \lambda\text{I}.\text{EXP} = \emptyset$.

(iii) $\text{HEAD} \cap \text{NOH} = \emptyset$.

(iv) $\text{HNF} \cap \text{NOH} = \emptyset$.

(v) $\text{HNF} \cup \text{NOH} = \text{EXP}$.

Proof:-

(i) -From the characterisation of 0.4.3, this is clear since the generating grammar is a restriction of that for HNF. Formally :-

-Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \in \text{HEAD})$. Clearly, $\mathcal{A}[\![x]\!]$ and $(\forall\mu \in \text{HD})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\nu)]\!])$. So, by structural induction, $(\forall\mu \in \text{HD})\mathcal{A}[\![\mu]\!]$.

-Now, let $\mathcal{B}[\![\mu]\!] \equiv (\mu \in \text{HNF})$. We have, $(\forall\mu \in \text{HD})\mathcal{B}[\![\mu]\!]$. Clearly, $(\forall\mu \in \text{NF})(\mathcal{B}[\![\mu]\!] \Rightarrow \mathcal{B}[\![\lambda x.\mu]\!])$. $\therefore$, $(\forall\mu \in \text{NF})\mathcal{B}[\![\mu]\!]$ - i.e. $\text{NF} \subset \text{HNF}$.

(ii) -Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \notin \lambda\text{I}.\text{EXP})$.

-Clearly, $\mathcal{A}[\![x]\!]$ and $(\forall\mu \in \text{HEAD})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\varepsilon)]\!])$.

-$\therefore$, by structural induction, $(\forall\mu \in \text{HEAD})\mathcal{A}[\![\mu]\!]$.

(iii) -Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \notin \text{NOH})$.

-Clearly, $\mathcal{A}[\![x]\!]$.

-Claim: $(\forall\mu \in \text{HEAD})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\varepsilon)]\!])$:-

-If $\mu(\varepsilon) \in \text{NOH}$, then either $\mu \in \text{NOH}$ - $\mathbb{X}$ to $\mathcal{A}[\![\mu]\!]$ - or $\mu \in \lambda\text{I}.\text{EXP}$

- X to part (ii).

-Hence, by structural induction, $(\forall\mu \in \mathrm{HEAD})\mathcal{A}[\![\mu]\!]$.

(iv) -Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \notin \mathrm{NOH})$.

-Then, $(\forall\mu \in \mathrm{HEAD})\mathcal{A}[\![\mu]\!]$, by part (iii).

-Clearly, $(\forall\mu \in \mathrm{HNF})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\lambda x.\mu]\!])$.

-Hence, by structural induction, $(\forall\mu \in \mathrm{HNF})\mathcal{A}[\![\mu]\!]$.

(v) -Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \in \mathrm{EXP})$.

-Clearly, $\mathcal{A}[\![x]\!]$ and $(\forall\mu \in \mathrm{HEAD})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\varepsilon)]\!])$.

-$\therefore$, by induction, $(\forall\mu \in \mathrm{HEAD})\mathcal{A}[\![\mu]\!]$. Thus, $\mathrm{HEAD} \subset \mathrm{EXP}$.

-Also, $(\forall\mu \in \mathrm{HNF})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\lambda x.\mu]\!])$.

-$\therefore$, by induction, $(\forall\mu \in \mathrm{HNF})\mathcal{A}[\![\mu]\!]$. Thus, $\mathrm{HNF} \subset \mathrm{EXP}$.

-Next, we have $\mathcal{A}[\![(\lambda x.\varepsilon)(\delta)]\!]$.

-And, $(\forall\mu \in \mathrm{NOH})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\varepsilon)]\!] \wedge \mathcal{A}[\![\lambda x.\mu]\!])$.

-$\therefore$ by induction, $(\forall\mu \in \mathrm{NOH})\mathcal{A}[\![\mu]\!]$. Thus, $\mathrm{NOH} \subset \mathrm{EXP}$.

-$\therefore$, $\mathrm{HNF} \cup \mathrm{NOH} \subset \mathrm{EXP}$.

-Conversely, let $\mathcal{B}[\![\varepsilon]\!] \equiv (\varepsilon \in \mathrm{HNF} \cup \mathrm{NOH})$.

-Then, $\mathcal{B}[\![x]\!]$, since $x \in I \subset \mathrm{HEAD} \subset \mathrm{HNF} \subset \mathrm{HNF} \cup \mathrm{NOH}$.

-Clearly, $\mathcal{B}[\![\varepsilon]\!] \Rightarrow \mathcal{B}[\![\lambda x.\varepsilon]\!]$.

-<u>Claim:</u> $\mathcal{B}[\![\varepsilon]\!]\wedge\mathcal{B}[\![\delta]\!] \Rightarrow \mathcal{B}[\![\varepsilon(\delta)]\!]$:-

-If $\varepsilon \in \mathrm{NOH}$, then $\varepsilon(\delta) \in \mathrm{NOH}$, and so we have the claim.

-Otherwise, $\varepsilon \in \mathrm{HNF}$, by $\mathcal{B}[\![\varepsilon]\!]$. <u>Either</u> $\varepsilon \in \lambda I.\mathrm{HNF}$, in which case $\varepsilon(\delta) \in \mathrm{NOH}$ and we have the claim, <u>or</u> $\varepsilon \in \mathrm{HEAD}$, in which case $\varepsilon(\delta) \in \mathrm{HEAD} \subset \mathrm{HNF}$ and we have the claim. (In fact, we do not need $\mathcal{B}[\![\delta]\!]$.)

-$\therefore$, by induction, $(\forall\varepsilon \in \mathrm{EXP})\mathcal{B}[\![\varepsilon]\!]$. Thus, $\mathrm{EXP} \subset \mathrm{HNF} \cup \mathrm{NOH}$.

-$\therefore$, $\mathrm{HNF} \cup \mathrm{NOH} = \mathrm{EXP}$.

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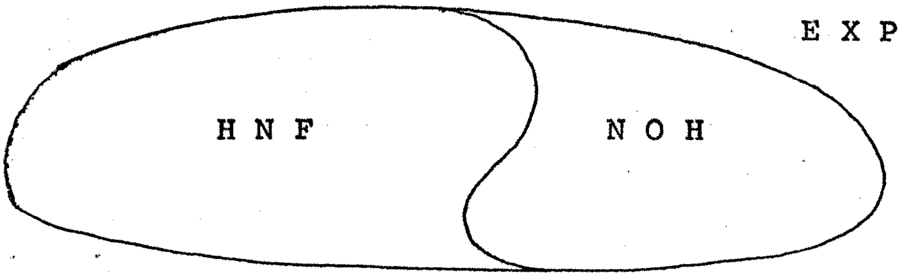

0.5.2:LEMMA:-

(i) $(\mu \in \text{HEAD}) \wedge (\mu \xrightarrow{\alpha} \mu') \Rightarrow ([x/y]\mu, \mu' \in \text{HEAD})$.

(ii) $(\mu \in \text{HNF}) \wedge (\mu \xrightarrow{\alpha} \mu') \Rightarrow ([x/y]\mu, \mu' \in \text{HNF})$.

(iii) $(\mu \in \text{NOH}) \wedge (\mu \xrightarrow{\alpha} \mu') \Rightarrow ([\varepsilon/y]\mu, \mu' \in \text{NOH})$.

Proof:-

(i) -Let $\mathcal{A}[\![\mu]\!] \equiv (\mu \xrightarrow{\alpha} \mu') \Rightarrow ([x/y]\mu, \mu' \in \text{HEAD})$.

-Clearly, $\mathcal{A}[\![x]\!]$ and $(\forall \mu \in \text{HEAD})(\mathcal{A}[\![\mu]\!] \Rightarrow \mathcal{A}[\![\mu(\varepsilon)]\!])$.

-So, by induction, $(\forall \mu \in \text{HEAD})\mathcal{A}[\![\mu]\!]$.

(ii) -For this we define a rank on HNF :-

$$\text{rank}\left(\left\{\begin{matrix}\mu, \mu \in \text{HEAD} \\ \lambda x.\mu, \mu \in \text{HNF}\end{matrix}\right\}\right) := \left\{\begin{matrix}0 \\ 1 + \text{rank}(\mu)\end{matrix}\right\}.$$

-Let $P(i) \equiv (\mu \in \text{HNF}) \wedge (\text{rank}(\mu) = i) \wedge (\mu \xrightarrow{\alpha} \mu') \Rightarrow$

$([x/y]\mu, \mu' \in \text{HNF}) \wedge (\text{rank}([x/y]\mu), \text{rank}(\mu') = i)$.

-We have $P(0)$, by part (i).

-Claim: $P(i) \Rightarrow P(i+1)$, $\forall i \geq 0$:-

-Let $\mu \xrightarrow{\alpha} \mu'$, where $\mu \in \text{HNF}$ and $\text{rank}(\mu) = i+1$.

-Then, $\mu = \lambda x.\nu$, where $\nu \in \text{HNF}$ and $\text{rank}(\nu) = i$.

-Either $\mu' = \lambda x.\nu'$ and $\nu \xrightarrow{\alpha} \nu'$, in which case $\nu' \in \text{HNF}$ with rank i, by $P(i)$, and so $\mu' \in \text{HNF}$ with rank $i+1$.

-Or $\mu' = \lambda y.[y/x]\nu$, where y is not free in ν, in which case $[y/x]\nu \in \text{HNF}$ with rank i, by $P(i)$, and so $\mu' \in \text{HNF}$ with rank $i+1$.

-Also, $[a/b]\mu = [a/b](\lambda x.\nu)$

$\xrightarrow{\alpha} \lambda c.[a/b][c/x]\nu$, by 0.3.13(ix), where $c \neq a,b$ and is not free in ν.

$\in \text{HNF}$ with rank $i+1$, using $P(i)$ twice.

-Hence, $[a/b]\mu \in \text{HNF}$ with rank $i+1$, by the first half of this claim, proved above.

-$\therefore$, by ordinary induction, $(\forall i \geq 0)P(i)$.

(iii) -For this we define a rank on NOH :-

$$\mathrm{rank}\left\{\begin{matrix} (\lambda x.\varepsilon)(\delta), & \varepsilon \notin \mathrm{NOH} \\ \mu(\varepsilon), & \mu \in \mathrm{NOH} \\ \lambda x.\mu, & \mu \in \mathrm{NOH} \end{matrix}\right\} := \left\{\begin{matrix} 0 \\ 1 + \mathrm{rank}(\mu) \\ 1 + \mathrm{rank}(\mu) \end{matrix}\right\}.$$

-Let $\mathcal{Q}(i) \equiv (\mu \in \mathrm{NOH})_\wedge(\mathrm{rank}(\mu) = i)_\wedge(\mu \xrightarrow{\alpha} \mu') \Rightarrow$

$([\varepsilon/y]\mu, \mu' \in \mathrm{NOH})_\wedge(\mathrm{rank}([x/y]\mu), \mathrm{rank}(\mu') = i)$.

-<u>Claim:</u> $\mathcal{Q}(0)$:-

-Let $(\lambda x.\varepsilon)(\delta) \xrightarrow{\alpha} \mu'$, where $\varepsilon \notin \mathrm{NOH}$.

-Then, $\varepsilon \in \mathrm{HNF}$, by 0.5.1(iv), and so $\lambda x.\varepsilon \in \mathrm{HNF}$.

-But, $\mu' = (\lambda x'.\varepsilon')(\delta')$, where $\lambda x.\varepsilon \xrightarrow{\alpha} \lambda x'.\varepsilon'$ and $\delta \xrightarrow{\alpha} \delta'$ and, so, $\mu' \in \mathrm{NOH}$. Further, $\lambda x'.\varepsilon' \in \mathrm{HNF}$, by part (ii), and, so, $\varepsilon' \notin \mathrm{NOH}$ and $\mathrm{rank}(\mu') = 0$.

-Now, clearly, $[\bar{\varepsilon}/y](\lambda x.\varepsilon)(\delta) \in \mathrm{NOH}$.

-But, $[a/y](\lambda x.\varepsilon) \in \mathrm{HNF}$, by part (ii), and, so, $\mathrm{rank}([a/y](\lambda x.\varepsilon)(\delta)) = 0$.

-<u>Claim:</u> $\mathcal{Q}(i) \Rightarrow \mathcal{Q}(i+1), \forall i \geq 0$:-

-Let $\mu \xrightarrow{\alpha} \mu'$, where $\mu \in \mathrm{NOH}$ and $\mathrm{rank}(\mu) = i+1$.

-If $\mu = \nu(\delta)$, where $\nu \in \mathrm{NOH}$ and $\mathrm{rank}(\nu) = i$, then $\mu' = \nu'(\delta')$, where $\nu \xrightarrow{\alpha} \nu'$, $\delta \xrightarrow{\alpha} \delta'$. By $\mathcal{Q}(i)$, $\nu' \in \mathrm{NOH}$ with rank i and, so, $\mu' \in \mathrm{NOH}$ with rank $i+1$. Also, $[\varepsilon/y]\mu = ([\varepsilon/y]\nu)([\varepsilon/y]\delta) \in \mathrm{NOH}$, by $\mathcal{Q}(i)$, and, if $\varepsilon = a \in I$, then it has rank $i+1$, by $\mathcal{Q}(i)$ again.

-If $\mu = \lambda x.\nu$, where $\nu \in \mathrm{NOH}$ and $\mathrm{rank}(\nu) = i$, then <u>either</u> $\mu' = \lambda x.\nu'$ and $\nu \xrightarrow{\alpha} \nu'$, in which case $\nu' \in \mathrm{NOH}$ with rank i, by $\mathcal{Q}(i)$, and so $\mu' \in \mathrm{NOH}$ with rank $i+1$; <u>or</u> $\mu' = \lambda y.[y/x]\nu$, in which case $[y/x]\nu \in \mathrm{NOH}$ with rank i, by $\mathcal{Q}(i)$, and so $\mu' \in \mathrm{NOH}$ with rank $i+1$. Also, $[\varepsilon/y](\lambda x.\nu) \xrightarrow{\alpha} \lambda z.[\varepsilon/y][z/x]\nu$, by 0.3.13(ix), $\in \mathrm{NOH}$, using $\mathcal{Q}(i)$ twice, and, if $\varepsilon = a \in I$, then we also get rank $i+1$. ∴, done.

-∴, by ordinary induction, $(\forall i \geq 0)\mathcal{Q}(i)$.

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<u>0.5.3:REMARK:-</u>

The above lemma is one of those "obvious" results about the λ-calculus that seems to want an awful lot of proving! We had to use an induction on rank in parts (ii) and (iii) since the straightforeward structural induction fails. Later (see section 4.1), we introduce a technique that enables us to use structural inductions in cases like these. In principle, we could replace all our structural inductions with various forms of rank inductions, but we prefer to use the former because they seem more natural and are more compact. To compare the techniques, look at the proofs of :-

$$(\mu \in \mathrm{NOH}) \Rightarrow ([\varepsilon/x]\mu \in \mathrm{NOH}),$$

in 0.5.2(iii) and 5.0.3.

<u>0.5.4:LEMMA:-</u>

$$(\mu \in \mathrm{HNF}) \wedge (\mu \xrightarrow{\beta,\eta} \delta) \Rightarrow (\delta \in \mathrm{HNF}).$$

<u>Proof:-</u>

-Let $A\llbracket\mu\rrbracket \equiv (\mu \xrightarrow{1\beta} \delta) \Rightarrow (\delta \in \mathrm{HEAD})$.

-Clearly, $A\llbracket x\rrbracket$.

-<u>Claim:</u> $A\llbracket\mu\rrbracket \Rightarrow A\llbracket\mu(\varepsilon)\rrbracket,\ \forall\mu \in \mathrm{HEAD}$:-

-By 0.5.1(ii), $\mu(\varepsilon)$ is not a β-redex.

-Let $\mu(\varepsilon) \xrightarrow{1\beta} \delta$.

-<u>Either</u> $\delta = \mu'(\varepsilon)$ and $\mu \xrightarrow{1\beta} \mu'$ <u>or</u> $\delta = \mu(\varepsilon')$ and $\varepsilon \xrightarrow{1\beta} \varepsilon'$.

-So, by $A\llbracket\mu\rrbracket$, we must have $\delta \in \mathrm{HEAD}$.

-$\therefore$, by induction, we have $(\forall\mu \in \mathrm{HEAD})A\llbracket\mu\rrbracket$.

-Let $A'\llbracket\mu\rrbracket \equiv (\mu \xrightarrow{1\beta} \delta) \Rightarrow (\delta \in \mathrm{HNF})$.

-We have, $(\forall\mu \in \mathrm{HEAD})A'\llbracket\mu\rrbracket$, since $\mathrm{HEAD} \subset \mathrm{HNF}$.

-<u>Claim:</u> $A'\llbracket\mu\rrbracket \Rightarrow A'\llbracket\lambda x.\mu\rrbracket,\ \forall\mu \in \mathrm{HNF}$:-

-If $\lambda x.\mu \xrightarrow{1\beta} \delta$, then $\delta \xrightarrow{\alpha} \lambda x.\mu'$ and $\mu \xrightarrow{1\beta} \mu'$.

-But, by $A'\llbracket\mu\rrbracket$, $\mu' \in \mathrm{HNF}$ and, so, $\lambda x.\mu' \in \mathrm{HNF}$.

-So, by 0.5.2(ii), $\delta \in \mathrm{HNF}$.

-$\therefore$, by induction, we have $(\forall\mu \in \mathrm{HNF})A'\llbracket\mu\rrbracket$.

-Hence, $(\mu \in \mathrm{HNF}) \wedge (\mu \xrightarrow{\beta} \delta) \Rightarrow (\delta \in \mathrm{HNF})$.

-Let $B\llbracket\mu\rrbracket \equiv (\mu \xrightarrow{1\eta} \delta) \Rightarrow (\delta \in \mathrm{HEAD})$.

-Clearly, $\mathcal{B}[\![x]\!]$ and, similarly to above, since $\mu(\varepsilon)$ cannot be an η-redex, $\mathcal{B}[\![\mu]\!] \Rightarrow \mathcal{B}[\![\mu(\varepsilon)]\!]$, $\forall\mu \in \mathrm{HEAD}$.

-$\therefore$, by induction, $(\forall\mu \in \mathrm{HEAD})\mathcal{B}[\![\mu]\!]$.

-Let $\mathcal{B}'[\![\mu]\!] \equiv (\mu \xrightarrow{1\eta} \delta) \Rightarrow (\delta \in \mathrm{HNF})$.

-As before, we have, $(\forall\mu \in \mathrm{HEAD})\mathcal{B}'[\![\mu]\!]$.

-<u>Claim:</u> $\mathcal{B}'[\![\mu]\!] \Rightarrow \mathcal{B}'[\![\lambda x.\mu]\!]$, $\forall\mu \in \mathrm{HNF}$:-

-Let $\lambda x.\mu \xrightarrow{1\eta} \delta$. If $\delta \xrightarrow{\alpha} \lambda x.\mu'$ and $\mu \xrightarrow{1\eta} \mu'$, then $\delta \in \mathrm{HNF}$, by $\mathcal{B}'[\![\mu]\!]$, as before.

-Otherwise, $\delta = \mu'$, where $\mu = \mu'x$ and x is not free in μ'. In this case, $\lambda x.\mu \in \mathrm{HNF} \Rightarrow \lambda x.\mu'x \in \mathrm{HNF} \Rightarrow \mu'x \in \mathrm{HNF} \Rightarrow \mu'x \in \mathrm{HEAD}$ $\Rightarrow \mu' \in \mathrm{HEAD} \Rightarrow \mu' \in \mathrm{HNF} \Rightarrow \delta \in \mathrm{HNF}$.

-$\therefore$, by induction, $(\forall\mu \in \mathrm{HNF})\mathcal{B}'[\![\mu]\!]$.

-Hence, $(\mu \in \mathrm{HNF})\wedge(\mu \xrightarrow{\eta} \delta) \Rightarrow (\delta \in \mathrm{HNF})$.

-And so, $(\mu \in \mathrm{HNF})\wedge(\mu \xrightarrow{\beta,\eta} \delta) \Rightarrow (\delta \in \mathrm{HNF})$.

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<u>0.5.5:DEF:-</u> (Wadsworth)

SOL := $\{\varepsilon \in \mathrm{EXP} \mid (\text{there exists } \varepsilon' \in \mathrm{HNF})(\varepsilon \xrightarrow{\beta,\eta} \varepsilon')\}$.

INSOL := EXP\SOL.

<u>0.5.6:EXAMPLES:-</u>

(i) Any expression with a normal form is in SOL.

(ii) Recall $\Delta := \lambda x.xx$ and let $T := \lambda x.xxx$. Then, $\Delta\Delta$, $\Delta\Delta\Delta$ and $TT \in \mathrm{INSOL}$.

(iii) $(\forall i \geq 0)(Y_i \in \mathrm{SOL})$.

<u>Proof:-</u>

(i) -Clear, by 0.5.1(i).

(ii) -Write ε^n for $\varepsilon\varepsilon\varepsilon......\varepsilon$ (n times).

-Then, $\{\varepsilon' | \Delta^2 \xrightarrow{\beta,\eta} \varepsilon'\} = \{\Delta^2\} \subset \mathrm{NOH}$.

-And, $\{\varepsilon' | \Delta^3 \xrightarrow{\beta,\eta} \varepsilon'\} = \{\Delta^3\} \subset \mathrm{NOH}$.

-And, $\{\varepsilon' | T^2 \xrightarrow{\beta,\eta} \varepsilon'\} = \{T^n | n \geq 2\} \subset \mathrm{NOH}$.

(iii) -Let $P(i) \equiv (Y_i \xrightarrow{\beta} \lambda f.f(\nu)$, for some $\nu \in \mathrm{EXP})$.

-Clearly, $P(0)$, since $Y_0 \xrightarrow{\beta} \lambda f.f(\Sigma\Sigma)$, by 0.4.8.

-Suppose $P(i)$, for some $i \geq 0$.

-Then, $Y_{i+1} = Y_i G \xrightarrow{\beta} (\lambda f.f(\nu))(G) \xrightarrow{\beta} G(\nu) \xrightarrow{\beta} \lambda f.f(\nu f)$.

-$\therefore$, by induction, $(\forall i \geq 0)P(i)$.

-But, $\lambda f.f(\nu) \in$ HNF and, so, $(\forall i \geq 0)(Y_i \in \text{SOL})$.

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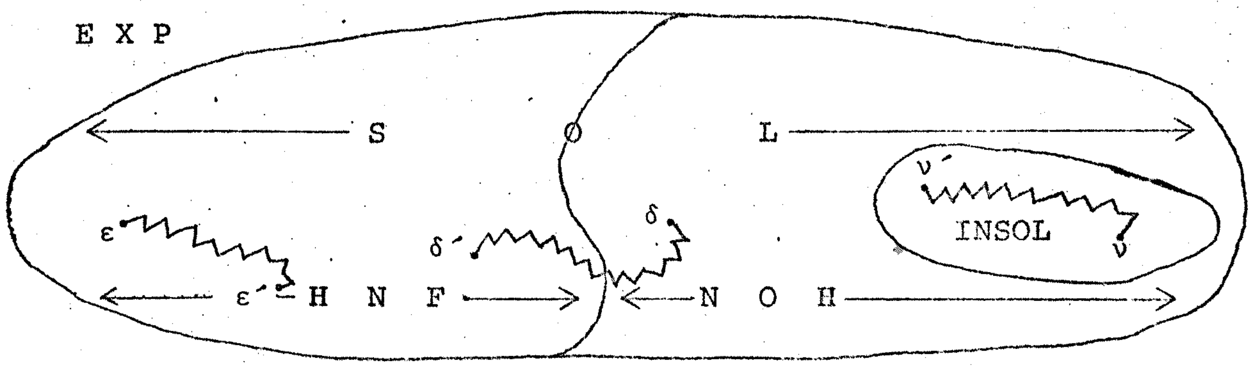
0.5.7:REMARK:-



The spikey lines represent β-η-reduction paths from the expressions ε, δ and ν to ε', δ' and ν', respectively. Since $\varepsilon \in$ HNF, its path must always remain inside HNF, by 0.5.4. Since $\nu \in$ INSOL, its path, by definition, must always remain inside NOH and, fairly obviously, inside INSOL. Similarly, we note that we cannot have a path from some $\sigma \in$ SOL to $\sigma' \in$ INSOL, since $\sigma \xrightarrow{\beta,\eta} \sigma'' \in$ HNF and so, by the Church-Rosser theorem for $\xrightarrow{\beta,\eta}$, $\sigma' \xrightarrow{\beta,\eta} \sigma'''$, where $\sigma'' \xrightarrow{\beta,\eta} \sigma'''$ which, by 0.5.4, is in HNF - $\cancel{X}$ to $\sigma' \in$ INSOL. Finally, if $\delta \in \text{SOL} \cap \text{NOH}$, then there is a path that eventually crosses over to HNF.

Next, we will show that we need only consider β-reduction sequences to determine elements of SOL. Again, this seems an obvious sort of result, particularly as it is quite easy to show that, if $\nu \in$ NOH and $\nu \xrightarrow{1\eta} \nu' \in$ HNF, then $\nu \xrightarrow{1\beta} \nu'$ and that by means of a head redex (see 0.5.10). However, it is not clear how to proceed in view of the problems described in the proof of 0.4.16(iii). The following proof is an adaption of the

Curry/Hindley/Seldin proof of 0.4.20; it is, however, simpler since the inductions can be performed "serially" rather than "nested".

0.5.8:LEMMA:-

(i) $(\varepsilon \xrightarrow{\eta} \delta \in \text{HEAD}) \Rightarrow (\varepsilon \xrightarrow{\beta} \varepsilon^* \in \text{HNF})$

(ii) $(\varepsilon \xrightarrow{\eta} \delta \in \text{HNF}) \Rightarrow (\varepsilon \xrightarrow{\beta} \varepsilon^* \in \text{HNF})$.

Proof:-

(i) -By induction on the (general) rank of ε, as defined in 0.4.18.

-Let $A(n) \equiv (\varepsilon \xrightarrow{\eta} \delta \in \text{HEAD}) \wedge (\text{rank}\llbracket\varepsilon\rrbracket \leq n)$

$\Rightarrow (\varepsilon \xrightarrow{\beta} \varepsilon^* \in \text{HNF})$.

-Clearly, $A(0)$, since $\text{rank}\llbracket\varepsilon\rrbracket = 0 \Rightarrow \varepsilon \in I \subset \text{HNF}$.

-Claim: $A(n-1) \Rightarrow A(n)$, for any $n > 0$:-

-Suppose $A(n-1)$.

-Let $\varepsilon \xrightarrow{\eta} \delta \in \text{HEAD}$ and $\text{rank}\llbracket\varepsilon\rrbracket = n$.

-If $\varepsilon \in \text{HNF}$, then trivial; so, suppose $\varepsilon \in \text{NOH}$.

-Case 1: $\varepsilon = \lambda x.\varepsilon'$ and $\varepsilon' \in \text{NOH}$:-

-Now,

$$\begin{tikzcd}
(\lambda x.\varepsilon')(x) \arrow[r, "\eta"] \arrow[d, "1\beta"'] & \delta x \arrow[d, "{\beta,\eta}"] \\
\varepsilon' \arrow[r, "\eta"] & \gamma
\end{tikzcd}$$

, by 0.4.16(v).

-Since $\delta \in \text{HEAD}$, we must have $\gamma = \delta' x \in \text{HEAD}$ and $\delta \xrightarrow{\beta,\eta} \delta'$.

-But, $\text{rank}\llbracket\varepsilon'\rrbracket = n-1$ and so $\varepsilon' \xrightarrow{\beta} \varepsilon'^* \in \text{HNF}$, by $A(n-1)$.

-Hence, $\varepsilon = \lambda x.\varepsilon' \xrightarrow{\beta} \lambda x.\varepsilon'^* = \varepsilon^* \in \text{HNF}$.

-Case 2: $\varepsilon = (\lambda x.\nu)\mu_1\mu_2\ldots\ldots\mu_s$, where $s \geq 1$:-

-We must have $\delta = \pi\mu_1'\mu_2'\ldots\ldots\mu_s'$, where $\lambda x.\nu \xrightarrow{\eta} \pi$ and $\mu_i \xrightarrow{\eta} \mu_i'$, for $1 \leq i \leq s$.

-Further, since $\delta \in \text{HEAD}$, so is $\pi \in \text{HEAD}$.

-As in case 1, we deduce that $\nu \xrightarrow{\eta} \pi' x$, where

$\pi \xrightarrow{\beta,\eta} \pi'$.

-Since $\lambda x.\nu \xrightarrow{\beta,\eta} \pi'$, x does not occur free in π'.

-Hence, $\pi' x$ has only one free occurrence of x.

-$\therefore$, ν has only one free occurrence of x, since η-reductions cannot cancel free x's.

-Let $\varepsilon' = ([\mu_1/x]\nu)\mu_2 \ldots\ldots \mu_s$ - i.e. $\varepsilon \xrightarrow{1\beta} \varepsilon'$.

-Then, $\text{rank}\llbracket\varepsilon'\rrbracket < n$, by 0.4.19(iii).

-Now,

$$\begin{tikzcd}
\varepsilon \arrow[r, "\eta"] \arrow[d, "1\beta"'] & \delta \arrow[d, "\beta,\eta"] \\
\varepsilon' \arrow[r, "\eta"] & \delta'
\end{tikzcd}$$

, by 0.4.16(v).

-Further, $\delta' \in$ HEAD, by the proof of 0.5.4.

-$\therefore$, by $A(n-1)$, $\varepsilon' \xrightarrow{\beta} \varepsilon^* \in$ HNF.

-Thus, $\varepsilon \xrightarrow{\beta} \varepsilon^* \in$ HNF.

-Since these are the only cases for ε, we have $A(n)$.

-$\therefore$, by induction, $(\forall n \geq 0)A(n)$.

(ii) -By induction on the (restricted to HNF) rank of δ, as defined in the proof of 0.5.2(ii). To avoid confusion with part (i) above, we shall rename this as the "arity" of δ.

-We use the following obvious properties which can be proved with a straightforward structural induction on HNF :-

$$(\varepsilon \in \text{HNF}) \wedge (\varepsilon \xrightarrow{1\beta} \varepsilon') \Rightarrow (\text{arity}\llbracket\varepsilon'\rrbracket = \text{arity}\llbracket\varepsilon\rrbracket)$$

$$(\varepsilon \in \text{HNF}) \wedge (\varepsilon \xrightarrow{1\eta} \varepsilon') \Rightarrow (\text{arity}\llbracket\varepsilon'\rrbracket \leq \text{arity}\llbracket\varepsilon\rrbracket).$$

-Let $P(n) \equiv (\varepsilon \xrightarrow{\eta} \delta \in \text{HNF}) \wedge (\text{arity}\llbracket\delta\rrbracket \leq n)$

$$\Rightarrow (\varepsilon \xrightarrow{\beta} \varepsilon^* \in \text{HNF}).$$

-By part (i), we have $P(0)$.

-<u>Claim:</u> $P(n-1) \Rightarrow P(n)$, for any $n > 0$:-

-Suppose $P(n-1)$.

-Let $\varepsilon \xrightarrow{\eta} \delta \in$ HNF and $\text{arity}\llbracket\delta\rrbracket = n$.

-Then, $\delta = \lambda a.\delta'$ and $\delta' \in$ HNF.

-If $\varepsilon \in \mathrm{HNF}$, then trivial; so, suppose $\varepsilon \in \mathrm{NOH}$.

-If $\varepsilon = (\lambda x.\nu)\mu_1\mu_2 \cdot\cdot\cdot\cdot\cdot \mu_s$, where $s \geq 1$, as in case 2 of part (i) above, then $\delta = \pi\mu_1'\mu_2' \cdot\cdot\cdot\cdot\cdot \mu_s'$ - $\cancel{X}$ to $\delta = \lambda a.\delta'$.

-∴, $\varepsilon = \lambda x.\varepsilon'$ and $\varepsilon' \in \mathrm{NOH}$.

-As in case 1 of part (i) above, we have $\varepsilon' \xrightarrow{\eta} \gamma$ and $(\lambda a.\delta')x \xrightarrow{\beta,\eta} \gamma$.

-<u>Case 1:</u> $\gamma = (\lambda a.\delta'')x$ and $\delta' \xrightarrow{\beta,\eta} \delta''$:-

-Let $\gamma' = [x/a]\delta''$.

-Now, $\delta' \in \mathrm{HNF} \Rightarrow \delta'' \in \mathrm{HNF} \Rightarrow \gamma' \in \mathrm{HNF}$, by 0.5.2 and 0.5.4.

-Further, $\mathrm{arity}\llbracket\gamma'\rrbracket = \mathrm{arity}\llbracket\delta''\rrbracket \leq \mathrm{arity}\llbracket\delta'\rrbracket = n-1$.

-<u>Case 2:</u>

$$\begin{tikzcd}
(\lambda a.\delta')x \arrow[r, "\beta{,}\eta"] & (\lambda a.\delta'')x \arrow[d, "1\beta \text{ or } 1\eta"] & \\
& {[x/a]\delta''} \arrow[r, "\beta{,}\eta"] & \gamma \text{ :-}
\end{tikzcd}$$

-Let $\gamma' = \gamma$.

-Again, $\gamma' \in \mathrm{HNF}$ and $\mathrm{arity}\llbracket\gamma'\rrbracket \leq n-1$.

-In either case, $\varepsilon' \xrightarrow{\beta,\eta} \gamma' \in \mathrm{HNF}$ and $\mathrm{arity}\llbracket\gamma'\rrbracket \leq n-1$.

-But, $\varepsilon' \xrightarrow{\beta} \omega \xrightarrow{\eta} \gamma'$, by 0.4.16(iii).

-∴, by $P(n-1)$, $\omega \xrightarrow{\beta} \omega^* \in \mathrm{HNF}$.

-∴, $\varepsilon = \lambda x.\varepsilon' \xrightarrow{\beta} \lambda x.\omega \xrightarrow{\beta} \lambda x.\omega^* = \varepsilon^* \in \mathrm{HNF}$.

-∴, $P(n)$.

-∴, by induction, $(\forall n \geq 0)P(n)$.

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0.5.9:THEOREM:-

$\text{SOL} = \{\varepsilon \in \text{EXP} \mid (\text{there exists } \varepsilon' \in \text{HNF})(\varepsilon \xrightarrow{\beta} \varepsilon')\}$.

Proof:-

-Let $\varepsilon \in \text{SOL}$.

-Then, $\varepsilon \xrightarrow{\beta,\eta} \varepsilon' \in \text{HNF}$.

-By 0.4.16(iii), $\varepsilon \xrightarrow{\beta} \gamma \xrightarrow{\eta} \varepsilon' \in \text{HNF}$.

-∴, by 0.5.8(ii), $\varepsilon \xrightarrow{\beta} \gamma \xrightarrow{\beta} \gamma^* \in \text{HNF}$.

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0.5.10:LEMMA:-

$(\nu \in \text{NOH}) \wedge (\nu \xrightarrow{1\beta} \nu') \wedge (\nu' \in \text{HNF}) \Rightarrow$ (the redex contracted in ν is leftmost-outermost).

Proof:-

-Let $\mathcal{A}[\![\nu]\!] \equiv (\nu \xrightarrow{1\beta} \nu') \wedge (\nu' \in \text{HNF}) \Rightarrow$ (the redex contracted in ν is the leftmost-outermost).

-Claim: $\mathcal{A}[\![(\lambda x.\varepsilon)(\delta)]\!]$:-

-Let $(\lambda x.\varepsilon)(\delta) \xrightarrow{1\beta} \nu'$ and $\nu' \in \text{HNF}$.

-Either $\nu' \xrightarrow{\alpha} (\lambda x.\varepsilon')(\delta)$ - ✗ - or $\nu' = (\lambda x.\varepsilon)(\delta')$ - ✗ - or $\nu' = [\delta/x]\varepsilon$ and the leftmost-outermost redex was contracted.

-Claim: $\mathcal{A}[\![\nu]\!] \Rightarrow \mathcal{A}[\![\nu(\varepsilon)]\!]$, $\forall \nu \in \text{NOH}$:-

-Let $\nu(\varepsilon) \xrightarrow{1\beta} \nu'$ and $\nu' \in \text{HNF}$.

-If $\nu' = \nu(\varepsilon')$, then ν' is still in NOH - ✗.

-If $\nu' = \mu(\varepsilon)$ and $\nu \xrightarrow{1\beta} \mu$, then $\mu \in \text{HEAD}$, since $\nu' \in \text{HNF}$.

-Hence, by $\mathcal{A}[\![\nu]\!]$, the leftmost-outermost redex of ν was contracted. But, this is the leftmost-outermost redex of $\nu(\varepsilon)$, since, if $\nu \in \lambda\text{I.EXP}$, then so is μ - ✗ to $\mu \in \text{HEAD}$, by 0.5.1(ii).

-The only other possibility is for $\nu(\varepsilon)$ to have been a β-redex which was the one contracted. But, this is leftmost-outermost.

-Claim: $\mathcal{A}[\![\nu]\!] \Rightarrow \mathcal{A}[\![\lambda x.\nu]\!]$, $\forall \nu \in \text{NOH}$:-

-Let $\lambda x.\nu \xrightarrow{1\beta} \nu'$ and $\nu' \in \text{HNF}$.

-Then, $\nu' \xrightarrow{\alpha} \lambda x.\mu$ and $\nu \xrightarrow{1\beta} \mu$.

-But, $\lambda x.\mu \in HNF$, by 0.5.2(ii), and so $\mu \in HNF$.

-Thus, by $\mathcal{A}\llbracket\nu\rrbracket$, the redex contracted is leftmost-outermost in ν, which is leftmost-outermost in $\lambda x.\nu$.

-$\therefore$, by structural induction, $(\forall\nu \in NOH)\mathcal{A}\llbracket\nu\rrbracket$.

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0.5.11:THEOREM:- (Wadsworth)

Let $\varepsilon \in SOL$. Then, the normal reduction sequence reaches an expression in HNF. Further, if we denote the first expression in HNF reached during the normal reduction sequence by ε^*, then :-

$$(\varepsilon \xrightarrow{\beta} \varepsilon')\wedge(\varepsilon' \in HNF) \Rightarrow (\varepsilon^* \xrightarrow{\beta} \varepsilon').$$

Proof:-

-Let $\varepsilon \in SOL$. If $\varepsilon \in HNF$, then trivial. Suppose not.

-Then, $\varepsilon \xrightarrow{\beta} \varepsilon' \in HNF$.

-By 0.4.27, there is a standard reduction sequence :-

$$\varepsilon = \varepsilon_0 \xrightarrow{1\beta} \varepsilon_1 \xrightarrow{1\beta} \ldots\ldots \xrightarrow{1\beta} \varepsilon_n = \varepsilon'.$$

-Let p be such that $\varepsilon_p \in NOH$ and $\varepsilon_{p+1} \in HNF$.

-Then, the redex contracted in ε_p is leftmost-outermost, by 0.5.10.

-But, if there were a redex, R_i, that was left-outside any of the redexes contracted in $\varepsilon_0, \varepsilon_1, \ldots\ldots, \varepsilon_{p-1}$, then it would have a unique residual that was left-outside of the one contracted in ε_p, by 0.4.25(ii) - $\cancel{X}$.

-$\therefore$, the sequence $\varepsilon \xrightarrow{\beta} \varepsilon_{p+1}$ is normal.

-Writing ε^* for ε_{p+1}, we see that $\varepsilon^* \xrightarrow{\beta} \varepsilon'$.

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0.5.12:COR:-

The predicate $P\llbracket\varepsilon\rrbracket$, $\equiv (\varepsilon \in SOL)$, is semi-decidable.

Proof:-

-Same as the proof of 0.4.29.

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0.5.13:COR:- (Wadsworth)

(i) $(\varepsilon \in \text{SOL}) \Longleftrightarrow (\lambda x.\varepsilon \in \text{SOL})$.

(ii) $(\varepsilon \in \text{INSOL}) \Rightarrow ([\delta/x]\varepsilon \in \text{INSOL}) \Rightarrow ((\lambda x.\varepsilon)(\delta) \in \text{INSOL})$.

(iii) $(\varepsilon \in \text{INSOL}) \Rightarrow (\varepsilon(\delta) \in \text{INSOL})$.

Proof:-

(i) -By 0.5.9, trivial.

(ii) -This is the tricky part.

-Let $P(i) \equiv (\varepsilon \in \text{NOH}) \wedge (\text{rank}(\varepsilon) = i) \wedge$(R is the leftmost-outermost redex in ε) $\wedge (\varepsilon \xrightarrow{\alpha} \varepsilon') \Rightarrow$ (the residual of R in ε' and $[\delta/x]\varepsilon$ is unique and still leftmost-outermost) $\wedge (\text{rank}([a/b]\varepsilon) = i)$.

-In $P(i)$, by "the residual of R in $[\delta/x]\varepsilon$" we mean "the residual of R relative to $(\lambda x.\varepsilon)(\delta) \xrightarrow{\beta} [\delta/x]\varepsilon$". Also, the "rank" is that defined on NOH in the proof of 0.5.2(iii).

-We leave the induction of $(\forall i \geq 0)P(i)$ as an exercise.

-Now, let $\varepsilon \in \text{NOH}$ and R be its leftmost-outermost redex.

-Then, by 0.4.14, we may construct the diagram :-

$$\begin{tikzcd}
(\lambda x.\varepsilon)(\delta) \arrow[r, "R"] \arrow[d] & (\lambda x.\varepsilon')(\delta) \arrow[d] \\
{[\delta/x]\varepsilon} \arrow[r, "R'"] & {[\delta/x]\varepsilon'},
\end{tikzcd}$$

where R′ is the residual of R which is unique and leftmost-outermost by the above.

-So, if $\varepsilon \in \text{INSOL}$, the normal reduction sequence from ε :-

$$\varepsilon = \varepsilon_0 \xrightarrow{1\beta} \varepsilon_1 \xrightarrow{1\beta} \varepsilon_2 \xrightarrow{1\beta} \ldots\ldots\ldots\ldots,$$

is non-terminating and never reaches HNF.

-By the above argument,

$$[\delta/x]\varepsilon_0 \xrightarrow{1\beta} [\delta/x]\varepsilon_1 \xrightarrow{1\beta} [\delta/x]\varepsilon_2 \xrightarrow{1\beta} \ldots\ldots\ldots\ldots,$$

is the normal reduction sequence from $[\delta/x]\varepsilon$.

-But, $(\forall i \geq 0)([\delta/x]\varepsilon_i \in \text{NOH})$, by 0.5.2(iii), since $\varepsilon_i \in \text{NOH}$.

-∴, by 0.5.11, $[\delta/x]\varepsilon \in \text{INSOL}$.

-Finally, $(\lambda x.\varepsilon)(\delta) \in$ INSOL, since it β-reduces to an element of INSOL, $[\delta/x]\varepsilon$, by remark 0.5.7.

(iii) -Let $\varepsilon \in$ INSOL. Again, consider the normal reduction sequence from ε, with notation as in part (ii).

-If $\varepsilon_i \notin \lambda$I.EXP, for all $i \geq 0$, then :-

$$\varepsilon_0(\delta) \xrightarrow{1\beta} \varepsilon_1(\delta) \xrightarrow{1\beta} \varepsilon_2(\delta) \xrightarrow{1\beta} \ldots\ldots\ldots\ldots,$$

is the non-terminating normal reduction sequence from $\varepsilon(\delta)$ and it remains in NOH, since $\varepsilon_i \in$ NOH, for all $i \geq 0$. Hence, $\varepsilon(\delta) \in$ INSOL, by 0.5.11.

-On the other hand, let p be the first number such that $\varepsilon_p \in \lambda$I.EXP.

-Then, $\varepsilon_p = \lambda x.\omega$ and $\omega \in$ NOH.

-Now, by part (ii) above, $(\lambda x.\omega)(\delta) \in$ INSOL.

-But, $\varepsilon(\delta) \xrightarrow{\beta} (\lambda x.\omega)(\delta)$, and so, again by remark 0.5.7, we have $\varepsilon(\delta) \in$ INSOL.

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0.5.14:REMARK:-

Theorem 0.5.9 shows that we need only look at β-reductions to decide if an expression is in SOL. Theorem 0.5.11 narrows this down to the deterministic procedure of checking the normal order sequence. It also says that the first HNF-form produced by this method is "minimal" in the sense that all other HNF-forms of the expression are reducible from it.

This might suggest that normal reductions provide an efficient way to get an HNF since it reaches this "minimum" one. However, normal reductions are inherently inefficient and we feel that whenever a normal reduction reaches an HNF, alternative paths to HNF may exist that are shorter. Further, the HNF thus produced will be "better" or "tell us more" than the "minimum" one found by the normal reduction.

As we mentioned in 0.4.26, we shall later be studying

β-reduction rules known as "inside-out" and a deterministic version called "i'th reductions". We shall see that these also provide a sufficient number of reductions to decide membership of SOL (6.8.2). These reductions are much more efficient than standard ones and may provide powerful tools for analysing reductions in general. To indicate what we mean, we shall prove again the important part (iii) of the above corollary: first using inside-out reductions :-

-Let $\varepsilon(\delta) \in$ SOL. Then, $\varepsilon(\delta) \longrightarrow \omega \in$ HNF, by 6.8.2(iii).

-$\therefore$, $\varepsilon(\delta) \longrightarrow \varepsilon'(\delta') \xrightarrow{\overset{\varepsilon'}{\nearrow}\quad\overset{\delta'}{\nearrow}} \omega$, by 6.1.9(i).

-By 6.1.12, $\varepsilon' \in$ NOH $\Rightarrow$ $\omega \in$ NOH - $\cancel{X}$.

-$\therefore$, $\varepsilon \longrightarrow \varepsilon' \in$ HNF and, so, $\varepsilon \in$ SOL.

and, secondly, using i'th reductions :-

-Let $\varepsilon(\delta) \in$ SOL. Then, $i<\varepsilon(\delta)> \in$ HNF, for some $i \geq 1$, by 6.8.2(iv).

-Now, $i<\varepsilon(\delta)> = Ap^i(i<\varepsilon>, i<\delta>)$.

-By 5.0.11, $i<\varepsilon> \in$ NOH $\Rightarrow$ $Ap^i(i<\varepsilon>, i<\delta>) \in$ NOH - $\cancel{X}$

-$\therefore$, $i<\varepsilon> \in$ HNF and, so, $\varepsilon \in$ SOL.

To finish this section, we relate SOL, which was defined from Wadsworth's notion of head normal form, with Barendregt's notion of "solvable" - [17].

0.5.15:DEF:- (Barendregt)

Let $\varepsilon \in$ EXP. Then, ε is SOLVABLE if there exist $\delta_1, \ldots\ldots, \delta_n \in$ EXP such that $\varepsilon\delta_1\delta_2\ldots\ldots\delta_n$ has a normal form.

0.5.16:LEMMA:-

(i) (ε is solvable) $\Rightarrow$ ($\varepsilon \in$ SOL).

(ii) ($\varepsilon \in$ SOL) $\not\Rightarrow$ (ε is solvable).

Proof:-

(i) -Let $\varepsilon \in$ INSOL.

-Then, by 0.5.13(iii), for all $\delta_1, \delta_2, \ldots\ldots, \delta_n \in$ EXP, we still have $\varepsilon\delta_1\delta_2\ldots\ldots\delta_n \in$ INSOL.

-In particular, $\varepsilon\delta_1\delta_2\ldots\ldots\delta_n$ does not have a normal form.

-$\therefore$, ε is not solvable.

(ii) - Consider $x(\Delta\Delta) \in$ HEAD. Then, $x(\Delta\Delta) \in$ SOL.

-But, $x(\Delta\Delta)\delta_1\delta_2\ldots\ldots\delta_n \xrightarrow{\beta} x(\Delta\Delta)\delta_1'\delta_2'\ldots\ldots\delta_n' \in$ NF. All reductions are of this form and, so, $x(\Delta\Delta)$ is not solvable.

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0.5.17:DEF:-

Let $\varepsilon \in$ EXP. Then, ε is <u>CLOSED</u> if $\{x \mid x$ is not free in $\varepsilon\} = I$.

0.5.18:LEMMA:-

(i) (ε is closed)$\wedge$($\varepsilon \xrightarrow{\beta} \varepsilon'$) $\Rightarrow$ (ε' is closed).

(ii) ($\varepsilon \in$ HEAD) $\Rightarrow$ (ε is not closed).

(iii) (ε is closed)$\wedge$($\varepsilon \in$ SOL) $\Rightarrow$ (ε is solvable).

Proof:-

(i) -By 0.3.13(x).

(ii) -Trivial structural induction on HEAD.

(iii) -Let ε be closed and in SOL. Then, $\varepsilon \xrightarrow{\beta} \varepsilon' \in$ HNF, where ε' is closed, by part (i), and so $\varepsilon' \notin$ HEAD, by part (ii).

-$\therefore$, ε' must be of the form :-

$$\lambda x_1.\lambda x_2.\ldots\ldots\ldots\lambda x_s.\lambda a.\lambda y_1.\ldots\ldots\ldots\lambda y_t.a\omega_1\omega_2\ldots\ldots\omega_n.$$

-<u>Define:</u> $K_i := \lambda c_1.\lambda c_2.\ldots\ldots\ldots\lambda c_i.b,\ \forall i \geq 0$.

-Then,
$$\begin{aligned}\varepsilon\delta_1\delta_2\ldots\ldots\delta_s K_n &\xrightarrow{\beta} \varepsilon'\delta_1\delta_2\ldots\ldots\delta_s K_n\\ &\xrightarrow{\beta} \lambda y_1'.\lambda y_2'.\ldots\ldots\lambda y_t'.K_n\omega_1'\omega_2'\ldots\ldots\omega_n'\\ &\xrightarrow{\beta} \lambda y_1'.\lambda y_2'.\ldots\ldots\lambda y_t'.b \in \text{NF}.\end{aligned}$$

-$\therefore$, ε is solvable.

⊨

0.5.19:COR:-

($\forall$ closed ε)(ε is solvable $\Leftrightarrow$ $\varepsilon \in$ SOL).

Proof:-

-By 0.5.16(i) and 0.5.18(iii).

$\dashv$

0.6:Lattice Theory:-

0.6.0:REMARK:-

The techniques of using lattice theory to build models of computing phenomena, in particular the λ-calculus, were realised by Scott. For a detailed description, including motivation, we refer to his papers - [18] - and to those of Reynolds - [19] - and Wadsworth - [20] . We confine ourselves here to a formal presentation of the ideas involved and brief motivation.

0.6.1:DEF:-

A relation, $\sqsubseteq$, on a set, S, is a PARTIAL ORDERING if it is :-

(i) REFLEXIVE : $(\forall a \in S)(a \sqsubseteq a)$

(ii) TRANSITIVE : $(\forall a,b,c \in S)((a \sqsubseteq b)\wedge(b \sqsubseteq c) \Rightarrow (a \sqsubseteq c))$ and

(iii) ANTI-SYMMETRIC: $(\forall a,b \in S)((a \sqsubseteq b)\wedge(b \sqsubseteq a) \Rightarrow (a = b))$.

Let X be a subset of S. Then, the LEAST UPPER BOUND of X, $\bigsqcup X$, is an element of S such that :-

(iv) $(\forall x \in X)(x \sqsubseteq \bigsqcup X)$ and

(v) $(\forall y \in S)((\forall x \in X)(x \sqsubseteq y) \Rightarrow (\bigsqcup X \sqsubseteq y))$.

The GREATEST LOWER BOUND of X, $\bigsqcap X$, is an element of S such that :-

(vi) $(\forall x \in X)(\bigsqcap X \sqsubseteq x)$ and

(vii) $(\forall y \in S)((\forall x \in X)(y \sqsubseteq x) \Rightarrow (y \sqsubseteq \bigsqcap X))$.

If X is a two element set, {x,y}, we write $x \sqcup y$ for $\bigsqcup X$ and $x \sqcap y$ for $\bigsqcap X$.

$<S,\sqsubseteq>$ is a LATTICE if, for all finite subsets, both the least upper bound and the greatest lower bound exist. It is a COMPLETE lattice if the least upper bound exists for all subsets.

0.6.2:LEMMA:-

(i) Let $\langle S, \sqsubseteq \rangle$ be a set with a partial ordering. Let X be a subset of X. Then, if either $\bigsqcup X$ or $\bigsqcap X$ exist, they are unique.

(ii) A finite lattice is complete.

(iii) In a complete lattice, greatest lower bounds of arbitrary subsets also exist.

(iv) In a complete lattice there exist unique top and bottom elements, $\top$ and $\bot$, such that $(\forall x \in S)(\bot \sqsubseteq x \sqsubseteq \top)$.

Proof:-

(i) -Let $\bigsqcup X$ exist and let l be another least upper bound.

-By 0.6.1(v), we have $\bigsqcup X \sqsubseteq l$ and $l \sqsubseteq \bigsqcup X$.

-$\therefore$, by anti-symmetry, 0.6.1(iii), $l = \bigsqcup X$.

-Similarly, for $\bigsqcap X$.

(ii) -Clear.

(iii) -Let $X \subseteq S$, a complete lattice.

-Then, $\bigsqcup \{y \mid (\forall x \in X)(y \sqsubseteq x)\}$ exists. But, this is $\bigsqcap X$.

(iv) -Let $\top = \bigsqcup S$ and $\bot = \bigsqcup \emptyset$.

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0.6.3:MOTIVATION:-

Within most "data-types" found in computing, there is a notion of "approximation" given by the "information content" of its elements. This corresponds precisely with that of a partial ordering. On simple types (e.g. natural numbers, truth values), this ordering is trivial (i.e. $x \sqsubseteq x$ only), but on the more complex ones (e.g. real numbers, character strings, sets, functions) the structure is correspondingly richer - see Reynolds [21].

The notion of taking least upper bounds is the "pooling" or "anding" of the information available and, so, we have complete lattices for our data-types to ensure we can always do this. Of course, this process might lead to inconsistencies and this is what the top element, $\top$, represents. The bottom element, $\bot$, represents what we usually have when we start a computation - i.e. no idea about the final outcome!

0.6.4:DEF:-

Let $<S,\sqsubseteq>$ and $<S',\sqsubseteq'>$ be sets with partial orderings. Then, a function, $f:S \rightarrow S'$, is MONOTONIC if $(\forall x,y \in S)((x \sqsubseteq y) \Rightarrow (f(x) \sqsubseteq' f(y)))$.

Let $X \subseteq S$. Then, X is DIRECTED subset if $(\forall x,y \in X)$(there exists $z \in X)(x,y \sqsubseteq z)$.

Let $<S,\sqsubseteq>$ and $<S',\sqsubseteq>$ be complete lattices. Then, a function, $f:S \rightarrow S'$, is CONTINUOUS if, for all directed subsets X of S :-

$$f(\bigsqcup X) = \bigsqcup f(X) = \bigsqcup\{f(x)\,|\,x \in X\}.$$

Further, f is COMPLETELY ADDITIVE if the above equation holds for any subset X.

0.6.5:LEMMA:-

(i) Completely additive $\Rightarrow$ continuous $\Rightarrow$ monotonic.

(ii) If S is finite, then continuous $\Leftrightarrow$ monotonic.

(iii) Let $<S,\sqsubseteq>$ be a complete lattice and $f:S \rightarrow S$ be monotonic. Then, f has a minimal fixed point, $\mu f \in S$ - i.e. :-

$$(f(\mu f) = \mu f)$$

$$\text{and } (\forall s \in S)((f(s) = s) \Rightarrow (\mu f \sqsubseteq s)).$$

(iv) If f is continuous in part (iii), then μf is given, constructively, by :-

$$\mu f = \bigsqcup\{f^i(\bot)\,|\,i \geq 0\}.$$

Proof:-

(i) -Trivial.

(ii) -Clearly, if X is directed and finite, then $\bigsqcup X \in X$.

-Hence, the result.

(iii) -μf is given by $\sqcap\{s \in S\,|\,f(s) \sqsubseteq s\}$. (This is the Tarski fixed-point theorem.)

(iv) -Trivial.

†

0.6.6:MOTIVATION:-

Programs define computable functions between data-types. If we give more information about the data to the program, it is sensible to expect more information about the answer - or, at least, not less or inconsistent information. Hence, we expect computable functions to be monotonic.

The simplest example of a directed set is a chain,

$$\{x_i | x_i \sqsubseteq x_{i+1}, \forall i \geq 0\}.$$

We will often have an "infinite" object represented by a chain of "finite" "approximations" which gradually "tend" to the object. This is because a computer can only hold a finite amount of information at a time. For instance, we could represent the real number π by giving a (directed) sequence of its decimal expansions,

$$\{3,3.1,3.14,3.142,......\},$$

where we read "3.14" as "somewhere between 3.135 and 3.145". Note that the boundaries are exactly and finitely specified. We will make precise this notion of "finite approximation" presently.

Now, the sensible way for a computer to evaluate a function on an infinite object, $\bigsqcup\{x_i | i \geq 0\}$, is to evaluate it on the finite approximations, x_i, and hope that this leads to the correct answer - i.e. that :-

$$f(\bigsqcup\{x_i | i \geq 0\}) = \bigsqcup\{f(x_i) | i \geq 0\}.$$

This hope is not forlorn if f is continuous and, so, continuity is another property we expect computable functions to have. The fact that they then have minimal fixed points enables us to characterise programming phenomena like looping and recursion. We shall not be concerned with this in this thesis, but there is a connection with the fixed-point combinators, Y_i - see section 7.1.

We do not expect computable functions to be completely additive.

0.6.7:DEF:- (Scott)

Let $<S,\sqsubseteq>$ be a complete lattice. Then, we define the set of subsets, OP, of S by $X \in OP$ if :-

(i) $(x \in X)_{\wedge}(x \sqsubseteq y) \Rightarrow (y \in X)$

and (ii) $(D \text{ is directed})_{\wedge}(\bigsqcup D \in X) \Rightarrow (D \cap X \neq \emptyset)$.

0.6.8:REMARK:-

We do not propose to discuss topology in this thesis. However, OP does form a set of open sets which gives S a T_0-topology under which the topological notion of continuity does correspond with that defined in 0.6.4. For the development of the theory of partial orderings, complete and continuous (see below) lattices from the point of view of T_0-topological spaces, we refer to Scott - [22]. We only defined the class OP so as to introduce the following concepts.

0.6.9:DEF:- (Scott)

Let $<S,\sqsubseteq>$ be a complete lattice. Then, we define the TOPOLOGICAL ORDERING, $\prec$, as follows :-

$(x \prec y)$ if (there exists $X \in OP)(X \subseteq \{z|x \sqsubseteq z\})_{\wedge}(y \in X)$.

Then, x is ISOLATED if $x \prec x$. Finally, the lattice is CONTINUOUS if,

$(\forall y \in S)(y = \bigsqcup\{x|x \prec y\})$.

0.6.10:MOTIVATION:-

In our data-types, we have many infinite objects. In order to be able to compute over them, as described in 0.6.6, we have to be able to represent them by a sequence of finite objects. The relation "$\prec$" can be read as "is a finite approximation of" - for instance, with sets, $X \prec Y$ iff X is a finite subset of Y; while, with partial functions, $f \prec g$ iff g is defined and agrees with f whenever f is defined and the domain of f is finite. Hence, we expect our data-types to be continuous.

0.6.11:LEMMA:- (Scott)

Let $\langle S, \sqsubseteq \rangle$ be a complete lattice. Then,

(i) $\perp \prec x$,

(ii) $x \prec y \Rightarrow x \sqsubseteq y$,

(iii) $x \prec y \sqsubseteq z \Rightarrow x \prec z$,

(iv) $x \sqsubseteq y \prec z \Rightarrow x \prec z$,

(v) $x \prec z$ and $y \prec z \Rightarrow x \sqcup y \prec z$,

(vi) x is isolated $\Leftrightarrow \{z \mid x \sqsubseteq z\} \in OP$,

(vii) if D is directed, then $x \prec \bigsqcup D \Leftrightarrow x \prec d$, for some $d \in D$.

Further, if S is continuous, then,

(viii) $x \prec y \Rightarrow x \prec z \prec y$, for some $z \in S$,

(ix) $(x \sqsubseteq y) \Leftrightarrow (z \prec x \Rightarrow z \prec y)$,

(x) $(x \prec y) \Leftrightarrow ((D \text{ is directed}) \wedge (y \sqsubseteq \bigsqcup D) \Rightarrow (x \sqsubseteq d) \wedge (d \in D))$,

(xi) the function, $f_x : S \longrightarrow S$

$y \longmapsto x \sqcap y$, is continuous.

Also, parts (viii) and (ix) together imply S is continuous.

Proof:-

-See Scott [23].

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0.6.12:DEF:-

Let $\langle S, \sqsubseteq \rangle$ be a set with a partial ordering. Then, it has the ASCENDING CHAIN CONDITION, ACC, if for all ascending chains,

$$s_0 \sqsubseteq s_1 \sqsubseteq s_2 \sqsubseteq s_3 \ldots\ldots\ldots\ldots\ldots,$$

there is a finite number, $n \geq 0$, such that $s_i = s_n$, for all $i \geq n$.

0.6.13:THEOREM:-

Let $\langle S, \sqsubseteq \rangle$ be a countable lattice with the ACC. Then, it is complete and, if D is a directed subset, then $\bigsqcup D \in D$. Thus, S is continuous with all its elements isolated. Further, if $\langle S', \sqsubseteq' \rangle$ is a complete lattice and $f : S \rightarrow S'$ is monotonic, then f is also continuous.

Proof:-

-Let $D = \{d_0, d_1, \ldots\ldots\} \subseteq S$.

-Define: $\mu_0 := d_0$ and $\mu_{i+1} := \mu_i \sqcup d_{i+1}$.

-Then, $\mu_0 \sqsubseteq \mu_1 \sqsubseteq \mu_2 \sqsubseteq \ldots\ldots$, clearly.

-By the ACC, there are only finitely many different μ_i's.

-Let π be the largest one.

-Clearly, $(\forall i \geq 0)(d_i \sqsubseteq \mu_i \sqsubseteq \pi)$.

-Suppose, $(\forall i \geq 0)(d_i \sqsubseteq \sigma)$.

-Then, by a trivial induction, $(\forall i \geq 0)(\mu_i \sqsubseteq \sigma)$ - i.e. $\pi \sqsubseteq \sigma$.

-$\therefore$, $\pi = \bigsqcup D$ and, so, S is complete.

-Now, suppose D is directed.

-But, $\pi = \mu_k$, for some $k \geq 0$.

$\quad = d_0 \sqcup d_1 \sqcup \ldots\ldots \sqcup d_k$.

-Since D is directed, there exists $n \geq 0$ such that $\pi \sqsubseteq d_n$.

-$\therefore$, $\pi = d_n \in D$ - i.e. $\bigsqcup D \in D$.

-Thus, part (ii) of definition 0.6.7 is always true.

-i.e. $(\forall s \in S)(\{z | s \sqsubseteq z\} \in OP)$.

-i.e. $(\forall s \in S)$(s is isolated), by 0.6.11(vi).

-i.e. $(\forall s \in S)(s = \bigsqcup\{x | x \prec s\})$, since $s \prec s$.

-i.e. S is continuous.

-Further, if $f : S \rightarrow S'$ is monotonic and D is directed in S, then $f(\bigsqcup D) \sqsupseteq \bigsqcup f(D)$.

-But, $f(\bigsqcup D) = f(d) \in f(D)$.

-$\therefore$, $f(\bigsqcup D) \sqsubseteq \bigsqcup f(D)$ - and, so, $f(\bigsqcup D) = \bigsqcup f(D)$ - i.e. f is continuous.

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0.6.14: REMARK:-

The ACC allows lattices to have "downwards pointing limit points" but not "upwards pointing" ones. They can be, however, infinitely "broad". The lattices we construct in chapter 1 have the stronger property of "finite depth" (see below), in which "d.p.l.p.'s" are not allowed either.

0.6.15:DEF:-

Let $<S,\sqsubseteq>$ be a set with a partial ordering. Then, it has FINITE DEPTH if there exists a positive integer n such that :-

$$s_0 \sqsubseteq s_1 \sqsubseteq \ldots\ldots \sqsubseteq s_n \Rightarrow s_i = s_j, \text{ for some } 0 \leq i < j \leq n.$$

In particular, it has a FINITE DEPTH OF n if n is the least such number.

0.6.16:LEMMA:-

Finite => finite depth => ACC.

Proof:-

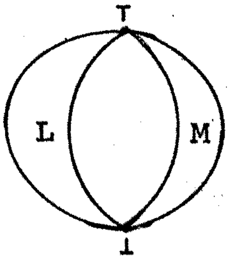
-Trivial.

†

0.6.17:DEF:-

Let A be a set and $<L,\sqsubseteq>$, $<M,\sqsubseteq'>$ be lattices. Then,

(i) L + M is the disjoint union of L and M, identifying the $\top$'s and $\bot$'s. We induce an ordering, $\sqsubseteq''$, on it as suggested by the picture :-



(ii) L × M is the cartesian product of L and M. Then, we define $(a,b) \sqsubseteq'' (c,d)$ iff $a \sqsubseteq c$ and $b \sqsubseteq' d$.

(iii) (A → M) is the set of functions from A to M. We define $f \sqsubseteq'' g$ iff $f(a) \sqsubseteq' g(a)$, for all $a \in A$.

(iv) {L → M} is the set of monotonic functions from L to M. Same ordering as in part (iii).

(v) *$[L \to M]$* is the set of continuous functions from L to M. Same ordering as in part (iii).

0.6.18:THEOREM:- (Scott)

(i) All the above orderings are partial orderings that make $L + M$, $L \times M$, $(A \to M)$, and $\{L \to M\}$ into lattices.

(ii) If L and M are complete, then so are $L + M$, $L \times M$, $(A \to M)$, $\{L \to M\}$ and $[L \to M]$.(*)

(iii) If L and M are continuous, then so are $L \times M$ and $[L \to M]$. Further, if $\top$ is isolated in both L and M, then $L + M$ is continuous and $\top$ is isolated in $L + M$ and $L \times M$.

(iv) Let S be another complete lattice. Then :-

$(f \in [L \times M \to S]) \iff (\forall l \in L)(f_l \in [M \to S]) \wedge (\forall m \in M)(f_m \in [L \to S])$,

where,

$$f_l : M \longrightarrow S \qquad \text{and } f_m : L \longrightarrow S$$
$$m \longmapsto f(l,m) \qquad\qquad l \longmapsto f(l,m) .$$

Proof:-

-See Scott [24].

†

0.6.19:DEF:-

Let $<L,\sqsubseteq>$ and $<M,\sqsubseteq'>$ be complete lattices. Then, L is a *RETRACTION* of M if there exist maps $f \in [L \to M]$ and $g \in [M \to L]$ such that $g \circ f = id_L$, where id_L is the identity map in $[L \to L]$. Further, L is a *PROJECTION* of M if we also have $f \circ g \sqsubseteq id_M \in [M \to M]$. In this case, we write :-

$$L \underset{g}{\overset{f}{\lhd}} M ,$$

although we usually omit the maps, f and g, when it is clear what they are. Note that f and g are not neccessarily unique. We call f the *inclusion* mapping and g the *projection* mapping.

(*) Also, the ACC and finite depth are preserved by + and ×.

0.6.20:THEOREM:-

Let L and M be complete continuous lattices. Then, $L \lhd L + M$, $L \lhd L \times M$ and $L \lhd [L \to L]$.

Proof:-

-First two parts, obvious.

-Define: $f : L \longrightarrow [L \to L]$, where $f_1 : L \longrightarrow L$.

$$1 \longmapsto f_1 \qquad\qquad x \longmapsto 1$$

-Define: $g : [L \to L] \longrightarrow L$.

$$h \longmapsto h(\perp)$$

-Then, $L \underset{g}{\overset{f}{\lhd}} [L \to L]$.

-N.B: there exist other suitable pairs, <f,g> - see Scott [54].

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0.6.21:DEF:-

Let $<(L_i, \phi_{i,i+1}, \phi_{i+1,i})>_{i=0}^{\infty}$ be a sequence of complete lattices such that,

$$L_i \lhd L_{i+1},$$

by $\phi_{i,i+1} \in [L_i \to L_{i+1}]$ and $\phi_{i+1,i} \in [L_{i+1} \to L_i]$.

Then, the INVERSE LIMIT, L_∞, is defined by :-

$$L_\infty := \{<l_0, l_1, l_2, \ldots\ldots\ldots\ldots> | l_i \in L_i \text{ and } l_i = \phi_{i+1,i}(l_{i+1})\}.$$

We define an ordering on L_∞ by :-

$$<l_i>_{i=0}^{\infty} \leq <l_i'>_{i=0}^{\infty} \text{ if } (\forall i \geq 0)(l_i \leq l_i').$$

We define $\phi_{i,j} \in [L_i \to L_j]$ by composition of the maps we already have, with $\phi_{i,i}$ being the identity. Further,

$$\phi_{i,\infty} : L_i \longrightarrow L_\infty$$
$$l_i \longmapsto <\phi_{i,j}(l_i)>_{j=0}^{\infty}$$

and,

$$\phi_{\infty,i} : L_\infty \longrightarrow L_i .$$
$$<l_j>_{j=0}^{\infty} \longmapsto l_i$$

Finally, the DIRECT LIMIT is $\bigcup_{i=0}^{\infty} \phi_{i,\infty}(L_i)$.

0.6.22:THEOREM:- (Scott)

(i) In the above definition, L_∞ is a complete lattice where, if S is any subset of L_∞, then :-

$$\bigsqcup S = \langle \bigsqcup_{j=0}^{\infty} \phi_{j,i}(\bigsqcup\{\phi_{\infty,j}(s) | s \in S\}) \rangle_{i=0}^{\infty},$$

and where $\{\phi_{j,i}(\bigsqcup\{\phi_{\infty,j}(s) | s \in S\}) | j \geq 0\}$ is a directed chain in L_i. In particular, if D is a directed subset of L_∞, then :-

$$\bigsqcup D = \langle \bigsqcup\{\phi_{\infty,i}(d) | d \in D\} \rangle_{i=0}^{\infty}.$$

(ii) The maps $\phi_{i,j}$, $\phi_{i,\infty}$ and $\phi_{\infty,i}$ are continuous.

(iii) For all $i \leq j$, $L_i \lhd L_j$ and $L_i \lhd L_\infty$.

(iv) For all $l \in L_\infty$, $l = \bigsqcup_{i=0}^{\infty} \phi_{i,\infty} \circ \phi_{\infty,i}(l)$, the limit of a directed chain.

(v) Let S be a complete lattice. Then,

$$(f \in [S \to L_\infty]) \iff (\forall i \geq 0)(\phi_{\infty,i} \circ f \in [S \to L_i]).$$

(vi) If L_i is continuous, for all $i \geq 0$, then so is L_∞.

Proof:-

(i),(ii),(iii) and (iv) -Trivial.

(v) $(\Rightarrow)$ -Trivial, since composition preserves continuity.

$(\Leftarrow)$ -Now,

$$f(s) = \bigsqcup_{i=0}^{\infty} \phi_{i,\infty} \circ \phi_{\infty,i} \circ f(s), \text{ by part (iv) above.}$$

-Then, f is continuous, by 0.6.18(ii), since it is the limit of continuous maps.

(vi) -See Scott [25].

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0.6.23:DEF:-

Let $\langle S, \sqsubseteq \rangle$ be a complete lattice. Then, a subset, B, of S is a SUB-BASIS of S if, for all $s \in S$:-

$$s = \bigsqcup\{b \in B | b \sqsubseteq s\}.$$

Further, it is a BASIS of S if it is closed under the taking of least upper bounds of finite subsets - i.e. if :-

$$(B' \subseteq B) \wedge (B' \text{ is finite}) \Rightarrow (\bigsqcup B' \in B).$$

We call the lattice COUNTABLY BASED if it has a countable basis.

0.6.24:LEMMA:-

Let $\langle S, \sqsubseteq \rangle$ be a complete lattice. Then,

(i) S is a basis of S,

(ii) S is countably based $\Leftrightarrow$ S has a countable sub-basis,

(iii) S is countably based $\Rightarrow$ S has cardinality at most that of the continuum,

(iv) If B is a basis of S, both $\{b \in B | b \sqsubseteq s\}$ and $\{b \in B | b \prec s\}$ are directed subsets of S,

(v) Let $B \subseteq S$. Then, S is a continuous lattice with sub-basis B $\Leftrightarrow$ $(\forall s \in S)(s = \bigsqcup\{b \in B | b \prec s\})$,

(vi) If $\langle L, \sqsubseteq' \rangle$ is a complete lattice and both S and L are countably based, then it does not necessarily follow that $[S \rightarrow L]$ is countably based.

Proof:-

(i),(ii) and (iii) -Trivial.

(iv) -First part is trivial and the second is true by 0.6.11(v).

(v) ($\Rightarrow$) -Let S be continuous and have a sub-basis B.

-Then, $s = \bigsqcup\{x | x \prec s\}$, by continuity.

$= \bigsqcup\{\bigsqcup\{b \in B | b \sqsubseteq x\} | x \prec s\}$

$= \bigsqcup\{b \in B | b \sqsubseteq x \prec s\} \sqsubseteq \bigsqcup\{b \in B | b \prec s\}$, by 0.6.11(iv).

$\sqsubseteq s$, by 0.6.11(ii). Hence, we have equality throughout.

(<=) -Now, $s = \bigsqcup\{b \in B | b \prec s\} \sqsubseteq \bigsqcup\{x | x \prec s\} \sqsubseteq s$, by 0.6.11(ii).

-Hence, we have equality and S is continuous.

-Also, $s = \bigsqcup\{b \in B | b \prec s\} \sqsubseteq \bigsqcup\{b \in B | b \sqsubseteq s\}$, again by 0.6.11(ii).

$\sqsubseteq s$.

-Hence, we have equality and B is a sub-basis.

(vi) -See Reynolds - [26] - for a counter-example.

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0.6.25:LEMMA:-

(i) Let L and M be complete continuous countably based lattices Then, so are $L \times M$, $[L \to M]$ and, provided the $\top$'s are isolated, $L + M$.

(ii) Let $L_i \lhd L_{i+1}$, for all $i \geq 0$, where each L_i is a complete continuous lattice with countable basis B_i such that $\phi_{i,i+1}(B_i) \subseteq B_{i+1}$. Then, the inverse limit, L_∞, is countably based by,

$$\bigcup_{i=0}^{\infty} \phi_{i,\infty}(B_i).$$

Hence, if L_i is countable for all $i \geq 0$, the direct limit forms a countable basis for the inverse limit.

Proof:-

-See Reynolds [27].

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0.6.26:MOTIVATION:-

We expect our data-types to be countably based since elements of a countable set are something that can be represented exactly and finitely in a computer - e.g. by using their numerical position in some enumeration. Then, by 0.6.24(v), the computer can get at any member of the data-type by a continuous approximation of these "finite" "basic" elements. Notice that, by 0.6.24(iii) and 0.6.25, the cardinality of our data-types is always bounded by the continuum, no matter to what heights of functionality we rise.

We have listed the properties of lattices we shall be using in this thesis. However, in section 7.5, we shall consider briefly semi-lattices, and so we end this section with them. We use a more richly structured notion of semi-lattice than that usually defined, in that we insist on l.u.b.'s if some upper bound exists.

0.6.27:DEF:-

Let $<S, \sqsubseteq>$ be a set with a partial ordering. Then, S is a SEMI-LATTICE if, for all non-empty finite subsets, the greatest lower bound exists and, provided some upper bound exists, so does the least upper bound.

Further, S is DIRECTEDLY COMPLETE if the least upper bound of all directed subsets exists. In this case, we define continuous functions, open sets, $\prec$, isolated elements and continuity in the same way as for complete lattices.

0.6.28:LEMMA:-

(i) If S is a semi-lattice, we can extend it to a lattice by adjoining a "top" element.

(ii) If S is a directedly complete semi-lattice, then the greatest lower bound of all non-empty subsets exists. Further, the lattice extension, $S \cup \{\top\}$, is complete.

Proof:-

(i) -In $S \cup \{\top\}$, we extend the partial ordering so that $s \sqsubseteq \top$, for all $s \in S$.

-Let $a,b \in S$. If a and b have a common upper bound in S, then $a \sqcup b$ exists. Otherwise, $\top$ is the only upper bound in $S \cup \{\top\}$ and, so, $a \sqcup b = \top$.

-Hence, $S \cup \{\top\}$ is a lattice.

(ii) -Let $X \subseteq S$, where S is a directedly complete semi-lattice and X is non-empty.

-Let $a,b \sqsubseteq x$, for some $x \in X$. Then, $a \sqcup b$ exists, since S is a semi-lattice, and $a \sqcup b \sqsubseteq x$.

-Thus, the set $\{a \mid (\forall x \in X)(a \sqsubseteq x)\}$ is directed - in fact, it is closed under $\sqcup$. (Note that if X were empty, there is no guarantee that $a \sqcup b$ exists.)

-Hence, $\bigsqcup\{a \mid (\forall x \in X)(a \sqsubseteq x)\}$ exists and, as in 0.6.2(iii), is $\sqcap X$.

-Now, let $X \subseteq S \cup \{\top\}$.

-If $X = \emptyset$ or $\top \in X$, then $\sqcap X$ is given by $\top$.

-Otherwise, X is a non-empty subset of S.

-So, by the above argument, $\sqcap X$ exists.

-$\therefore$, $(\forall X \subseteq S \cup \{\top\})(\sqcap X$ exists$)$.

-$\therefore$, $(\forall X \subseteq S \cup \{\top\})(\sqcup X$ exists and is given by $\sqcap\{a \mid (\forall x \in X)(x \sqsubseteq a)\})$.

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0.6.29:LEMMA:-

Let $<S,\sqsubseteq>$ be a lattice and let $S' \subseteq S$ and be closed under $\sqsupseteq$ - i.e. :-

$$(a \in S') \wedge (b \sqsubseteq a) \Rightarrow (b \in S').$$

Then,

(i) S' is a semi-lattice.

Now, suppose S is complete and S' is directedly complete. Then,

(ii) (X is open in S) => ($X \cap S'$ is open in S'),

(iii) $(\forall x',y' \in S')((x' \prec y'$ in $S) \Rightarrow (x' \prec y'$ in $S'))$.

Further, if S is continuous, then,

(iv) S' is continuous,

(v) (X' is open in S') <=> (there exists X open in S)($X' = X \cap S'$),

(vi) $(\forall x',y' \in S')((x' \prec y'$ in $S) \Leftrightarrow (x' \prec y'$ in $S'))$.

Proof:-

(i) -We assume that $S' \neq \emptyset$ - otherwise all parts are trivial.

-Then, $\bot \in S'$.

-Let X be a finite subset of S'.

-If X has an upper bound, z say, then $\sqcup X$ in S is such that $\sqcup X \sqsubseteq z$.

Thus, $\bigsqcup X \in S'$.

-Now, suppose X is non-empty and take $\bigsqcap X$ in S. Then, there is an $x \in X$ such that $\bigsqcap X \sqsubseteq x$ and, so, $\bigsqcap X \in S'$ since $X \subseteq S'$.

(ii) -Let X be open in S.

-Let $x \in X \cap S'$ and $x \sqsubseteq y \in S'$. Then, $y \in X \cap S'$.

-Let D be directed in S' and $\bigsqcup D \in X \cap S'$.

-Then, D is directed in S and $\bigsqcup D \in X$.

-$\therefore$, $D \cap X \neq \emptyset$ - i.e. (there exists $d \in D$)($d \in X$).

-But, $d \sqsubseteq \bigsqcup D \in S'$, and so $d \in S'$.

-$\therefore$, $D \cap (X \cap S') \neq \emptyset$.

-$\therefore$, $X \cap S'$ is open in S'.

(iii) -Let $x', y' \in S'$ and $x' \prec y'$ in S.

-Then, there exists X, open in S, such that $y' \in X$ and $X \subseteq \{z \in S \mid x' \sqsubseteq z\}$.

-By part (ii), $X \cap S'$ is open. But, $y' \in X \cap S'$ and $X \cap S' \subseteq \{z' \in S' \mid x' \sqsubseteq z'\}$.

-$\therefore$, $x' \prec y'$ in S'.

(iv)-Suppose S is continuous and let $s' \in S'$.

$$
\begin{aligned}
\text{-Then, } s' &= \bigsqcup\{x \mid x \prec s' \text{ in } S\} \\
&\sqsubseteq \bigsqcup\{x \mid x \prec s' \text{ in } S'\}, \text{ by part (iii).} \\
&\sqsubseteq s', \text{ by 0.6.11(ii).}
\end{aligned}
$$

-Hence, we have equality throughout and S' is continuous.

(v) (<=) -By part (ii).

(=>) -Let X' be open in S'.

-Let X be its closure in S under $\sqsubseteq$.

-Clearly, if $x \in X$ and $x \sqsubseteq y$, then $y \in X$.

-Let D be directed in S and $\bigsqcup D \in X$.

-Then, $x' \sqsubseteq \bigsqcup D$, for some $x' \in X'$, by the construction of X.

-Let $D' := \{d \sqcap x' \mid d \in D\}$.

-Then, D' is directed and, by 0.6.11(xi), $\bigsqcup D' = (\bigsqcup D) \sqcap x' = x'$.

-$\therefore$, (there exists $d' \in D'$)($d' \in X'$).

-$\therefore$, (there exists $d \in D$)($d \sqcap x' \in X'$).

-$\therefore$, (there exists $d \in D$)($d \in X$), since $d \sqcap x' \sqsubseteq d$.

-$\therefore$, X is open in S.

-Clearly, $X' = X \cap S'$, since X' is closed under $\sqsubseteq$ in S'.

(vi) ($\Rightarrow$)-By part (iii).

($\Leftarrow$) -Let $x', y' \in S'$ and $x' \prec y'$ in S'.

-Then, there exists X', open in S', such that $y' \in X'$ and $X' \subseteq \{z' \in S' | x' \sqsubseteq z'\}$.

-Define X as in part (v) and, then, X is open in S, $y' \in X$ and $X \subseteq \{z \in S | x' \sqsubseteq z\}$.

-$\therefore$, $x' \prec y'$ in S.

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0.6.30:MOTIVATION:-

So, we see that in restricting our data-types to semi-lattices we make them like the "bottom halves" of lattices. The "incorrect" or "inconsistent" elements are not represented and, so, only "consistent" information is allowed to be "pooled". This may be a sensible thing to do. However, since the theory goes through for lattices, we might as well use them - we can always "prune" them down to a suitable semi-lattice afterwards, as is done in 7.5.

0.7:Semantics:-

0.7.0:DEF:-

An ordered pair, $<S,f>$, is a SEMANTICS of the λ-calculus if S is a set and $f : EXP \longrightarrow S$. It induces a SEMANTIC EQUIVALENCE, $\equiv_{<S,f>}$, on EXP by :-

$$\epsilon \equiv_{<S,f>} \delta \text{ if } f[\![\epsilon]\!] = f[\![\delta]\!].$$

We define an ordering, $\leq$, on semantics by inclusion of the induced semantic equivalences - i.e. :-

$$\langle S,f\rangle \le \langle S',f'\rangle \text{ if } \equiv_{\langle S,f\rangle} \subseteq \equiv_{\langle S',f'\rangle},$$

where we are considering the semantic equivalences as subsets of EXP × EXP. If we have $\langle S,f\rangle \le \langle S',f'\rangle$ and $\langle S',f'\rangle \le \langle S,f\rangle$, then we say the two semantics are <u>EQUIVALENT</u> and write $\langle S',f'\rangle \equiv \langle S,f\rangle$.

We say the semantics $\langle S',f'\rangle$ is <u>DERIVABLE</u> from $\langle S,f\rangle$ if there is a function, $g : S \to S'$, such that $f' = g \circ f$. Further, if S and S′ are topological spaces and g is continuous, then $\langle S',f'\rangle$ is <u>CONTINUOUSLY DERIVABLE</u> from $\langle S,f\rangle$.

<u>0.7.1:LEMMA:-</u>

(i) A semantic equivalence is an equivalence relation on EXP.

(ii) $\equiv$ is an equivalence relation on semantics.

(iii) $\le$ is a partial ordering on semantics modulo $\equiv$.

(iv) Let ρ be an equivalence relation on EXP and $[\rho]$ be the map that produces equivalence classes. Then, $\langle EXP/\rho, [\rho]\rangle$ is a semantics of the λ-calculus.

(v) $\langle S,f\rangle \equiv \langle EXP/\equiv_{\langle S,f\rangle}, [\equiv_{\langle S,f\rangle}]\rangle$.

(vi) If $\langle S,f\rangle$ is a semantics, then f(EXP) and $EXP/\equiv_{\langle S,f\rangle}$ are isomorphic as sets. Further, if S has a partial ordering, then we can induce a partial ordering on $EXP/\equiv_{\langle S,f\rangle}$ such that the isomorphism is monotonic in both directions. Also, if f(EXP) is a directedly complete semi-lattice, then so is $EXP/\equiv_{\langle S,f\rangle}$ and the isomorphism is continuous in both directions.

(vii) Derivable $\Longleftrightarrow \ge$.

(viii) $\langle EXP, id\rangle$, where $id[\![\varepsilon]\!] := \varepsilon$, and $\langle \{*\}, const\rangle$, where $const[\![\varepsilon]\!] := *$, are the minimal and maximal semantics - i.e. :-

$$\langle EXP, id\rangle \le \langle S,f\rangle \le \langle \{*\}, const\rangle.$$

(ix) $$\langle EXP/\xrightarrow{\alpha}, [\xrightarrow{\alpha}]\rangle \le \langle EXP/\underline{cnv}, [\underline{cnv}]\rangle \le \langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}, [\beta\text{-}\eta\text{-}\underline{cnv}]\rangle.$$

<u>Proof:-</u>

(i),(ii),(iii),(iv) and (v) -Trivial.

(vi) -<u>Define</u> $p : EXP/\equiv_{<S,f>} \longrightarrow f(EXP)$.

$$[\varepsilon] \longmapsto f[\![\varepsilon]\!]$$

-Clearly, p is well-defined, 1-1 and onto.

-Hence, p is a set isomorphism with inverse p^{-1}.

-If S has a partial ordering, $\sqsubseteq$, we induce an ordering, $\sqsubseteq_{<S,f>}$, on $EXP/\equiv_{<S,f>}$ by :-

$$\varepsilon \sqsubseteq_{<S,f>} \delta \quad \text{if } f[\![\varepsilon]\!] \sqsubseteq f[\![\delta]\!].$$

-Clearly, this ordering is well-defined and makes p and p^{-1} monotonic.

-Let f(EXP) be directedly complete. Let D be a directed set in $EXP/\equiv_{<S,f>}$. <u>Claim:</u> $\bigsqcup D$ is given by $p^{-1}(\bigsqcup p(D))$:-

-Since p is monotonic, p(D) is directed in f(EXP) and so $p^{-1}(\bigsqcup p(D))$ exists.

-Now, $d \in D \Rightarrow p(d) \in p(D) \Rightarrow p(d) \sqsubseteq \bigsqcup p(D) \Rightarrow d \sqsubseteq p^{-1}(\bigsqcup p(D))$, since p^{-1} is monotonic.

-And, $(\forall d \in D)(d \sqsubseteq z) \Rightarrow (\forall d \in D)(p(d) \sqsubseteq p(z))$
$\Rightarrow (\bigsqcup p(D) \sqsubseteq p(z))$
$\Rightarrow (p^{-1}(\bigsqcup p(D)) \sqsubseteq z)$.

-Hence, claim is established.

-$\therefore$, $EXP/\equiv_{<S,f>}$ is directedly complete.

-Clearly, $p(\bigsqcup D) = p \circ p^{-1}(\bigsqcup p(D)) = \bigsqcup p(D)$.

-$\therefore$, p is continuous.

-Let S' be directed in f(EXP).

-Then, $p^{-1}(\bigsqcup S') = p^{-1}(\bigsqcup p \circ p^{-1}(S')) = \bigsqcup p^{-1}(S')$.

-Hence, p^{-1} is continuous.

(vii) -Clearly, derivable $\Rightarrow$ $\geq$.

-Suppose $<S,f> \leq <S',f'>$. We have :-

$$\begin{tikzcd}
S \supseteq f(EXP) \arrow[dr, "p^{-1}"'] & & EXP \arrow[ll, "f"'] \arrow[rr, "f'"] \arrow[dl, "{[\ ]}"] \arrow[dr, "{[\ ]'}"] & & f'(EXP) \subseteq S' \\
& EXP/\equiv_{<S,f>} \arrow[rr, "\theta"] & & EXP/\equiv_{<S',f'>} \arrow[ur, "p'"'] &
\end{tikzcd}$$,

where we define θ by $\theta([\varepsilon]) := [\varepsilon]'$.

-Now, θ is well-defined, since if $\varepsilon \equiv_{<S,f>} \delta$ then $\varepsilon \equiv_{<S',f'>} \delta$, by the definition of $\leq$.

-Define $g : S \longrightarrow S'$

$$s \longmapsto \begin{Bmatrix} p' \circ \theta \circ p^{-1}(s), \text{ if } s \in f(EXP) \\ \text{anything, if } s \notin f(EXP) \end{Bmatrix}.$$

-Then, $g \circ f\llbracket\varepsilon\rrbracket = p' \circ \theta \circ p^{-1} \circ f\llbracket\varepsilon\rrbracket = p' \circ \theta([\varepsilon]) = p'([\varepsilon]') = f'\llbracket\varepsilon\rrbracket$.

-Hence, $<S',f'>$ is derivable from $<S,f>$.

(viii) and (ix) -Trivial.

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0.7.2:DEF:-

Let $<S,f>$ be a semantics. Then, it is :-

an α-MODEL if $<EXP/\xrightarrow{\alpha}, [\xrightarrow{\alpha}]> \leq <S,f>$,

a (β-)MODEL if $<EXP/cnv, [cnv]> \leq <S,f>$

and a β-η- MODEL if $<EXP/\beta\text{-}\eta\text{-}cnv, [\beta\text{-}\eta\text{-}cnv]> \leq <S,f>$.

The semantics is SUBSTITUTIVE if $\equiv_{<S,f>}$ is a substitutive relation on EXP.

The semantics is NORMAL if, whenever ε has a normal form and δ does not, then $\varepsilon \not\equiv_{<S,f>} \delta$.

The semantics is SOLVABLE if, whenever $\varepsilon \in SOL$ and $\delta \in INSOL$, then $\varepsilon \not\equiv_{<S,f>} \delta$.

0.7.3:LEMMA:-

All the semantics defined so far are substitutive and, except for $<\{*\}, const>$, normal and solvable.

Proof:-

-Trivial.

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0.7.4:REMARK:-

The trivial semantics, <EXP,id>, is only worth investigating if all we are interested in is the syntax of λ-expressions. The minimal α-model, $<EXP/\xrightarrow{\alpha},[\xrightarrow{\alpha}]>$, tells us a bit more than the pure syntax in that we are informed that α-convertible expressions mean the same thing. Then, the minimal model and β-η-model tell more and more about the meaning of λ-expressions. We shall not really be interested in any semantics that are not, at least, models.

Of course, we do not always gain insight by considering larger (under $\leq$) semantics. In the extreme case, the degenerate semantics, <{*},const>, swamps all meaning with the statement :"all expressions are equal". Our intuition tells us that some are more equal than others - for instance, expressions with a normal form have got nothing to do with those in INSOL - hence, the definitions of normality and solvability.

We expect our semantics to be substitutive since, replacing parts of programs with other parts that have the same meaning, should not affect the overall meaning of the whole program.

Still, the minimal model, based only on β-conversion, does not satisfy our intuition, even though it is normal, solveable and substitutive. We have said before that our intuition is not able to distinguish between elements of INSOL and the same is true for the fixed-point combinators, Y_i; but $<EXP/\underline{cnv},[\underline{cnv}]>$ distinguishes!

We can crudely resolve part of this problem by deriving from the minimal model :-

$$<EXP/\underline{cnv}_{v}INSOL,[\underline{cnv}_{v}INSOL]>,$$

where we have declared all the elements of INSOL equivalent. However, although we have retained normality and solvability we have lost substitutivity! For, while $\Delta\Delta \equiv \Delta\Delta\Delta$, since they are both in INSOL, $x(\Delta\Delta) \not\equiv x(\Delta\Delta\Delta)$, since they are both in SOL and are not

β-convertible.

We can crudely resolve the rest of the problem by deriving, further, the model :-

$$<\text{EXP}/\underline{\text{cnv}}_{\vee}\sim, [\underline{\text{cnv}}_{\vee}\sim]>,$$

where we have declared equivalent all elements that do not have a normal form - i.e. $\varepsilon \sim \delta$ if neither ε nor δ have a normal form. This certainly equivalences the Y_i's but, clearly, we have gone too far - we do not want $Y \equiv \Delta\Delta$: the model is unsolvable! We have also lost substitutivity : $Y(\lambda x.y) \xrightarrow{\beta} y \in \text{NF}$ but $(\Delta\Delta)(\lambda x.y) \in$ INSOL and, so, $Y(\lambda x.y)\ \cancel{\underline{\text{cnv}}_{\vee}\sim}\ (\Delta\Delta)(\lambda x.y)$.

Decent semantics should lie somewhere in between these two and a good place to start looking is their substitutive closure/restrictions.

0.7.5:DEF:-

Let ρ be any relation on EXP. Then, $\hat{\rho}$ is its <u>SUBSTITUTIVE CLOSURE</u> if it is the smallest super-relation of ρ that is substitutive. Further, $\check{\rho}$ is its <u>SUBSTITUTIVE RESTRICTION</u> if it is the largest sub-relation of ρ that is substitutive.

0.7.6:LEMMA:-

(i) $(\varepsilon\ \hat{\rho}\ \delta) \iff (\varepsilon = C[\varepsilon'])\wedge(\delta = C[\delta'])\wedge(\varepsilon'\ \rho\ \delta')$.

(ii) $(\varepsilon\ \check{\rho}\ \delta) \iff (\forall \text{ contexts } C[\ ])(C[\varepsilon]\ \rho\ C[\delta])$.

(iii) (ρ is transitive) $\not\Rightarrow$ ($\hat{\rho}$ is transitive).

(iv) (ρ is a partial ordering) $\Rightarrow$ ($\check{\rho}$ is a partial ordering).

(v) (ρ is an equivalence relation) $\Rightarrow$ ($\check{\rho}$ is an equivalence relation).

Proof:-

-Trivial.

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0.7.7:EXAMPLE:-

Let § be the transitive closure of the substitutive

$\underline{cnv}_V$INSOL. Then,

(i) $\S$ is an equivalence relation,

(ii) $\langle EXP/\underline{cnv}_V INSOL, [\underline{cnv}_V INSOL]\rangle \leq \langle EXP/\S, [\S]\rangle$ (*)

and (iii) $\langle EXP/\S, [\S]\rangle$ is a substitutive normal solvable model in which all elements of INSOL are equivalent.

<u>Proof:-</u>

(i) and (ii) -Trivial.

(iii) -$\langle EXP/\S, [\S]\rangle$ is substitutive by definition and it is a model, equivalencing all elements of INSOL, by part (ii).

-We will show later that it is $\leq$ a normal solvable semantics and, so, it is itself normal and solvable (0.7.9(x)).

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<u>0.7.8:REMARK:-</u>

We do not think, however, that this model solves the problem with the fixed-point combinators. We <u>sketch</u> a <u>possible</u> proof of this. First, establish a Church-Rosser property for $\S$. Let $\xrightarrow{1*}$ be the substitutive closure of INSOL (N.B. ε INSOL δ iff $\varepsilon, \delta \in$ INSOL) and let $\xrightarrow{*}$ be the transitive closure of $\xrightarrow{1*}$. Let $\P$ be the transitive closure of $\xrightarrow{\beta}_V \xrightarrow{*}$. Clearly,

(i) $\S$ is the transitive closure of the symmetric closure of $\P$ - i.e. :-

$(\varepsilon \ \S \ \delta) \iff$ (there exist $\varepsilon_0, \varepsilon_1, \ldots\ldots, \varepsilon_n$ such that $\varepsilon = \varepsilon_0$ and $\delta = \varepsilon_n$ and, for all $0 \leq i < n$, <u>either</u> $\varepsilon_i \ \P \ \varepsilon_{i+1}$ <u>or</u> $\varepsilon_{i+1} \ \P \ \varepsilon_i$).

The next two stages in this argument can be deduced trivially, later, by reference to another semantics in which INSOL is characterised (0.7.9(x)).

(ii) (δ, δ' and $C[\delta] \in$ INSOL) $\Rightarrow$ ($C[\delta'] \in$ INSOL),

(iii) $(\delta \xrightarrow{\beta} \delta') \wedge (C[\delta] \in \text{INSOL}) \Rightarrow (C[\delta'] \in \text{INSOL})$.

For the next two stages, we need a notion of <u>rank</u> on λ-expre-

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(*) Barendregt calls this system H_λ in his thesis.

ssions as given in 0.4.18. Then, we can prove :-

(iv) $(\varepsilon \xrightarrow{1*} \varepsilon') \Rightarrow ([\delta/a]\varepsilon \xrightarrow{1*} [\delta/a]\varepsilon')$,

(v) $(\delta \xrightarrow{1*} \delta') \Rightarrow ([\delta/a]\varepsilon \xrightarrow{*} [\delta'/a]\varepsilon)$,

by means of a "course-of-values" induction on the <u>rank</u> of ε (N.B.1: the proof of part (iv) also requires 0.5.13(ii) - N.B.2: alternatively, as mentioned in 0.5.3, we could prove these with a structural induction on ε, enriched with the technique of COVO, as described in section 4.1).

Now, we can get at,

(vi) $(\varepsilon \xrightarrow{1*} \delta) \wedge (\varepsilon \xrightarrow{1*} \gamma) \Rightarrow (\delta \xrightarrow{1*} \omega) \wedge (\gamma \xrightarrow{1*} \omega)$,

(vii) $(\varepsilon \xrightarrow{1\beta} \delta) \wedge (\varepsilon \xrightarrow{1*} \gamma) \Rightarrow (\delta \xrightarrow{*} \omega) \wedge (\gamma \xrightarrow{1\beta} \omega)$,

by structural inductions on ε, using parts (ii),(iii),(iv) and (v). From these, we can immediately deduce,

(viii) $(\varepsilon \xrightarrow{*} \delta) \wedge (\varepsilon \xrightarrow{*} \gamma) \Rightarrow (\delta \xrightarrow{*} \omega) \wedge (\gamma \xrightarrow{*} \omega)$,

(ix) $(\varepsilon \xrightarrow{1\beta} \delta) \wedge (\varepsilon \xrightarrow{*} \gamma) \Rightarrow (\delta \xrightarrow{*} \omega) \wedge (\gamma \xrightarrow{1\beta} \omega)$,

(x) $(\varepsilon \xrightarrow{\beta} \delta) \wedge (\varepsilon \xrightarrow{*} \gamma) \Rightarrow (\delta \xrightarrow{*} \omega) \wedge (\gamma \xrightarrow{\beta} \omega)$,

(xi) $(\varepsilon \P \delta) \wedge (\varepsilon \P \gamma) \Rightarrow (\delta \P \omega) \wedge (\gamma \P \omega)$,

(xii) $(\varepsilon \S \delta) \Leftrightarrow (\varepsilon \P \omega) \wedge (\delta \P \omega)$,

where the last part comes from parts (i) and (xi). Next, we claim,

(xiii) $(Y_0 \xrightarrow{\beta} C[\delta]) \Rightarrow (\delta \in SOL)$,

(xiv) $(Y_1 \xrightarrow{\beta} C[\delta]) \Rightarrow (\delta \in SOL)$. (<u>Conjecture</u>)

Now, part (xiii) is trivial, since $Y_0 \xrightarrow{\beta} \varepsilon$ implies that $\varepsilon \xrightarrow{\alpha} \lambda f.f^n(\Sigma\Sigma)$, for some $n \geq 0$. We feel that Y_1 (and probably all the Y_i's) has a similar property. By the way, it is not true in general that an expression with no unsolvable sub-expressions (like Y_1) only has reductions with the same property - i.e. unsolvable sub-expressions can be created during a β-reduction : $\Sigma\Delta \xrightarrow{\beta} f(\Delta\Delta)$.

Anyway, conjecturing part (xiv) to be true, we must have,

(xv) $(Y_0 \P \varepsilon) \Rightarrow (Y_0 \xrightarrow{\beta} \varepsilon)$,

(xvi) $(Y_1 \P \varepsilon) \Rightarrow (Y_1 \xrightarrow{\beta} \varepsilon)$,

and, hence,

(xvii) $(Y_0 \not\S Y_1)$,

since, otherwise, by parts (xii),(xv) and (xvi), we would have $Y_0 \underline{\text{cnv}} Y_1$ - X to 0.4.8.

Let us tackle this problem from the other direction, namely with the substitutive restriction of $\underline{\text{cnv}}_{\vee}{\sim}$; this leads to greater success.

<u>0.7.9:EXAMPLE:-</u>

Let ρ be a relation on EXP defined by :-

$$(\varepsilon \ \rho \ \delta) \text{ if } (\varepsilon \xrightarrow{\beta} \nu \ \in \text{NF}) \Rightarrow (\delta \xrightarrow{\beta} \nu).$$

Then,

(i) (ε does not have a normal form) $\Rightarrow (\forall\delta \in \text{EXP})(\varepsilon \ \rho \ \delta)$,

(ii) ρ is reflexive and transitive,

(iii) $(\varepsilon \ \underline{\text{cnv}}_{\vee}{\sim} \ \delta) \Leftrightarrow (\varepsilon \ \rho \ \delta)_{\wedge}(\delta \ \rho \ \varepsilon)$.

In this case, we write $\varepsilon << \delta$ for $\varepsilon \ \check{\rho} \ \delta$ and say that ε <u>IS EXTENDED BY</u> δ. The substitutive restriction of $\underline{\text{cnv}}_{\vee}{\sim}$ is written $\approx$ and called <u>EXTENSIONAL EQUIVALENCE</u>. Then,

(iv) $(\varepsilon \approx \delta) \Leftrightarrow (\varepsilon << \delta)_{\wedge}(\delta << \varepsilon)$,

(v) $<\text{EXP}/\approx,[\approx]>$ is a substitutive normal solvable model and $<<$ is a well-defined partial ordering on it,

(vi) Expressions with a normal form are maximal - i.e. :-

$$(\varepsilon \xrightarrow{\beta} \nu \ \in \text{NF})_{\wedge}(\varepsilon << \delta) \Rightarrow (\varepsilon \ \underline{\text{cnv}} \ \delta) \Rightarrow (\varepsilon \approx \delta),$$

(vii) Elements of INSOL are minimal - i.e. :-

$$(\varepsilon \in \text{INSOL}) \Rightarrow (\forall\delta \in \text{EXP})(\varepsilon << \delta).$$

Hence, $(\forall\varepsilon,\delta \in \text{INSOL})(\varepsilon \approx \delta)$, and $(\varepsilon \in \text{INSOL}) \Leftrightarrow (\varepsilon \approx \Delta\Delta)$,

(viii) $(\forall i \geq 0)(Y_i \approx Y_0)$,

(ix) Hence, $<\text{EXP}/\S,[\S]> \leq <\text{EXP}/\approx,[\approx]>$,

(x) We can resolve the forward references given in 0.7.7(iii) and at stages (ii) and (iii) of remark 0.7.8.

Proof:-

(i),(ii),(iii) and (iv) -Trivial, using 0.7.6(ii).

(v) -By definition, $\approx$ is substitutive.

-By 0.7.6(iv) and part (iv), << is a well-defined partial ordering.

-Also, $\varepsilon \; \underline{cnv} \; \delta \iff (\forall \text{ contexts } C[\;])(C[\varepsilon] \; \underline{cnv} \; C[\delta])$

$\implies (\forall \text{ contexts } C[\;])(C[\varepsilon] \; \underline{cnv}_v{}^{\sim} \; C[\delta])$

$\iff \varepsilon \approx \delta.$

-Hence, the semantics is a model.

-If ε has a normal form and δ does not have one, then ε not$(\underline{cnv}_v{}^{\sim})$ δ and, so, $\varepsilon \not\approx \delta$. Thus, the model is normal.

-Let $\varepsilon \in$ SOL and $\delta \in$ INSOL. Let C[ ] be the context which first closes ε and then applies the construction of 0.5.18(iii) - i.e. C[ ] is of the form : $(\lambda x_1 . \lambda x_2 . \ldots\ldots . \lambda x_r . [\;])(\delta_1)(\delta_2)\ldots\ldots(\delta_s)(K_n)$
Then, $C[\varepsilon]$ has a normal form but, by 0.5.13(i) and (iii), $C[\delta] \in$ INSOL. So, in this case, $C[\varepsilon]$ not$(\underline{cnv}_v{}^{\sim})$ $C[\delta]$, which implies $\varepsilon \not\approx \delta$.
Hence, the model is solvable.

(vi) -Now, $\varepsilon << \delta \implies \varepsilon \; \rho \; \delta$.

-So, if $\varepsilon \xrightarrow{\beta} \nu \in \text{NF}$, then $\delta \xrightarrow{\beta} \nu$.

-Hence, $\varepsilon \; \underline{cnv} \; \delta$ and, by part (v), $\varepsilon \approx \delta$.

(vii) -See Morris [29] or Wadsworth [30] or 6.8.4(i).

(viii) -As with 0.7.7(iii), it will be trivial to show this result later when we produce a semantics, $\leq$ <EXP/$\approx$,[$\approx$]>, in which these fixed-point combinators have already been made equivalent (6.8.3).

(ix) -By parts (v) and (vii), <EXP/$\approx$,[$\approx$]> is seen to be a semantics that is $\geq$ <EXP/$\underline{cnv}_v$INSOL,[$\underline{cnv}_v$INSOL]>. Since, by part (v), it is substitutive, we deduce the result.

(x) -By part (ix), <EXP/§,[§]> must be normal and solvable , since, otherwise, <EXP/$\approx$,[$\approx$]> would not be - $\not\gtrless$ to part (v).

-In both stages (ii) and (iii) of remark 0.7.8, we have $\delta \approx \delta'$.

-Hence, $C[\delta] \simeq C[\delta']$ and, so, $C[\delta'] \in$ INSOL. $\ddagger$

0.7.10:REMARK:-

Anyway, we see that we have been successful with this approach in overcoming the objections raised in 0.7.4 concerning the minimal model. Further, we have introduced a partial ordering, <<, which has elements of INSOL minimal and expressions with a normal form maximal - i.e. this ordering agrees with our intuition about the "information content" of λ-expressions.

We hope the reader does not mind forward references such as in 0.7.9(viii). The material in this section is being introduced for background interest only and will not be made use of in the main part of this thesis.

For the sake of completeness, we give next Morris' original extensional equivalence. This is the same as 0.7.9 except that we throw in η-reductions as well. Its properties are much the same as 0.7.9, but it also has the very nice characterisation that forms the last part of the following.

0.7.11:EXAMPLE:- (Morris)

Let ρ be a relation on EXP defined by :-

$$(\varepsilon\ \rho\ \delta) \text{ if } (\varepsilon \xrightarrow{\beta,\eta} \nu \in \text{NF}) \Rightarrow (\varepsilon\ \beta\text{-}\eta\text{-}\underline{\text{cnv}}\ \delta).$$

Then,

(i) (ε does not have a normal form) $\Rightarrow (\forall \delta \in \text{EXP})(\varepsilon\ \rho\ \delta)$,

(ii) ρ is reflexive and transitive,

(iii) $(\varepsilon\ \beta\text{-}\eta\text{-}\underline{\text{cnv}}_{\vee}^{\sim}\ \delta) \Leftrightarrow (\varepsilon\ \rho\ \delta) \wedge (\delta\ \rho\ \varepsilon)$.

In this case, we write $\varepsilon\ \beta\text{-}\eta\text{-}{<<}\ \delta$ for $\varepsilon\ \overset{\vee}{\rho}\ \delta$ and say that ε IS β-η- EXTENDED BY δ. The substitutive restriction of $\beta\text{-}\eta\text{-}\underline{\text{cnv}}_{\vee}^{\sim}$ is written $\beta\text{-}\eta\text{-}\simeq$ and called β-η-EXTENSIONAL EQUIVALENCE. Then,

(iv) $(\varepsilon\ \beta\text{-}\eta\text{-}\simeq\ \delta) \Leftrightarrow (\varepsilon\ \beta\text{-}\eta\text{-}{<<}\ \delta) \wedge (\delta\ \beta\text{-}\eta\text{-}{<<}\ \varepsilon)$,

(v) $\langle \text{EXP}/\beta\text{-}\eta\text{-}\simeq, [\beta\text{-}\eta\text{-}\simeq] \rangle$ is a substitutive normal solvable β-η-model and β-η-<< is a well-defined partial ordering on it,

(vi) Expressions with a (β-η-)normal form are maximal - i.e. :-

$$(\varepsilon \xrightarrow{\beta,\eta} \nu \in NF) \wedge (\varepsilon\ \beta\text{-}\eta\text{-}{<<}\ \delta) \Rightarrow (\varepsilon\ \beta\text{-}\eta\text{-}\underline{cnv}\ \delta) \Rightarrow (\varepsilon\ \beta\text{-}\eta\text{-}{\approx}\ \delta),$$

(vii) Elements of INSOL are minimal - i.e. :-

$$(\varepsilon \in INSOL) \Rightarrow (\forall\delta \in EXP)(\varepsilon\ \beta\text{-}\eta\text{-}{<<}\ \delta).$$

Hence, $(\forall\varepsilon,\delta \in INSOL)(\varepsilon\ \beta\text{-}\eta\text{-}{\approx}\ \delta)$,

(viii) $\langle EXP/{\approx},[{\approx}]\rangle \leq \langle EXP/\beta\text{-}\eta\text{-}{\approx},[\beta\text{-}\eta\text{-}{\approx}]\rangle$,

(ix) $(\forall i \geq 0)(Y_i\ \beta\text{-}\eta\text{-}{\approx}\ Y_0)$,

(x) $\langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL,[\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL]\rangle \leq \langle EXP/\beta\text{-}\eta\text{-}{\approx},[\beta\text{-}\eta\text{-}{\approx}]\rangle$.

Hence, defining β-η-§ in the obvious way, the substitutive β-η-model it gives is $\leq$ that given by β-η-$\approx$, and is therefore normal and solvable.

(xi) $\langle EXP/\beta\text{-}\eta\text{-}{\approx},[\beta\text{-}\eta\text{-}{\approx}]\rangle$ is the maximal normal substitutive model.

<u>Proof:-</u>

(i),(ii),(iii),(iv),(v),(vi) and (vii) -Same as the corresponding parts of 0.7.9, bearing in mind that ε has a normal form iff ε has a β-η-normal form (0.4.20), the original definition of SOL (0.5.5) and theorem 0.5.9.

(viii) -Clear, by 0.7.6(ii) and $\underline{cnv} \Rightarrow \beta\text{-}\eta\text{-}\underline{cnv}$.

(ix) <u>Either</u> from 0.7.9(viii) and part (viii) above <u>or</u> we can show this directly using the model described in 0.7.22.

(x) -By parts (v) and (vii), as in the proof of 0.7.9(ix).

(xi) -Let $\langle S,f\rangle$ be a normal substitutive model.

-Now, by definition, $\langle EXP/\beta\text{-}\eta\text{-}{\approx},[\beta\text{-}\eta\text{-}{\approx}]\rangle$ is the maximal substitutive sub-semantics of $\langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V{\sim},[\beta\text{-}\eta\text{-}\underline{cnv}_V{\sim}]\rangle$, so all we need to show is that $\langle S,f\rangle \leq \langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V{\sim},[\beta\text{-}\eta\text{-}\underline{cnv}_V{\sim}]\rangle$:-

-Let ε not$(\beta\text{-}\eta\text{-}\underline{cnv}_V{\sim})$ δ.

-<u>Either</u> ε has a normal form and δ does not : in this case, $f\llbracket\varepsilon\rrbracket \neq f\llbracket\delta\rrbracket$, since $\langle S,f\rangle$ is normal.

-<u>Or</u> ε and δ both have normal form and $\varepsilon\ \beta\text{-}\eta\text{-}\underline{c\not{n}v}\ \delta$: in this case, we quote the following theorem of Böhm [31] - see also

Curry [77] - and as interpreted by Wadsworth [32] :-

> "if ε and δ both have normal forms but ε β-η-$\underline{c\not{n}v}$ δ, then there exists a context C[ ] such that $C[\varepsilon] \xrightarrow{\beta} x$ and $C[\delta] \xrightarrow{\beta} y$ and where $x \neq y$." (Bohm's theorem)

Take the context, C[ ], provided by the above and suppose that $f[\![\varepsilon]\!] = f[\![\delta]\!]$. Let Γ be any expression that does not have a normal form. Then,

$$f[\![\Gamma]\!] = f[\![(\lambda x.C[\varepsilon])(\Gamma)]\!], \text{ since } <S,f> \text{ is a model.}$$

$$= f[\![(\lambda x.C[\delta])(\Gamma)]\!], \text{ since } <S,f> \text{ is substitutive.}$$

$$= f[\![y]\!], \text{ since } <S,f> \text{ is a model - but this is a}$$

contradiction to the normality of <S,f>. Hence, $f[\![\varepsilon]\!] \neq f[\![\delta]\!]$.

$-\therefore, \ \varepsilon \equiv_{<S,f>} \delta \Rightarrow \varepsilon$ β-η-$\underline{cnv}_v{}^{\sim}$ δ.

$-\therefore$, <EXP/β-η-$\simeq$,[β-η-$\simeq$]> is the maximal normal substitutive model and, so, since it is solvable by part (v), so also will be any other normal substitutive model.

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0.7.12:REMARK:-

All the semantics we have looked at so far have had an implic construction - all that has been done is the taking of (highly undecidable) equivalence relations based on "syntactic" conversion rules of the λ-calculus. The following semantics are more truly "semantic" in nature in that an explicit construction is given. They are mostly going to lie somewhere in the region bounded by <EXP/§,[§]> and <EXP/β-η-§,[β-η-§]> on the one hand, and <EXP/$\simeq$,[$\simeq$ and <EXP/β-η-$\simeq$,[β-η-$\simeq$]> on the other. The first two were discovere by Scott.

0.7.13:EXAMPLE:- (Scott)

Let D_0 be a complete continuous countably based lattice.(*) Then

$$D_{i+1} := [D_i \rightarrow D_i], \text{ for all } i \geq 0.$$

Also,

(*) D_0 should have more than one element.

$$\begin{array}{rl} \phi_{0,1} : D_0 & \longrightarrow D_1, \\ x & \longmapsto k_x \end{array} \quad \text{and where } \begin{array}{rl} k_x : D_0 & \longrightarrow D_0, \\ d & \longmapsto x \end{array}$$

and,

$$\begin{array}{rl} \phi_{1,0} : D_1 & \longrightarrow D_0. \\ x & \longmapsto x(\perp) \end{array}$$

Further, for all $i \geq 1$,

$$\begin{array}{rl} \phi_{i,i+1} : D_i & \longrightarrow D_{i+1}, \\ d_i & \longmapsto \phi_{i-1,i} \circ d_i \circ \phi_{i,i-1} \end{array}$$

and,

$$\begin{array}{rl} \phi_{i+1,i} : D_{i+1} & \longrightarrow D_i \\ d_{i+1} & \longmapsto \phi_{i,i-1} \circ d_{i+1} \circ \phi_{i-1,i} \end{array}$$

Then, for all $i \geq 0$,

(i) D_i is a complete continuous countably based lattice,

(ii) $D_i \lhd D_{i+1}$.

Let D_∞ be the inverse limit of this system. We define an application on D_∞ in the "Curry" form :-

$$\begin{array}{rl} Ap_{D_\infty} : D_\infty & \longrightarrow (D_\infty \longrightarrow D_\infty), \\ \langle d_i \rangle_{i=0}^{\infty} & \mapsto Ap_{D_\infty}(\langle d_i \rangle_{i=0}^{\infty}) \end{array}$$

where,

$$\begin{array}{rl} Ap_{D_\infty}(\langle d_i \rangle_{i=0}^{\infty}) : D_\infty & \longrightarrow D_\infty. \\ \langle x_i \rangle_{i=0}^{\infty} & \mapsto \bigsqcup_{i=0}^{\infty} \phi_{i,\infty} \circ d_{i+1}(x_i) \end{array}$$

Now,

(iii) D_∞ is a complete continuous countably based lattice,

(iv) Ap_{D_∞} is completely additive and doubly strict,$^{(*)}$

(v) $Ap_{D_\infty}(\langle d_i \rangle_{i=0}^{\infty})$ is continuous - i.e. $Ap_{D_\infty} \in [D_\infty \rightarrow [D_\infty \rightarrow D_\infty]]$,

(vi) Ap_{D_∞} is injective, surjective and has a continuous inverse given by :-

$$\begin{array}{rl} Ap_{D_\infty}^{-1} : [D_\infty \rightarrow D_\infty] & \longrightarrow D_\infty, \\ f & \longmapsto \bigsqcup_{i=1}^{\infty} \phi_{i,\infty}(f_i) \end{array}$$

(*) See footnote on page 88.

where $f_i := \phi_{\infty,i-1} \circ f \circ \phi_{i-1,\infty} \in D_i$,

(vii) Hence, we have :-

$$(x \equiv y) \iff (Ap_{D_\infty}(x) \equiv Ap_{D_\infty}(y)),$$

and, in particular, "extensionality" - i.e. :-

$$(x = y) \iff (Ap_{D_\infty}(x) = Ap_{D_\infty}(y)),$$

(viii) Also, Ap_{D_∞} is an isomorphism such that :-

$$D_\infty \doteq [D_\infty \to D_\infty].$$

In order to define our semantics, we find it necessary to introduce the notion of <u>environments</u> to take care of free variables. We define ENV := $(I \to D_\infty)$. We also define a <u>modifying operator</u>, $[\ /\ ]$, as follows :-

$$[\ /\ ] : D_\infty \times I \longrightarrow (ENV \to ENV),$$
$$(\delta, x) \longmapsto [\delta/x]$$

where,

$$[\delta/x] : ENV \longrightarrow ENV,$$
$$\rho \longmapsto [\delta/x]\rho$$

where,

$$[\delta/x]\rho : I \longrightarrow D_\infty$$
$$y \longmapsto \begin{Bmatrix} \rho(y), \text{ if } x \neq y \\ \delta, \text{ if } x = y \end{Bmatrix}.$$

Then, we note,

(ix) $[\ /\ ]$ is completely additive on its first argument - i.e. :-

$$[\bigsqcup D/x] = \bigsqcup\{[\delta/x] \mid \delta \in D\}, \text{ for } \underline{\text{any}}\ D \subseteq D_\infty.$$

In particular, if D is directed, then so is $\{[\delta/x] \mid \delta \in D\}$,

(x) $[\delta/x]$ is completely additive and, again, if R is directed in ENV, then so is $\{[\delta/x]\rho \mid \rho \in R\}$.

Now, we are ready to define our semantic function,

$$\mathcal{D} \in (EXP \to (ENV \to D_\infty)),$$

inductively on the structure of EXP, by means of the following three equations :-

$$(D1)\quad \mathcal{D}[\![x]\!](\rho) \quad := \rho(x)$$

$$(D2)\quad \mathcal{D}[\![\lambda x.\varepsilon]\!](\rho) := Ap^{-1}_{D_\infty}(\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho))$$

$$(D3)\quad \mathcal{D}[\![\varepsilon(\delta)]\!](\rho) := Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho))(\mathcal{D}[\![\delta]\!](\rho)).$$

Before proceeding, we must check that the above equations make sense : in particular D2 - in order for $Ap^{-1}_{D_\infty}$ to be defined, we have to have its argument continuous :-

(xi) $\mathcal{D}[\![\varepsilon]\!]$ is continuous - i.e. $\mathcal{D} \in (EXP \to [ENV \to D_\infty])$ - and, hence, $(\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho)) \in [D_\infty \to D_\infty]$,

(xii) $<[ENV \to D_\infty],\mathcal{D}>$ is a semantics of the λ-calculus,

(xiii) The above semantics is substitutive. Further, the partial ordering induced from $[ENV \to D_\infty]$ onto EXP as in 0.7.1(vi) is also substitutive,

(xiv) $<[ENV \to D_\infty],\mathcal{D}>$ is a β-η-model,

(xv) If ε is closed, then $\mathcal{D}[\![\varepsilon]\!]$ is a constant map in $[ENV \to D_\infty]$. Hence, we can consider closed λ-expressions as being modelled just by D_∞,

(xvi) Let ε and δ both have β-η-normal forms. Then,

$$\mathcal{D}[\![\varepsilon]\!] \sqsubseteq \mathcal{D}[\![\delta]\!] \Rightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{cnv}\ \delta \Rightarrow \mathcal{D}[\![\varepsilon]\!] = \mathcal{D}[\![\delta]\!].$$

In particular,

$$\mathcal{D}[\![\varepsilon]\!] = \mathcal{D}[\![\delta]\!] \Leftrightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{cnv}\ \delta,$$

(xvii) The fixed-point combinators, Y_i, are equivalenced,

(xviii) Y is the same as the lattice-theoretic minimal fixed-point operator μ (see 0.6.5(iv)), in the sense that :-

$$\mathcal{D}[\![Y(\varepsilon)]\!](\rho) = \mu(Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho))),$$

(xix) $<[ENV \to D_\infty],\mathcal{D}>$ is <u>not</u> normal.

<u>Proof:-</u>

(i) -By 0.6.25(i).

(ii) -By 0.6.20, $D_0 \lhd D_1$. The rest fall by induction.

(iii) -By 0.6.25(ii).

(iv),(v) and (vi) -See Scott [33].

(vii) and (viii) -By parts(v) and (vi).

(ix) and (x) -Trivial.

(xi) -This is something that was not done in Wadsworth [34], and so we set it out here.

-Let $\mathcal{A}[\![\varepsilon]\!] \equiv (\mathcal{D}[\![\varepsilon]\!] \in [\mathrm{ENV} \to D_\infty])$.

-Claim: $\mathcal{A}[\![\varepsilon]\!] \Rightarrow (\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho) \in [D_\infty \to D_\infty])$:-

-Let D be directed in D_∞.

-Then, $\mathcal{D}[\![\varepsilon]\!]([\bigsqcup D/x]\rho) = \mathcal{D}[\![\varepsilon]\!](\bigsqcup\{[\delta/x]\rho \mid \delta \in D\})$, by part (ix).

$= \bigsqcup\{\mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho) \mid \delta \in D\}$, by $\mathcal{A}[\![\varepsilon]\!]$.

-Claim: $\mathcal{A}[\![x]\!]$:-

-Let R be directed in ENV.

-Then, $\mathcal{D}[\![x]\!](\bigsqcup R) = (\bigsqcup R)(x) = \bigsqcup\{\rho(x) \mid \rho \in R\} = \bigsqcup\{\mathcal{D}[\![x]\!](\rho) \mid \rho \in R\}$.

-Claim: $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda x.\varepsilon]\!]$:-

-Let R be directed in ENV.

-Then, $\mathcal{D}[\![\lambda x.\varepsilon]\!](\bigsqcup R) = Ap_{D_\infty}^{-1}(\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\bigsqcup R))$, which is properly defined, by $\mathcal{A}[\![\varepsilon]\!]$ and the first claim above.

$= Ap_{D_\infty}^{-1}(\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!](\bigsqcup\{[\delta/x]\rho \mid \rho \in R\}))$, by part (x).

$= Ap_{D_\infty}^{-1}(\lambda\delta \in D_\infty . \bigsqcup\{\mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho) \mid \rho \in R\})$, by $\mathcal{A}[\![\varepsilon]\!]$.

$= Ap_{D_\infty}^{-1}(\bigsqcup\{\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho) \mid \rho \in R\})$, clearly, and with the set remaining directed.

$= \bigsqcup\{Ap_{D_\infty}^{-1}(\lambda\delta \in D_\infty . \mathcal{D}[\![\varepsilon]\!]([\delta/x]\rho)) \mid \rho \in R\}$, by part (vi).

$= \bigsqcup\{\mathcal{D}[\![\lambda x.\varepsilon]\!](\rho) \mid \rho \in R\}$.

-Claim: $\mathcal{A}[\![\varepsilon]\!] \wedge \mathcal{A}[\![\delta]\!] \Rightarrow \mathcal{A}[\![\varepsilon(\delta)]\!]$:-

-Let R be directed in ENV.

-Then, $\mathcal{D}[\![\varepsilon(\delta)]\!](\bigsqcup R) = Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\bigsqcup R))(\mathcal{D}[\![\delta]\!](\bigsqcup R))$

$= Ap_{D_\infty}(\bigsqcup\{\mathcal{D}[\![\varepsilon]\!](\rho) \mid \rho \in R\})(\bigsqcup\{\mathcal{D}[\![\delta]\!](\rho') \mid \rho' \in R\})$, by $\mathcal{A}[\![\varepsilon]\!] \wedge \mathcal{A}[\![\delta]\!]$

$= (\bigsqcup\{Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho)) \mid \rho \in R\})(\bigsqcup\{\mathcal{D}[\![\delta]\!](\rho') \mid \rho' \in R\})$, by (iv).

$= \bigsqcup\{Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho))(\bigsqcup\{\mathcal{D}[\![\delta]\!](\rho') \mid \rho' \in R\}) \mid \rho \in R\}$

$= \bigsqcup\{Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho))(\mathcal{D}[\![\delta]\!](\rho')) \mid \rho, \rho' \in R\}$, by part (v).

$= \bigsqcup\{Ap_{D_\infty}(\mathcal{D}[\![\varepsilon]\!](\rho''))(\mathcal{D}[\![\delta]\!](\rho'')) \mid \rho'' \in R\}$, since R is directed.

$$= \bigsqcup\{\mathcal{D}[\![\varepsilon(\delta)]\!](\rho) \mid \rho \in R\}.$$

-Hence, by structural induction, $(\forall \varepsilon \in EXP)\, \mathcal{A}[\![\varepsilon]\!]$.

(xii) -By part (xi).

(xiii) -Trivial.

(xiv) and (xv) -See Wadsworth [35].

(xvi) -Let ε and δ both have β-η-normal forms but ε β-η-$\underline{c\not{n}v}$ δ.

-As in 0.7.11(xi), Bohm's theorem tells us that there exists a context C[ ] such that $C[\varepsilon] \xrightarrow{--\beta} x$ and $C[\delta] \not\xrightarrow{\beta} y$.

-So, by part (xiii), if $\mathcal{D}[\![\varepsilon]\!] \sqsubseteq \mathcal{D}[\![\delta]\!]$, then $\mathcal{D}[\![C[\varepsilon]]\!] \sqsubseteq \mathcal{D}[\![C[\delta]]\!]$ and so, by part (xiv), $\mathcal{D}[\![x]\!] \sqsubseteq \mathcal{D}[\![y]\!] \rightarrow \not{\times}$; since D_∞ has more than one element.

-The rest is trivial.

(xvii) and (xviii) -See Wadsworth [36] and Park [37].

(xix) -Let $F := \lambda f.\lambda x.\lambda y.x(fy)$.

-Then, $F(I) \xrightarrow{\beta,\eta} I$ and so, by part (xviii), $\mathcal{D}[\![Y(F)]\!] \sqsubseteq \mathcal{D}[\![I]\!]$.

-But, it is not too difficult to show the reverse inequality - see Wadsworth [38]. Hence, $\mathcal{D}[\![I]\!] = \mathcal{D}[\![Y(F)]\!]$.

-But, Y(F) does not have a normal form since it is in the λ-I-calculus and the subexpression Y does not have normal form - see Curry [39].

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0.7.14:REMARK:-

Note that ENV is itself a complete lattice (0.6.18(i)) and so, therefore, are [ENV → ENV] and [ENV → D_∞] (0.6.18(ii)). We can also see that ENV has a countable sub-basis, $\{f_{i,j} \mid i,j \geq 0\}$, where :-

$$f_{i,j} : I \longrightarrow D_\infty,$$
$$n \longmapsto \begin{Bmatrix} b_j, \text{ if } n = i. \\ \bot, \text{ if } n \neq i. \end{Bmatrix}$$

and where $\{b_0, b_1, b_2, \ldots\ldots\ldots\ldots\}$ is an ennumeration of a countable basis of D_∞. Hence, ENV is countably based (0.6.24(ii)). Further, we can deduce that ENV is continuous as follows : extend I to the complete continuous countably based lattice I′ by adjoining a bottom and top ; then, $[I' \to D_\infty]$ is a c.c.c. lattice (0.6.25(i)) ; clearly, $(I \to D_\infty)$ is isomorphic to the set of doubly-strict element of $[I' \to D_\infty]$, which is a retraction ; but, all retracts of continuous lattices, according to a theorem of Scott [40], are themselves continuous lattices. Thus, ENV is a c.c.c. lattice and we are very pleased to find that the whole of the underlying set of this model, $[ENV \to D_\infty]$, is one also.

We made no restrictions on the initial c.c.c. lattice, D_0, except that it has more than one element. Indeed, the two element lattice $\{\top,\bot\}$ will do, although, for reasons of the use to which these lattices are put, Scott prefers the "truth table" lattice $\{\top,$ true, false, $\bot\}$, in which true and false are incomparable. It is extremely probable that the choice of the initial D_0 does not chang the final semantics - i.e. they are all equivalent under $\equiv$ - but the author has not seen a "concrete" proof of this.

Until 0.7.13(xix), $<[ENV \to D_\infty], \mathcal{D}>$ had been showing all the good properties of the ones based upon $\approx$ and β-η-$\approx$. The discovery of the non-normality of the model caused a bit of a shock! Since it seemed reasonable to consider expressions with a normal form analogous to programs that terminate, a semantics that sometimes fails to distinguish normal from non-normal forms might be thought to be inappropriate for certain applications like termination problems. However, the blame has now been firmly laid at the feet of λ-calculus itself, which, after all, must allow sentences like "Y(F) = I to be adjoined to it without throwing up any inconsistencies. We shall continue to explain this away later (0.7.27).

Perhaps, the more subtle property of solvability is more crucial to decent semantics. We shall see that the above model has

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0.7.16:REMARK:-

So, all we have to do to show solvability is to establish that all members of INSOL are $\perp$ in the model. This corresponds with the properties (vii) of the extensional equivalence models (0.7.9 and 0.7.11).

0.7.17:DEF:-

Let $\Omega \notin I$. Then,

$$\Omega\text{-EXP} ::= \Omega \mid I \mid \lambda I.\Omega\text{-EXP} \mid (\Omega\text{-EXP})(\Omega\text{-EXP})$$

If $\varepsilon \in \Omega\text{-EXP}$, then :-

$$\lambda x.\Omega \xrightarrow{1\omega} \Omega$$

$$\text{and } \Omega(\varepsilon) \xrightarrow{1\omega} \Omega.$$

As in 0.3.10, define $\xrightarrow{\omega}$ as the substitutive and transitive closure of $\xrightarrow{1\omega}$.

Extend the definition of D, so that it applies to Ω-expressions by adding the extra equation :-

(DO´) $D'[\![\Omega]\!] := \perp$.

Define an "approximation" function, $\sim$: EXP $\longrightarrow$ Ω-EXP, by :-

$$\tilde{x} := x,$$

$$\widetilde{\lambda x.\varepsilon} := \lambda x.\tilde{\varepsilon}$$

$$\text{and } \widetilde{\varepsilon(\delta)} := \left\{\begin{matrix} \Omega, \text{ if } \varepsilon \in \lambda I.\text{EXP} \\ \tilde{\varepsilon}(\tilde{\delta}), \text{ if not} \end{matrix}\right\}.$$

Then, we have :-

$$\begin{tikzcd}
\text{EXP} \arrow[rr, "D"] \arrow[dr, "\sim"'] & & {[\text{ENV} \to D_\infty]} \\
& \Omega\text{-EXP} \arrow[ur, "D'"'] &
\end{tikzcd}$$

0.7.18:LEMMA:-

(i) Ω-expressions always have an ω-normal form.

(ii) $\langle [\text{ENV} \to D_\infty], D' \rangle$ is an ω-model of the λ-Ω-calculus - i.e.,

$$(\varepsilon \xrightarrow{\omega} \delta) \Rightarrow (D'[\![\varepsilon]\!] = D'[\![\delta]\!]).$$

(iii) $(\varepsilon \in \mathrm{NOH}) \Rightarrow (\tilde{\varepsilon} \xrightarrow{\omega} \Omega) \Rightarrow (D'[\![\tilde{\varepsilon}]\!] = \bot)$.

(iv) $<[\mathrm{ENV} \to D_\infty], D' \circ \sim>$ is a semantics of the λ-calculus that is "approximate" to $<[\mathrm{ENV} \to D_\infty], D>$ - i.e. :-

$$(\forall \varepsilon \in \mathrm{EXP})(D'[\![\tilde{\varepsilon}]\!] \sqsubseteq D[\![\varepsilon]\!]).$$

Note, however, that it is not substitutive.

(v) $(\varepsilon \xrightarrow{\beta} \delta) \Rightarrow (D'[\![\tilde{\varepsilon}]\!] \sqsubseteq D'[\![\tilde{\delta}]\!])$.

(vi) Hence, for all $\varepsilon \in \mathrm{EXP}$, $\bigsqcup\{D'[\![\tilde{\delta}]\!] \mid \varepsilon \xrightarrow{\beta} \delta\} \sqsubseteq D[\![\varepsilon]\!]$, where the set is directed.

Proof:-

(i) -Obvious, since $\xrightarrow{1\omega}$ strictly shortens the expression.

(ii) -Simple structural induction on $\varepsilon \in \Omega\text{-EXP}$ to show modelling of just the substitutive closure of $\xrightarrow{1\omega}$. Hence, the transitive closure.

(iii) -First $\Rightarrow$ by a trivial structural induction on NOH.

-Second $\Rightarrow$ by part (ii).

(iv) -Trivial structural induction on EXP.

-It is not substitutive since $D'[\![\tilde{y}]\!] = D[\![y]\!] = D[\![\lambda x.yx]\!] = D'[\![\widetilde{\lambda x.yx}]\!]$ but $D'[\![\widetilde{ya}]\!] \neq \bot = D'[\![\widetilde{(\lambda x.yx)(a)}]\!]$.

(v) -Trivial structural induction on EXP.

(vi) -If $\varepsilon \xrightarrow{\beta} \delta$, then $D'[\![\tilde{\delta}]\!] \sqsubseteq D[\![\delta]\!]$, by part (iv).

$= D[\![\varepsilon]\!]$, by 0.7.13(xiv).

-Hence, the inequality.

-The set is directed since, if $\varepsilon \xrightarrow{\beta} \delta$ and $\varepsilon \xrightarrow{\beta} \gamma$, then, by the Church-Rosser theorem for $\xrightarrow{\beta}$, there exists an ξ such that $\delta \xrightarrow[*]{\beta} \xi$ and $\gamma \xrightarrow{\beta} \xi$ and, so, by part (v), $D'[\![\tilde{\delta}]\!] \sqsubseteq D'[\![\tilde{\xi}]\!]$ and $D'[\![\tilde{\gamma}]\!] \sqsubseteq D'[\![\tilde{\xi}]\!]$.

‡

0.7.19:THEOREM:- (Wadsworth)

For all $\varepsilon \in \mathrm{EXP}$, $D[\![\varepsilon]\!] = \bigsqcup\{D'[\![\tilde{\delta}]\!] \mid \varepsilon \xrightarrow{\beta} \delta\}$.

Proof:-

-Non-trivial - see Wadsworth [41].

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0.7.20:COR:-

(i) $(\varepsilon \in \mathrm{SOL}) \iff (D[\![\varepsilon]\!] \neq \bot)$.

(ii) $\langle [\mathrm{ENV} \to D_\infty], D \rangle$ is solvable.

(iii) $\langle \mathrm{EXP}/\beta\text{-}\eta\text{-}\S, [\beta\text{-}\eta\text{-}\S] \rangle \leq \langle [\mathrm{ENV} \to D_\infty], D \rangle$

$\not\leq \langle \mathrm{EXP}/\beta\text{-}\eta\text{-}\approx, [\beta\text{-}\eta\text{-}\approx] \rangle$.

Proof:-

(i) $(\Rightarrow)$ -By 0.7.15(iii).

$(\Leftarrow)$ -Let $\varepsilon \in \mathrm{INSOL}$.

-Then, if $\varepsilon \xrightarrow{\beta} \delta$, $\delta \in \mathrm{NOH}$, by definition.

-$\therefore$, $D[\![\varepsilon]\!] = \bigsqcup\{D'[\![\delta]\!] \mid \varepsilon \xrightarrow{\beta} \delta\}$, by 0.7.19.

$= \bot$, by 0.7.18(iii).

(ii) -By part (i).

(iii) -First inequality by part (i) and 0.7.13(xiii) and (xiv).

-Second non-inequality by 0.7.11(v) and 0.7.13(xix).

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0.7.21:REMARK:-

We think of the symbol Ω as being "undefined". Thus, the ω-rules were designed to be consistent with this intuition. The Ω-expression $\tilde{\varepsilon}$ can be thought of as the "information immediately available" from the expression ε - i.e. it answers the question : "if we cannot be bothered to evaluate any β-redexes and simply treat them as unknown quantities, what have we got in ε ?"

So, Wadsworth's theorem says that, in the $\langle [\mathrm{ENV} \to D_\infty], D \rangle$-semantics, expressions are represented by the limit of the representations of the information immediately available from all the expressions reducible from it. This is a very good and sensible property for semantics to have and later (7.6.1) we will make a definition out of the theorem and christen it "continuity".

Next, we present a model which is normal. The proof of the counter-example in 0.7.13(xix) depends upon the property of "extensionality", 0.7.13(vii), and so it is not surprising that the new model is not an η-model. However, of course, normal η-models do exist - e.g. $\langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}, [\beta\text{-}\eta\text{-}\underline{cnv}]\rangle$.

<u>0.7.22:EXAMPLE:-</u> (Scott)

Let D^*_0 be a complete continuous countably based lattice with an isolated $\top$. We call D^*_0 the <u>atomic lattice</u>. Then,

$$D^*_{i+1} := D^*_0 + [D^*_i \to D^*_i]^{\top},$$

where, if L is any lattice, then $L^{\top}$ is the same lattice with an extra $\top$ adjoined. (We need this so as to ensure that the lattices we construct always have an isolated $\top$ so as to ensure that their lattice sums always remain continuous - see 0.6.18(iii).) Also,

$$\phi^*_{0,1} : D^*_0 \longrightarrow D^*_1$$
$$d_0 \longmapsto d_0 \in D^*_0 \subseteq D^*_1$$

and,

$$\phi^*_{1,0} : D^*_1 \longrightarrow D^*_0$$
$$d_1 \longmapsto \begin{Bmatrix} \bot, \text{ if } d_1 \in [D^*_0 \to D^*_0] \\ d_1, \text{ if } d_1 \in D^*_0 \end{Bmatrix}.$$

Further, for all $i \geq 1$,

$$\phi^*_{i,i+1} : D^*_i \longrightarrow D^*_{i+1}$$
$$d_i \longmapsto \begin{Bmatrix} \phi^*_{i-1,i} \circ d_i \circ \phi^*_{i,i-1}, \text{ if } d_i \in [D^*_{i-1} \to D^*_{i-1}] \\ d_i, \text{ if } d_i \in D^*_0 \end{Bmatrix}$$

and,

$$\phi^*_{i+1,i} : D^*_{i+1} \longrightarrow D^*_i$$
$$d_{i+1} \longmapsto \begin{Bmatrix} \phi^*_{i,i-1} \circ d_{i+1} \circ \phi^*_{i-1,i}, \text{ if } d_{i+1} \in [D^*_i \to D^*_i] \\ d_{i+1}, \text{ if } d_{i+1} \in D^*_0 \end{Bmatrix}$$

Then, for all $i \geq 0$,

(i) D^*_i is a complete continuous countably based lattice with an isolated $\top$,

(ii) The above maps are well-defined and make $D_i^* \lhd D_{i+1}^*$.

Let D_∞^* be the inverse limit of this system. Define the predicate : $\mathrm{Atom}(\langle d_i\rangle_{i=0}^\infty) \equiv (\forall i \geq 0)(d_i = d_0 \in D_0^*)$. Then, clearly, $\mathrm{not}(\mathrm{Atom}(\langle d_i\rangle_{i=0}^\infty)) \equiv (\forall i \geq 1)(d_i \in [D_{i-1}^* \to D_{i-1}^*])\ \&\ (\langle d_i\rangle_{i=0}^\infty \neq \bot)$. Then,

$$\mathrm{Ap}^* : D_\infty^* \longrightarrow (D_\infty^* \longrightarrow D_\infty^*)^\top$$

$$\langle d_i\rangle_{i=0}^\infty \mapsto \left\{\begin{array}{l} \top,\ \text{if } \langle d_i\rangle_{i=0}^\infty = \top \\ \bot,\ \text{if } \langle d_i\rangle_{i=0}^\infty \neq \top \text{ and } \mathrm{Atom}(\langle d_i\rangle_{i=0}^\infty) \\ \mathrm{Ap}^*(\langle d_i\rangle_{i=0}^\infty),\ \text{if not}(\mathrm{Atom}(\langle d_i\rangle_{i=0}^\infty)) \end{array}\right\},$$

where, if $\mathrm{not}(\mathrm{Atom}(\langle d_i\rangle_{i=0}^\infty))$,

$$\mathrm{Ap}^*(\langle d_i\rangle_{i=0}^\infty) : D_\infty^* \longrightarrow D_\infty^*.$$

$$\langle x_i\rangle_{i=0}^\infty \mapsto \bigsqcup_{i=0}^\infty \phi_{i,\infty}^* \circ d_{i+1}(x_i)$$

(N.B. we should write $\mathrm{Ap}_{D_\infty}^*$ instead of Ap^* to remain strictly logical with our notation.)

Now,

(iii) D_∞^* is a complete continuous countably based lattice with an isolated $\top$,

(iv) Ap^* is well-defined, completely additive and doubly strict,

(v) $\mathrm{Ap}^* \in [D_\infty^* \to [D_\infty^* \to D_\infty^*]^\top]$,

(vi) While Ap^* is not injective, it is surjective. If we define Ap^{*-1} in the same way as in 0.7.13(vi) and extend it so that it maps the extra $\top$ to $\top \in D_\infty^*$, it is continuous and makes :-

$$[D_\infty^* \to D_\infty^*]^\top \lhd D_\infty^*,$$

(vii) We have "partial extensionality" - i.e. if not(Atom(x)) and not(Atom(y)) :-

$$(x \sqsubseteq y) \iff (\mathrm{Ap}^*(x) \sqsubseteq \mathrm{Ap}^*(y))$$

and,

$$(x = y) \iff (\mathrm{Ap}^*(x) = \mathrm{Ap}^*(y)).$$

Otherwise, if Atom(x) then <u>either</u> $\mathrm{Ap}^*(x) = \bot$ (if $x \neq \top$) <u>or</u> $\mathrm{Ap}^*(x) =$

$\top$ (if $x = \top$),

(viii) Define,

$$\theta^* : D^*_\infty \longrightarrow D^*_0 + [D^*_\infty \to D^*_\infty]^{\top}.$$

$$d \longmapsto \left\{ \begin{array}{l} \phi^*_{\infty,0}(d), \text{ if Atom}(d) \\ Ap^*(d), \text{ if not(Atom}(d)) \end{array} \right\}$$

Then, θ^* is an isomorphism with a well-defined continuous inverse :-

$$\theta^{*-1} : D^*_0 + [D^*_\infty \to D^*_\infty]^{\top} \longrightarrow D^*_\infty.$$

$$\left\{ \begin{array}{l} d_0 \in D^*_0 \\ f \in [D^*_\infty \to D^*_\infty]^{\top} \end{array} \right\} \longmapsto \left\{ \begin{array}{l} \phi^*_{0,\infty}(d_0) \\ Ap^{*-1}(f) \end{array} \right\}$$

So, we have :-

$$D^*_\infty \cong D^*_0 + [D^*_\infty \to D^*_\infty]^{\top}.$$

Next, we define ENV* and $\mathcal{D}^* \in (\text{EXP} \to (\text{ENV}^* \to D^*_\infty))$ in the same way as in 0.7.13, except that we use Ap^*, Ap^{*-1} and D^*_∞ instead of $Ap_{D_\infty}, Ap^{-1}_{D_\infty}$ and D_∞.

(ix), (x), (xi), (xii) and (xiii) Same as the corresponding parts of the previous model (0.7.13), with suitable *'s attached. Thus, our semantics is $<[\text{ENV}^* \to D^*_\infty], \mathcal{D}^*>$ and we write $\equiv^*$ and $\leqq^*$ for the induced equivalence relation and partial ordering (modulo $\equiv^*$) on EXP,

(xiv) $<[\text{ENV}^* \to D^*_\infty], \mathcal{D}^*>$ is a model but <u>not</u> an η-model. However,

$$(\varepsilon \xrightarrow{\eta} \delta) \Rightarrow (\varepsilon \leqq^* \delta).$$

There are cases when η-reductions are modelled, not only when the reductions could have been β's - e.g. :-

$$(\varepsilon \in \lambda I.\text{EXP}) \wedge (x \text{ is not free in } \varepsilon) \Rightarrow (\lambda x.\varepsilon(x) \equiv^* \varepsilon),$$

but also in their own right - e.g. $x(\lambda y.xy) \equiv^* xx$ while $\lambda y.xy \not\leqq^* x$,

(xv) This is 0.7.13(xv)*,

(xvi) Let ε and δ have β-η-normal forms. Then,

$$\mathcal{D}^*[\![\varepsilon]\!] \leqq \mathcal{D}^*[\![\delta]\!] \Rightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{\text{cnv}} \ \delta.$$

In particular,

$$\mathcal{D}^*[\![\varepsilon]\!] = \mathcal{D}^*[\![\delta]\!] \Rightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{\text{cnv}} \ \delta,$$

(xvii) $(\forall i \not\geq 0)(Y_0 \sqsubseteq^* Y_i)$,

(xviii) Defining $\mu^* : [D^*_\infty \to D^*_\infty]^\top \longrightarrow D^*_\infty$

$$d \longmapsto \begin{Bmatrix} \top, \text{ if } d = \text{(the extra) } \top \\ \mu d, \text{ if } d \in [D^*_\infty \to D^*_\infty] \end{Bmatrix},$$

$$\mathcal{D}^* \llbracket Y(\varepsilon) \rrbracket(\rho) = \mu^*(Ap^*(\mathcal{D}^* \llbracket \varepsilon \rrbracket(\rho))),$$

(xix) Let F be as in the proof of 0.7.13(xix). Then, $Y(F) \not\sqsubseteq^* I$,

(xx) $(\sigma \in SOL) \Rightarrow (\mathcal{D}^* \llbracket \sigma \rrbracket \neq \bot)$.

Now, extend $\mathcal{D}^*$ to $\mathcal{D}^{*\prime}$ in the same way as $\mathcal{D}$ was extended to $\mathcal{D}'$ in 0.7.17. Then,

(xxi) $<[ENV^* \to D^*_\infty], \mathcal{D}^{*\prime}>$ is an ω-model of the λ-Ω-calculus,

(xxii) $(\varepsilon \in NOH) \Rightarrow (\mathcal{D}^{*\prime} \llbracket \tilde{\varepsilon} \rrbracket = \bot)$,

(xxiii) $<[ENV^* \to D^*_\infty], \mathcal{D}^{*\prime} \circ\sim>$ is a semantics of the λ-calculus that is "approximate" to $<[ENV^* \to D^*_\infty], \mathcal{D}^*>$,

(xxiv) $(\varepsilon \xrightarrow{\beta} \delta) \Rightarrow (\mathcal{D}^{*\prime} \llbracket \tilde{\varepsilon} \rrbracket \sqsubseteq \mathcal{D}^{*\prime} \llbracket \tilde{\delta} \rrbracket)$,

(xxv) Hence, for all $\varepsilon \in EXP$, $\bigsqcup\{\mathcal{D}^{*\prime} \llbracket \tilde{\delta} \rrbracket \mid \varepsilon \xrightarrow{\beta} \delta\} \sqsubseteq \mathcal{D}^* \llbracket \varepsilon \rrbracket$, where the set is directed.

We also have Wadsworth's theorem,

(xxvi) For all $\varepsilon \in EXP$, $\mathcal{D}^* \llbracket \varepsilon \rrbracket = \bigsqcup\{\mathcal{D}^{*\prime} \llbracket \tilde{\delta} \rrbracket \mid \varepsilon \xrightarrow{\beta} \delta\}$,

(xxvii) $(\varepsilon \in SOL) \Leftrightarrow (\mathcal{D}^* \llbracket \varepsilon \rrbracket \neq \bot)$,

(xxviii) $<[ENV^* \to D^*_\infty], \mathcal{D}^*>$ is solvable.

(xxix) $<[ENV^* \to D^*_\infty], \mathcal{D}^*>$ is normal,

(xxx) $<EXP/\S, [\S]> \leq <[ENV^* \to D^*_\infty], \mathcal{D}^*> \leq <EXP/\beta\text{-}\eta\text{-}\simeq, [\beta\text{-}\eta\text{-}\simeq]>$.

Proof:-

(i),...,(xiii) -Similar to 0.7.13 -see Wadsworth [42].

(xiv) -Modelship is proved in the same way as for $<[ENV \to D_\infty], \mathcal{D}>$.

-We do not have an η-model but increases in value after η-reductions since the function space is a projection of the semantic space : in D_∞ they were isomorphic.

-For details of this and the examples cited - see Wadsworth [43].

(xv) -See 0.7.13(xv).

(xvi) -We are able to prove this in the same way as 0.7.13(xvi), since Böhm's theorem says we need only β-reductions to reduce the expressions in the contexts to their respective variables. This is just as well, since $<D_\infty^*, D^*>$ does not model η-reductions. This also explains why this result is weaker than the corresponding one for $<D_\infty, D>$. See also remark 0.7.23.

(xvii) and (xviii) -Similar to 0.7.13.

(xix) -See Wadsworth [45].

(xx) -Similar to 0.7.15(iii), noting that $Ap^{*-1}(f) = \bot$ iff $f = \bot$.

(xxi),...,(xxv) -Similar to 0.7.18.

(xxvi) -Similar to 0.7.19.

(xxvii) -Similar to 0.7.20(i).

(xxviii) -By part (xxvii).

(xxix)-See Wadsworth [46].

(xxx) -First ≤ since we have a substitutive model that equivalences members of INSOL.

-Second ≤, by 0.7.11(xi), since we have a normal substitutive model.

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0.7.23: REMARK:-

Thus, $<[ENV^* \to D_\infty^*], D^*>$ has most of the good points of

$<[ENV \to D_\infty], D>$ and has normality as well. Unfortunately, it has introduced some unpredictable quirks with regards when it models η-reductions. The model is not an η-model, but neither does it have all the properties we would like to see associated with β-models. In particular, 0.7.22(xxviii) is not really the result we would like to see corresponding to 0.7.13(xvi). What we wanted was that, whenever ε and δ have normal forms, then :-

$$D^*[\![\varepsilon]\!] \sqsubseteq D^*[\![\delta]\!] \Rightarrow \varepsilon \ \underline{\text{cnv}}\ \delta \Rightarrow D^*[\![\varepsilon]\!] = D^*[\![\delta]\!],$$

so that,

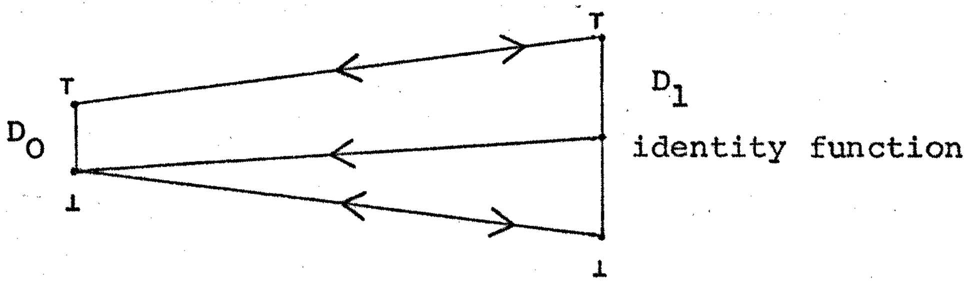
$$D^*[\![\varepsilon]\!] = D^*[\![\delta]\!] \Leftrightarrow \varepsilon \ \underline{\text{cnv}}\ \delta.$$

But this desire is quite simply denied by the example in 0.7.22(xiv) if we take $x(\lambda y.xy)$ for ε and xx for δ.

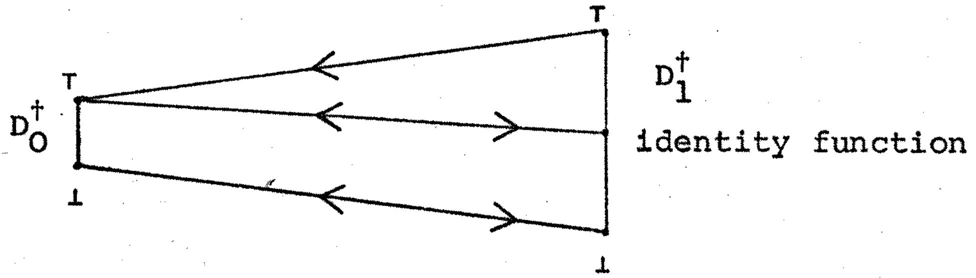
Next, we present a pathological version of 0.7.13, discovered by Park, in which things go drastically wrong.

0.7.24:EXAMPLE:- (Park)

Let $D_0^\dagger$ be the two point lattice, $\{\top, \bot\}$. Construct $D_\infty^\dagger$ exactly as in 0.7.13 except that we define the initial retraction, $D_0^\dagger \lhd D_1^\dagger$, differently. In 0.7.13, this was defined as :-



But, here we define it as :-



This is the only alternative way of making a projection. Now, define $Ap^\dagger$ and $D^\dagger \in (EXP \to [ENV^\dagger \to D_\infty^\dagger])$ as prescribed in 0.7.13. That things have gone very wrong can be deduced from the result :-

$$D^{\dagger}[\![\Delta\Delta]\!] \neq \bot.$$

In fact, it is possible, by starting with a more complicated $D_{0}^{\dagger\dagger}$ and choosing a suitable initial projection, to produce a system $D_{\infty}^{\dagger\dagger}$ in which $D^{\dagger\dagger}[\![\Delta\Delta]\!] = \top$!

Amongst the consequences are the facts that Y cannot equal the μ operator in the sense of 0.7.13(xviii) - otherwise, since 0.7.15 part (i) is still valid for this model, its part (ii) would lead to a contradiction - and that Wadsworth's theorem, 0.7.19, cannot hold since :-

$$\bigsqcup\{D^{\dagger\prime}[\![\delta]\!] \mid \Delta\Delta \xrightarrow{\beta} \delta\} = \bigsqcup\{D^{\dagger\prime}[\![\Delta\Delta]\!]\} = D^{\dagger\prime}[\![\Omega]\!]$$
$$= \bot \neq D^{\dagger}[\![\Delta\Delta]\!].$$

However, $<[ENV^{\dagger} \rightarrow D_{\infty}^{\dagger}], D^{\dagger}>$ is a substitutive β-η-model, even though a very strange one.

Proof:-

-See Park [47].

‡

0.7.25:REMARK:-

There is a well known theorem - see Scott [48] - that says that any complete continuous countably based lattice can be embedded as a retraction of P(ω) - the set of subsets of the natural numbers, itself a complete continuous lattice countably based by the finite subsets. Briefly, any element is uniquely represented by the limit of the basis elements topologically less than ($\prec$) it, so all we have to do is ennumerate that basis. For an account of this embedding in a particular instance, see section 7.4.

Therefore, it is not surprising that P(ω) should provide the domain for models of the λ-calculus. The following example defines a semantic function directly from EXP. It was discovered originally by Plotkin - [49] - and later, independently, by Scott - [50].

0.7.26:EXAMPLE:- (Plotkin/Scott)

Let $\{e_0, e_1, e_2, \ldots\ldots\ldots\ldots\}$ be a (recursive) enumeration of the finite subsets of ω. Everything relies upon the continuous lattice property of $P(\omega)$ - namely that all sets are characterised by their finite subsets :-

$$(S \subseteq \omega) \Rightarrow (S = \cup\{e_n | e_n \subseteq S\}).$$

We will define an application function that retracts $P(\omega)$ on to its continuous function space. Then, to construct the model, we just use the technique of environments exactly as in 0.7.13.

We find it easier to define the reverse application, Ap^{-1}, first. Let $f \in [P(\omega) \to P(\omega)]$ and $S \in P(\omega)$. Then,

$$\begin{aligned} f(S) &= f(\cup\{e_n | e_n \subseteq S\}) \\ &= \cup\{f(e_n) | e_n \subseteq S\} \\ &= \cup\{e_m | e_m \subseteq f(e_n) \text{ and } e_n \subseteq S\}. \end{aligned}$$

Hence, f is characterised by the set of pairs, $\{<e_n, e_m> | e_m \subseteq f(e_n)\}$. Define another ennumeration of pairs,

$$\nu : \omega \times \omega \longrightarrow \omega,$$

and we are ready to define :-

$$\begin{aligned} Ap_\omega^{-1} : P(\omega) \to P(\omega) &\longrightarrow P(\omega) \\ f &\longmapsto \{\nu<n,m> | e_m \subseteq f(e_n)\}. \end{aligned}$$

It is clear that we should then define :-

$$\begin{aligned} Ap_\omega : P(\omega) &\longrightarrow (P(\omega) \to P(\omega)), \\ S &\longmapsto Ap_\omega(S) \end{aligned}$$

where,

$$\begin{aligned} Ap_\omega(S) : P(\omega) &\longrightarrow P(\omega) \\ X &\longmapsto \cup\{e_m | e_n \subseteq X \text{ and } \nu<n,m> \in S\}. \end{aligned}$$

Then,

(i) Ap_ω is completely additive, Ap_ω^{-1} is continuous and both are doubly strict,

(ii) $Ap_\omega \in [P(\omega) \to [P(\omega) \to P(\omega)]]$,

(iii) $Ap_{\omega} \circ Ap_{\omega}^{-1}(f) = f$,

(iv) $Ap_{\omega}^{-1} \circ Ap_{\omega}(S) \supseteq S$.

Next, we define ENV^{ω} and $P \in (EXP \to (ENV^{\omega} \to P(\omega)))$ in the same way as in 0.7.13. Then,

(v) $P[\![\varepsilon]\!] \in [ENV^{\omega} \to P(\omega)]$,

(vi) $<[ENV^{\omega} \to P(\omega)], P>$ is a well-defined substitutive semantics of the λ-calculus. The induced partial ordering, $\sqsubseteq^{\omega}$ (modulo the semantic equivalence $\equiv^{\omega}$), is also substitutive,

(vii) $<[ENV^{\omega} \to P(\omega)], P>$ is a model but not an η-model. However,

$$(\varepsilon \xrightarrow{\eta} \delta) \Rightarrow (\varepsilon \sqsupseteq^{\omega} \delta).$$

As with D_{∞}^{*}, we have :-

$$(\varepsilon \in \lambda I.EXP) \wedge (x \text{ is not free in } \varepsilon) \Rightarrow (\lambda x.\varepsilon(x) \equiv^{\omega} \varepsilon).$$

But, unlike D_{∞}^{*}, we have :-

$$x(\lambda y.xy) \not\sqsubseteq^{\omega} xx.$$

(viii) Let ε and δ have normal forms. Then,

$$P[\![\varepsilon]\!] \sqsubseteq P[\![\delta]\!] \Rightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{cnv}\ \delta$$

and so,

$$P[\![\varepsilon]\!] = P[\![\delta]\!] \Rightarrow \varepsilon \ \beta\text{-}\eta\text{-}\underline{cnv}\ \delta,$$

(ix) $P[\![Y(\varepsilon)]\!](\rho) = \mu(Ap_{\omega}(P[\![\varepsilon]\!](\rho)))$,

(x) $Ap_{\omega}(P[\![I]\!](\rho))$ = identity function on $P(\omega)$ and, so, $P[\![\Delta\Delta]\!] = \perp$,

(xi) $(\sigma \in SOL) \Rightarrow (P[\![\sigma]\!] \neq \perp)$.

<u>Proof:-</u>

(i),...,(iv) -Straightforward - see Scott [51].

(v),...,(xi) -Similar to the corresponding results of 0.7.22.

-Part (vii) is interesting. The reason why η-reductions <u>decrease</u> the representation in $P(\omega)$ is because $Ap_{\omega}^{-1} \circ Ap_{\omega}$ is "greater than" the identity function ; in D_{∞}^{*}, it was "less than".

-<u>Claim:</u> $x(\lambda y.xy) \not\sqsubseteq^{\omega} xx$:-

-Let $S = \{\nu<n,n> | n \geq 0\}$.

-Then, $Ap_{\omega}(S) = Id_{P(\omega)}$, the identity function on $P(\omega)$.

-However, $Ap_{\omega}^{-1}(Id_{P(\omega)}) = \{\nu\langle n,m\rangle | e_m \subseteq e_n\}$, which is *strictly* larger than S.

-Let $\rho \in ENV^{\omega}$ such that $\rho(x) = S$.

-Then,
$$\begin{aligned} P[\![x(\lambda y.xy)]\!](\rho) &= Ap_{\omega}(P[\![x]\!](\rho))(P[\![\lambda y.xy]\!](\rho)) \\ &= Ap_{\omega}(S)(Ap_{\omega}^{-1}(\lambda\pi \in P(\omega).P[\![xy]\!]([\pi/y]\rho))) \\ &= Ap_{\omega}^{-1}(\lambda\pi \in P(\omega).Ap_{\omega}(\rho(x))(\pi)) \\ &= Ap_{\omega}^{-1}\circ Ap_{\omega}(S) \\ &\supset S \\ &= Ap_{\omega}(S)(S) \\ &= P[\![xx]\!](\rho). \end{aligned}$$

$\dashv$

0.7.27:REMARK:-

There are a number of obvious questions to be answered about the above model. The most important one, we feel, is whether Wadsworth's theorem applies. If it does, then $\langle[ENV^{\omega} \to P(\omega)],P\rangle$ will be solvable and normal, the fixed-point combinators will be equivalenced and we will be able to place the semantics between $\langle EXP/\S,[\S]\rangle$ and $\langle EXP/\beta\text{-}\eta\text{-}\simeq,[\beta\text{-}\eta\text{-}\simeq]\rangle$. Is it $\leq \langle EXP/\simeq,[\simeq]\rangle$? Unfortunately, we have not had time in this thesis to consider these questions. By the way, in [52], Scott calls Ap_{ω} "*fun*" and Ap_{ω}^{-1} "*graph*". One last note we find interesting : since $[ENV^{\omega} \to P(\omega)]$ is itself a c.c.c. lattice, it is embedded as a retract of $P(\omega)$ and, so, we may consider $P(\omega)$ as being, on its own, the underlying set of the model.

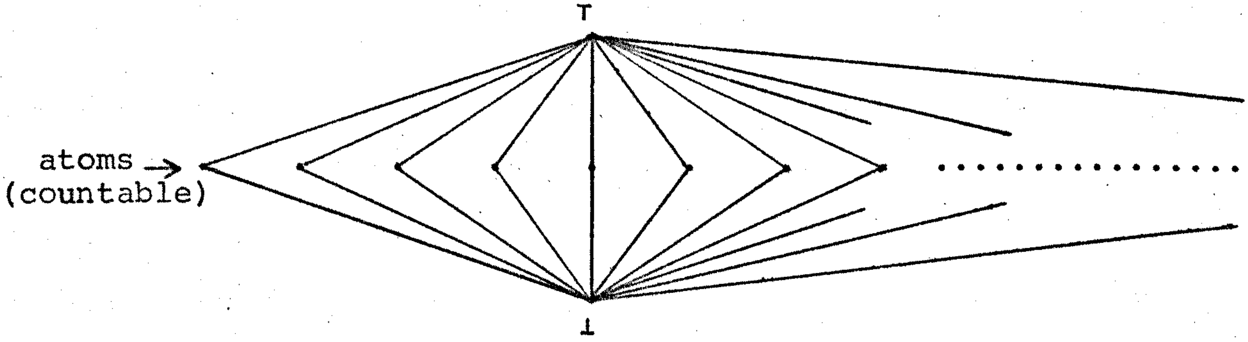
The last four models are all of a similar nature and we call their type "*Scott-models*". Briefly, to build a Scott-model, all we need is a lattice, L, whose continuous function space, $[L \to L]$, can be embedded as a retract in L by means of "application" functions, Ap_L and Ap_L^{-1}. Then, forming the set of "environments" ENV^L as $(I \to L)$, we can produce a well-defined semantics, $\langle[ENV^L \to L],L\rangle$, by means of the equations :-

$$\begin{array}{rl} \mathcal{L}[\![x]\!](\rho) & := \rho(x), \\ \mathcal{L}[\![\lambda x.\varepsilon]\!](\rho) & := Ap_L^{-1}(\lambda \ell \in L.\mathcal{L}[\![\varepsilon]\!]([\ell/x]\rho)) \\ \text{and } \mathcal{L}[\![\varepsilon(\delta)]\!](\rho) & := Ap_L(\mathcal{L}[\![\varepsilon]\!](\rho))(\mathcal{L}[\![\delta]\!](\rho)). \end{array}$$

These semantics will always be a model and the induced relations, $\equiv_L$ and $\sqsubseteq_L$, will be substitutive. η-conversion depends on how $Ap_L^{-1} \circ Ap_L$ behaves. In D_∞ and $D_\infty^\dagger$, this was the identity and we got an η-model; while in D_∞^* it was smaller than the identity and in $P(\omega)$ it was greater, with corresponding effects on the semantics of η-reductions.

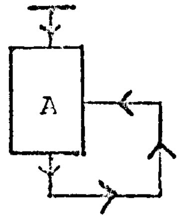
The model constructed in this thesis is not a Scott-model in the above sense. Its motivation is to capture the essential features about semantics that allow Wadsworth's theorem to work. Decent semantics should allow this theorem since it reflects very basic and natural ideas about the meaning of λ-expressions (see 0.7.21). Clearly, semantics that are $\leq \langle EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL, [\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL]\rangle$ do not have it, since amongst its implications is the equivalencing of the fixed-point combinators, $\{Y_i | i \geq 0\}$. We do not want to consider semantics $\geq \langle EXP/\underline{cnv}_V{}^\sim, [\underline{cnv}_V{}^\sim]\rangle$, since they make $Y \equiv \Delta\Delta$. The Scott-models defined by D_∞ and D_∞^* have it, $P(\omega)$ might, but $D_\infty^\dagger$ does not. Although the λ-calculus was invented to represent functions, it remains in fact a purely formal syntactical system, representing only <u>computable</u> functions. Thus, it would be a surprise if Scott-models, constructed from set-theoretic <u>continuous</u> function spaces (N.B. continuous does not imply computable - in fact, $\mathcal{D}[\![EXP]\!]$ is spread very thinly across $[ENV \to D_\infty]$), were minimal in this very basic respect.

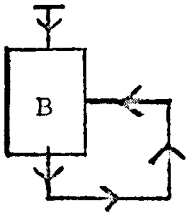
What we need is something more closely related to EXP, and, preferably, still with an explicit construction. This rules out the extensional equivalence models, one of which, in any case, lies on the wrong side of the D_∞^*-Scott-model for minimality. The key is provided by Scott's paper : "The Lattice of Flow Diagrams" - [53]. In this paper, the elements of the lattices are all "syntactical" entities. We start with two "atomic" lattices isomorphic to :-



In one, the "atoms" are the function symbols allowed in the flow diagrams and, in the other, they are the predicate symbols. In this first lattice, we can represent flow diagrams of length one - i.e. just consisting of one function box. We construct, then, larger and larger lattices from these basic ones by purely syntactical means (e.g. cross-products), so as to be able to represent longer and longer loop-free flow diagrams. These lattices fit together as projections,enabling us to take the inverse limit in which we can represent <u>all</u> flow diagrams. If we make a small change in Scott's construction we can produce an object from which useful analogies for the λ-calculus can be drawn. We show that this change makes no essential difference to the (simple) flow diagram semantics. To explain and justify this, we assume that the reader is familiar with [53] and its notation which we use for the next three pages.

Consider the following two diagrams :-





These can be thought to correspond to the unsolvable λ-expressions YA and YB. Now, we have spent some time arguing that sensible semantics ought to equivalence such objects. However, in the flow diagram lattice, E, they are, respectively, the minimal fixed points of the continuous functions $(\lambda x \in E.A;x)$ and $(\lambda x \in E.B;x)$, which we shall write as $(\mu x \in E.A;x)$ and $(\mu x \in E.B;x)$. Since we used "pure" cross-products in the construction of E, we have :-

$$A;\perp = (A,\perp) \neq (\perp,\perp) = \perp$$

Thus, $(\mu x \in E.A;x) \neq (\mu x \in E.B;x)$. However, take any <u>simple</u> semantics, $\mathcal{V} \in [E \to [S \to S]]$. We have some state lattice, S, and function/predicate semantics, $\mathcal{F} \in [F \to [S \to S]]$ and $\mathcal{B} \in [B \to [S \to T]]$, that are <u>simple</u> in the sense that $\mathcal{F}\llbracket\perp\rrbracket = \perp$ and $\mathcal{B}\llbracket\perp\rrbracket = \perp^{(*)}$ - see page 41, section 6 of [53]. We have :-

$$\mathcal{V}\llbracket\mu x \in E.A;x\rrbracket = \mathcal{V}\llbracket\bigsqcup_{n=0}^{\infty} A^n;\perp\rrbracket$$

$$= \bigsqcup_{n=0}^{\infty} \mathcal{V}\llbracket A^n;\perp\rrbracket, \text{ since } \mathcal{V} \text{ is continuous.}$$

$$= \bigsqcup_{n=0}^{\infty} \mathcal{V}\llbracket A^n\rrbracket \circ \mathcal{V}\llbracket\perp\rrbracket = \bigsqcup_{n=0}^{\infty} \mathcal{V}\llbracket A^n\rrbracket \circ \perp = \perp.$$

Thus, the two diagrams are semantically equivalent to bottom.

(*) Such lattice functions are called <u>STRICT</u>. If we also have the top element preserved, they are called <u>DOUBLY STRICT</u>.

However, they would fail to be equivalent if we chose a non-strict(*) F or B. This is sometimes necessary when we wish to describe side-effects. For example, suppose $\llbracket A\rrbracket$ and $\llbracket B\rrbracket$ are $\llbracket\underline{\text{print}}\ \text{"A"}\rrbracket$ and $\llbracket\underline{\text{print}}\ \text{"B"}\rrbracket$ and S is the lattice of strings over some alphabet (with an extra top), partially ordered by "initial part of". If the program goes into an infinite loop, we do not lose the current output. Thus,

$$F\llbracket\bot\rrbracket = \text{identity}.$$

Also, because F must be monotonic and we cannot rewrite previous output, we must have :-

$$F\llbracket f\rrbracket \sqsupseteq \text{identity},$$

for all $f \in F$. So, in these semantics, the diagrams are not equivalent. There is something unsavoury here in that the semantics of the loop is only the least fixed point above the identity function, while in all simple semantics :-

$$V\llbracket \mu x \in E.A;x\rrbracket = \mu x \in [S \rightarrow S].V\llbracket A\rrbracket \circ x.$$

In the clinical world of the λ-β-K-calculus, we do not have such nasty, noisy things as line-printers and, so, we are perfectly justified in keeping our semantics "simple". In this case, we have, loosely speaking :-

$$V\llbracket A;\bot\rrbracket = V\llbracket P \rightarrow \bot,\bot\rrbracket = \bot,$$

in their respective lattices. Therefore, it makes no difference semantically if we build in these equivalences during the construction of E - i.e. construct $D_n;D_n$ as $(D_n \times D_n)/\sim$, where $\sim$ is an equivalence relation such that $(d_n,\bot) \sim (\bot,\bot)$ etc.... It is easy to show that all the D_n's remain complete lattices and that the projection functions work properly and continuously so that

---

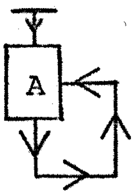
(*) See previous page.

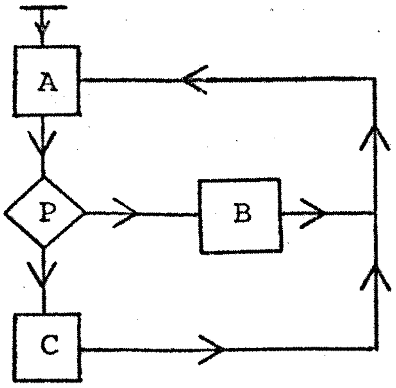
the inverse limit, E, can be constructed exactly as before. E is now much more economical in that infinite loops do not require the whole of the inverse limit in order to be represented, but can be handled merely by the bottom element of the original "atomic" lattice. Further, E now directly reflects our feeling :-

$$(\mu x \in E.A;x) \equiv YA \equiv YB \equiv (\mu x \in E.B;x).$$

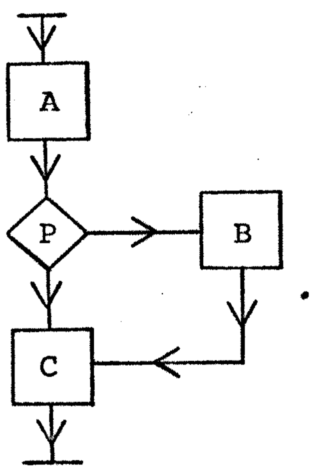
We can now classify flow diagrams according to our new E as follows :-

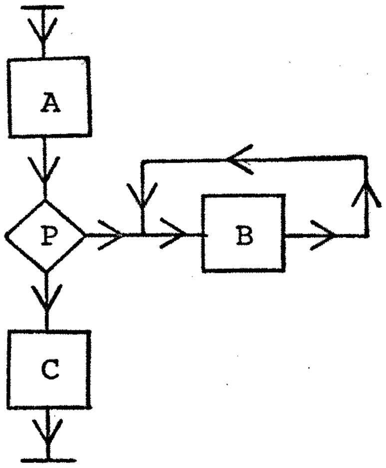
(i) diagrams that "never terminate" are represented by the $\perp$ element - e.g. :-



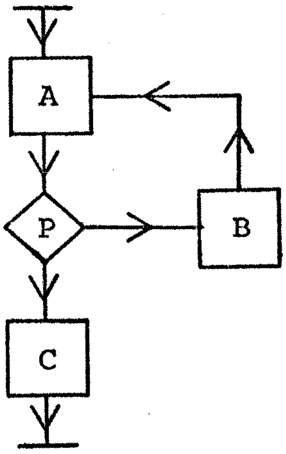


(ii) "loop-free" diagrams except for some "non-terminating" branches always fit in at some finite stage in the inverse limit - e.g. :-





(iii) the rest can be unravelled to infinitely long "loop-free" diagrams that fit into the limit - e.g. :-



We draw an analogy between the above three cases and λ-expressions that are :-

(i) in INSOL - e.g. ΔΔ - ("bottom" λ-expressions),

(ii) in SOL and reduces to an HNF, all of whose sub-parts are either in HNF or INSOL - e.g. x(ΔΔ), anything with a normal form - ("finite" λ-expressions),

(iii) in SOL but does not reduce as above - e.g. Y - ("infinite" λ-expressions).

In this case, we can see why the non-normality of models of the λ-calculus may not be such a bad thing after all. Consider the following two schema/flow-diagrams (acknowledgements to Wadsworth for this example) :-

$f(x) \equiv x$ (c.f. I)

$g(x) \equiv$ <u>if</u> zero(x) <u>then</u> x <u>else</u> succ(g(pred(x)) (c.f. Y(F))

Now, f is loop-free of length 1 and so is in "normal form" of type (ii), while g is recursive but its branches do terminate sometimes and so it is of type (iii). In the purely syntactical lattice of flow-diagrams, the two schema are not equivalent - the model distinguishes between all three types (i.e. it is solvable and normal). Clearly, however, there are models derivable from it, based on function spaces, in which f and g are equivalent. Nobody minds the non-normality of these models - on the contrary, that is their whole point - although there would be trouble if they failed to distinguish type (i) from the other two (i.e. were not solvable).

How, then, should we construct our "Lattice of λ-Expressions"? Obviously, we should start with an "atomic" lattice, $I' := I\cup\{\top,\bot\}$. There, we can represent expressions of length 1 - i.e. variables. We might be tempted to continue as follows :-

$$E_0 := I'$$

$$E_{i+1} := E_i + (I \times E_i) + (E_i \times E_i),$$

following the context-free grammar definition of EXP. We could define an initial projection $E_0 \lhd E_1$, by mapping all non-$\top$ elements of $(I \times E_0) + (E_1 \times E_1)$ to $\bot$ and leaving E_0 alone ; the remaining projections, $E_i \lhd E_{i+1}$, could then be defined inductively in the obvious way. Clearly, we can represent all λ-expressions in the inverse limit - but, all we need, in fact, is the direct limit! This does not help very much as there are no interesting limit points in the image of EXP which might lead to new equivalences. In fact, the semantics produced in this way is no better than the trivial one, <EXP,id>. We could improve the situation by defining a construction $\lambda I.E_i$, which is $(I \times E_i)$ with α-convertible forms equivalenced together, thus getting the minimal α-model. (N.B. this α-equivalence on $(I \times E_i)$ is very quickly decidable and, so, does not really alter the explicit nature of the construction). We could try to build in more meaning by defining $E_i(E_i)$, which is $E_i \times E_i$ with all elements $(\perp,\varepsilon_i)$ equivalenced and elements $(\top,\varepsilon_i)$ equivalenced. This is certainly more interesting but, since neither $\perp$ nor $\top$ will be in the image of EXP, makes no difference!

We are on the wrong track. Referring back to our analogy with flow-diagrams gives us the guidance we need. We have :-

"loop-free" ↔ "normal form"

"non-terminating" ↔ "unsolvable"

The finite lattices in the inverse limit flow-diagram lattice were constructed to accomodate larger and larger loop-free diagrams. Therefore, we should build our finite lattices only to accomodate larger and larger normal forms. To tailor them thus, we simply follow the context-free definition of normal forms (0.4.3). Since this consists of two equations, we construct a pair of simultaenous inverse limits, E_∞ and A_∞, from :-

$$E_0 := A_0 := I'$$
$$E_{i+1} := \lambda I.E_i + A_i$$
$$A_{i+1} := I' + A_i(E_{i+1}).$$

Examples of the sort of representations of λ-expressions we hope to get in E_∞ follow :-

$x \rightarrow \langle x,x,x,x,x,x,x,x,x,x,x,x,x,x,x,x,x,x,x,........\rangle$

$xy \rightarrow \langle \perp,\perp,xy,xy,xy,xy,xy,xy,xy,xy,xy,xy,xy,........\rangle$

$\lambda x.y \rightarrow \langle \perp,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,........\rangle$

$\lambda x.xy \rightarrow \langle \perp,\perp,\perp,\lambda x.xy,\lambda x.xy,\lambda x.xy,\lambda x.xy,\lambda x.xy,........\rangle$

$f(yy) \rightarrow \langle \perp,\perp,f(\perp),f(yy),f(yy),f(yy),f(yy),f(yy),.......\rangle$

$\lambda y.f(yy) \rightarrow \langle \perp,\perp,\perp,\lambda y.f(\perp),\lambda y.f(yy),\lambda y.f(yy),.............\rangle$

$b(\lambda c.dd) \rightarrow \langle \perp,b(\perp),b(\perp),b(\perp),b(\lambda c.dd),b(\lambda c.dd),...........\rangle$

$(\lambda y.xy)(b) \rightarrow \langle \perp,\perp,xb,xb,xb,xb,xb,xb,xb,xb,xb,xb,xb,........\rangle$

$\Delta\Delta \rightarrow <\perp,\perp,\perp,\perp,\perp,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots>$

$x(\Delta\Delta) \rightarrow <\perp,\perp,x(\perp),x(\perp),x(\perp),\ldots\ldots\ldots\ldots\ldots\ldots>$

$Y(f) \rightarrow <\perp,\perp,f(\perp),f(f(\perp)),f(f(f(\perp))),f(f(f(f(\perp)))),\ldots>$

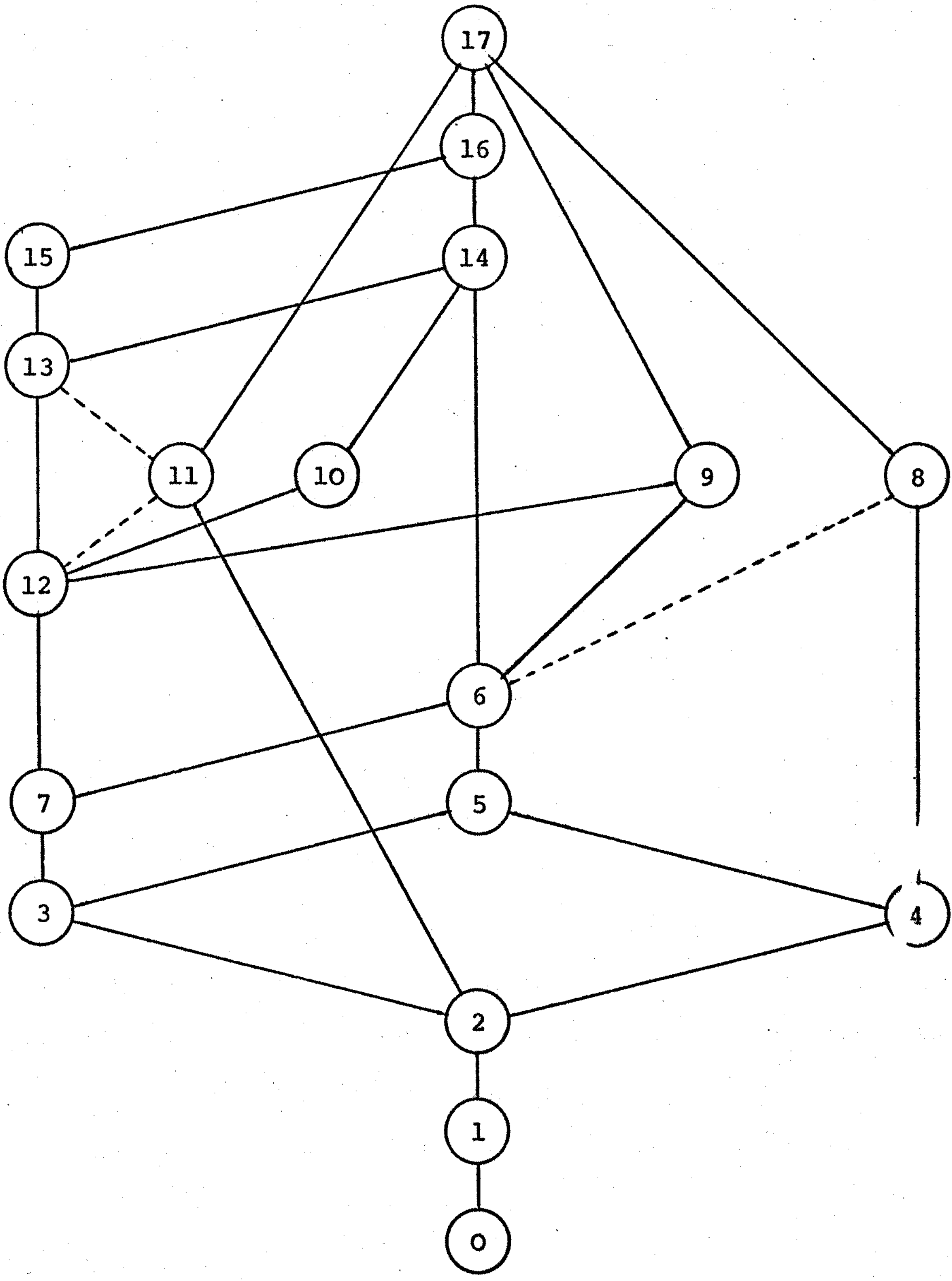
Note that we will certainly not have an η-model. In fact, x and λy.xy will be incomparable - the same is true for x(λy.xy) and xx. Hence, we can immediately deduce that this semantics is not derivable from the Scott-D_∞,D_∞^*-models and that it is not continuously derivable from the Scott-P(ω)-model. Thus, we have a candidate for minimality with respect to Wadsworth's theorem.

Note, also, that expressions are naturally represented by the limit of some of their approximate reductions - namely their coordinates. These coordinates do not provide all the approximate reductions possible. To get Wadsworth's theorem, which we shall need to prove "modellship", we have to show that they provide a "dense" subset of them - i.e. their limit is the same as the limit of all possible approximate reductions. This is non-trivial : we have to delve into the properties of certain "evaluation mechanisms" to establish it.

Just as syntax does not tell us the whole story about λ-calculus, neither does semantics. There is also what Wadsworth called pragmatics, which concerns itself with, amongst other things, evaluation mechanisms. These fields of study are not distinct, but are richly inter-dependent. In this thesis, we find that we have to go into the pragmatics of λ-calculus ("inside-out reductions") in order to establish a basic property ("modelship") of a semantics (E_∞) that is essentially syntactical in its nature.

0.7.28:SUMMARY:-

We hope that this section on the semantics of λ-K-β-η-calculus has provided sufficient information for the motivation of the construction of the E_∞ model and, also, for a framework in which to view it. We summarise the semantics we have discussed by means of the following lattice :-



where 0 is <EXP,id>,

1 is <EXP/α,[α]>,

2 is <EXP/$\underline{\text{cnv}}$,[$\underline{\text{cnv}}$]>,

3 is <EXP/$\underline{\text{cnv}}_{\text{v}}$INSOL,[$\underline{\text{cnv}}_{\text{v}}$INSOL]>,

4 is $<EXP/\beta\text{-}\eta\text{-}\underline{cnv},[\beta\text{-}\eta\text{-}\underline{cnv}]>$,

5 is $<EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL,[\beta\text{-}\eta\text{-}\underline{cnv}_V INSOL]>$,

6 is $<EXP/\beta\text{-}\eta\text{-}\S,[\beta\text{-}\eta\text{-}\S]>$,

7 is $<EXP/\S,[\S]>$,

8 is $<[ENV^{\dagger} \rightarrow D_{\infty}^{\dagger}],\mathcal{D}^{\dagger}>$,

9 is $<[ENV \rightarrow D_{\infty}],\mathcal{D}>$,

10 is $<[ENV^{*} \rightarrow D_{\infty}^{*}],\mathcal{D}^{*}>$,

11 is $<[ENV^{\omega} \rightarrow P(\omega)],\mathcal{P}>$,

12 is $<E_{\infty},\mathcal{E}>$,

13 is $<EXP/\simeq,[\simeq]>$,

14 is $<EXP/\beta\text{-}\eta\text{-}\simeq,[\beta\text{-}\eta\text{-}\simeq]>$,

15 is $<EXP/\underline{cnv}_V\sim,[\underline{cnv}_V\sim]>$,

16 is $<EXP/\beta\text{-}\eta\text{-}\underline{cnv}_V\sim,[\beta\text{-}\eta\text{-}\underline{cnv}_V\sim]>$

and 17 is $<\{*\},const>$.

In the above lattice, the relations ($7 \leq 12 \leq 13$) will be proved later in 6.8.3 and ($12 \leq 10, 12 \leq 9$) in 7.6.14. The "dotted" relations ($12 \leq 11 \leq 13, 6 \leq 8$) are only conjectures. All the semantics, excepting {3,5,15,16}, are substitutive. Above 1, they are all α-models; above 2, they are all models; above 4, they are all β-η-models; above 3, they all equivalence elements of INSOL; above 12, they all equivalence the fixed-point combinators (see 4.0.5) below 14, they are all normal and solvable. Also, 14 is the maximal normal substitutive model and we will show that 12 is the minimal continuous (≡ Wadsworth's theorem is true - see section 7.6) semantics.

<u>0.7.29:POSTSCRIPT:-</u> (Added in print)

There are, of course, many more semantics than we have discussed here and it should be one aim of computer science theory to build a comprehensive classification of them all. Each new semantics gives us new insights into the subject, sometimes in entirely unexpected ways (e.g. 9 and its non-normality or 12 and inside-out reductions). Further curiosities might be uncovered by a study of 8. Another worthwhile pursuit might be to see how the semantics can cope with the addition of extra constants and reductions to the λ-calculus (e.g. numbers and pairs and their associated functions).

There are also many other relations between the above 17 semantics to consider. For instance, are they all different? Levy has pointed out that $7 \not\leq 12$: because $\lambda f.(\lambda y.f^2(yy))(\lambda y.f^2(yy))$ and $\lambda f.f((\lambda y.f^2(yy))(\lambda y.f^2(yy)))$ are different in 7 but are both equivalent to Y in 12. In this case, are there any interesting semantics between 7 and 12? What about between 6 and 14??? Are there any positive relationships omitted? Yes, there are! The diagram on page 93 is not quite a lattice since neither $12_{\cup}6$ nor $14_{\cap}9$ are indicated. However, in as yet unpublished work, Wadsworth and J.M.E.Hyland (Christ Church, Oxford) have shown, independently, that 9 is the substitutive restriction of the maximal solvable semantics, $<\{\perp,\top\},(\lambda\varepsilon \in \text{EXP}.\ \varepsilon \in \text{SOL} \rightarrow \top,\perp)>$ - i.e. "9 is the maximal solvable substitutive semantics", although they do not quite put it like that! Thus, $14 \leq 9$. Also, Hyland has proved Wadsworth's theorem in 11 (i.e. "11 is continuous"), and, so, we will have $12 \leq 11 \leq 14$.

The process of taking the substitutive restrictions and substitutive (transitive) closures of suitably chosen elementary non-substitutive semantics seems to yield quite powerful results. Further, if we start with a (β-η-)model, the process is sure to yield a (β-η-)model. We also get very pleasing computational analogies : "if two procedures, A and B, are related in some way and we enclose them, in turn, within any larger procedure, C[ ], then the overall procedures, C[A] and C[B], are similarly related."

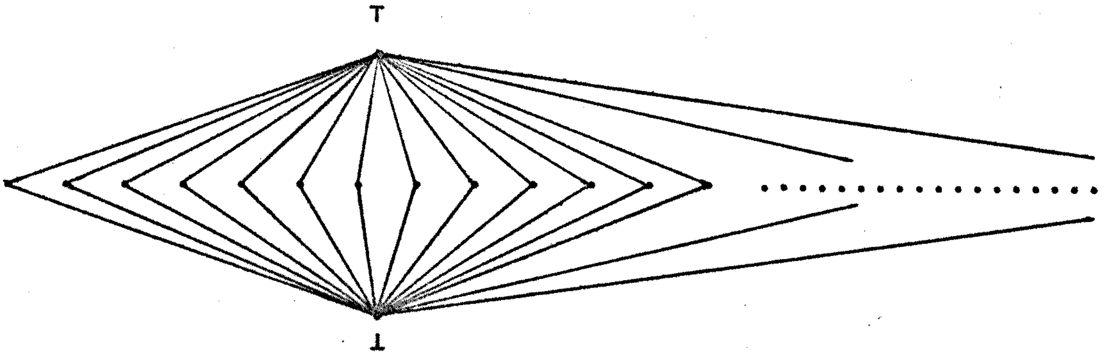
It seems, therefore, highly instructive to pursue this approach and we note that such characterisations of 11 and 12 have been made by Hyland. We wonder about the substitutive restrictions of 3 and 5. Finally, in view of the above characterisation of 9, we may also be wondering what is the maximal normal substitutive semantics - i.e. the substitutive restriction of the maximal normal semantics, $\langle\{\top,\bot\},(\lambda\varepsilon \in EXP.\varepsilon \text{ has normal form} \to \top,\bot)\rangle$. For a variety of obvious reasons, it is a (β-η-)model and, so, by 0.7.11(xi), it must be 14! Hence, we have a simpler characterisation of 14 together with a strengthened Morris' theorem to the effect that "14 is the maximal normal substitutive semantics".

1:THE FINITE LATTICES, E_n AND A_n.

1.0:Constructing the Base:-

1.0.0:DEF:-

Let I´ be the countable simple atomic lattice :-



Let I be the set of "atoms", $I'\backslash\{\top,\bot\}$. (We shall use symbols like $a,b,c,x,y,z,x_1,x_2,\ldots$etc... for elements of I - the context in which they are used preventing confusion with the variables of EXP.)

1.0.1:LEMMA:-

I´ is a complete continuous lattice of finite depth, 3, with all its elements isolated.

Proof:-

-Clearly, it has a finite depth of 3.

-∴, by 0.6.13 and 0.6.16, we have the result.

ǂ

1.0.2:DEF:-

Let x,y ∊ I. Then,

$$[x/y]: I' \longrightarrow I'$$

$$i \longmapsto \begin{Bmatrix} i, \text{ if } y \neq i \\ x, \text{ if } y = i \end{Bmatrix}.$$

1.0.3:LEMMA:-

(i) $[x/x]$ is the identity map on I'.

(ii) $[x/y]i = \begin{Bmatrix}\top \\ \bot\end{Bmatrix} \Longleftrightarrow i = \begin{Bmatrix}\top \\ \bot\end{Bmatrix}$.

(iii) $(a \neq d)\wedge(b \neq c,d) \Rightarrow$ ($[a/b]$ and $[c/d]$ commute).

(iv) $[x/y]$ is monotone and so continuous.

Proof:-

-Straightforward checking of all possible cases.

-In part (iv), monotone $\Rightarrow$ continuous, by 0.6.13, 0.6.16 and 1.0.1.

ǂ

1.0.4:DEF:-

Let $x \in I$. Then, x is NOT FREE IN $y \in I'$ if $x \neq y$.

1 0.5:LEMMA:-

(i) x is not free in $\top, \bot \in I'$.

(ii) $(x \neq y) \Rightarrow$ (y is not free in $[x/y]z$).

(iii) (y is not free in $z \in I'$) $\Rightarrow$ ($[x/y]z = z$).

(iv) $(x \neq a)\wedge$(x is not free in z) $\Rightarrow$ (x is not free in $[a/b]z$).

(v) (y is not free in w) $\Rightarrow$ ($[x/y][y/z]w = [x/z]w$).

(vi) (x is not free in y,z) $\Rightarrow$ (x is not free in $y \sqcup z, y \sqcap z$).

(vii) Let $x,a \in I$. Then,

$$[x/a]\begin{Bmatrix} y \sqcup z \\ y \sqcap z \end{Bmatrix} \begin{Bmatrix} \sqsupseteq \\ \sqsubseteq \end{Bmatrix} \begin{Bmatrix} [x/a]y \sqcup [x/a]z \\ [x/a]y \sqcap [x/a]z \end{Bmatrix}.$$

Further, if x is not free in $y,z \in I'$, there is equality.

(viii) $|\{x \mid x \text{ is free in } z \in I'\}| \leq 1$.

Proof:-

(i),(ii),(iii) and (iv) - Trivial.

(v) and (vi) -Straightforward checking of all possible cases.

(vii) -First part is a consequence of 1.0.3(iv).

-The equality part is more straightforward checking.

(viii) -By "x is free" we mean "not(x is not free)".

-Clearly, $|\{x \mid x \text{ is free in } z \in I'\}| = 0$ or 1.

ǂ

1.0.6:DEF:- (Base Lattices)

Let $E_0 := A_0 := I'$.

1.0.7:REMARK:-

We are trying to construct lattices of normal forms. A syntax for them is given in 0.4.3. In following these rules, we find we have to construct two sequences of lattices in parallel. Hence, the need for two base lattices.

We need all of the following induction hypothesis to carry it through.

1.0.8:DEF:- (Induction Hypothesis - G)

$G(i) \equiv$ (E_i, A_i are complete continuous lattices of finite depth with all their elements isolated)

& ([x/x] is the identity function on E_i, A_i)

& (($a \neq d) \wedge (b \neq c,d$) $\Rightarrow$ ([a/b] and [c/d] commute))

& ([x/y] is a continuous function on E_i, A_i)

& (($x \neq y$) $\Rightarrow$ (y is not free in $[x/y]\eta$, where $\eta \in E_i, A_i$))

& ($[x/y]\eta = \begin{Bmatrix} \top \\ \bot \end{Bmatrix} \Leftrightarrow \eta = \begin{Bmatrix} \top \\ \bot \end{Bmatrix}, \forall \eta \in E_i, A_i$)

& (x is not free in $\begin{Bmatrix} \top \\ \bot \end{Bmatrix}$ in E_i, A_i)

& ((y is not free in $\eta \in E_i, A_i$) $\Rightarrow$ ($[x/y]\eta = \eta$)

& (($x \neq a) \wedge$ (x is not free in $\eta \in E_i, A_i$) $\Rightarrow$ (x is not free in $[a/b]\eta$))

& ((y is not free in $\eta \in E_i, A_i$) $\Rightarrow$ ($[x/y][y/z]\eta = [x/z]\eta$))

& ((x is not free in $\eta, \eta' \in E_i, A_i$) $\Rightarrow$ (x is not free in $\eta \sqcup \eta'$, $\eta \sqcap \eta' \in E_i, A_i$))

& ($[x/a]\begin{Bmatrix} \eta \sqcup \eta' \\ \eta \sqcap \eta' \end{Bmatrix} \begin{Bmatrix} \sqsupseteq \\ \sqsubseteq \end{Bmatrix} \begin{Bmatrix} [x/a]\eta \sqcup [x/a]\eta' \\ [x/a]\eta \sqcap [x/a]\eta' \end{Bmatrix}$) $\wedge$ (if x is not free in η, η', we have equality)

& ($|\{x \mid x \text{ is free in } \eta \in E_i, A_i\}| < \infty$).

1.0.9:THEOREM:-

$G(0)$.

Proof:-

-By 1.0.1,1.0.3 and 1.0.5.

$\nmid$

1.1:Constructing the Rest:-

1.1.0:DEF:-

Suppose $G(i)$, for some $i \geq 0$. Then, α_{i+1} is a relation on $I \times E_i$ such that :-

$(x,\varepsilon)\ \alpha_{i+1}\ (y,\eta)$ if (there exists $z \in I$)(z is not free in ε,η)

$$([z/x]\varepsilon = [z/y]\eta).$$

1.1.1:REMARK:-

The α_i's are to model the α-conversion of λ-calculus.

1.1.2:LEMMA:-

$$(x,\varepsilon)\ \alpha_{i+1}\ (y,\eta) \iff (\forall z \in I)([z/x]\varepsilon = [z/y]\eta).$$

Proof:-

($\Rightarrow$) -We have $z \in I$ and not free in $\varepsilon,\eta \in E_i$ such that $[z/x]\varepsilon = [z/y]\eta$.

-$\therefore$, for any $z' \in I$, $[z'/x]\varepsilon = [z'/z][z/x]\varepsilon$, by $G(i)$.

$$= [z'/z][z/y]\eta$$

$$= [z'/y]\eta, \text{ by } G(i).$$

($\Leftarrow$) -$|\{z|z \text{ is free in } \varepsilon \text{ or } \eta\}| \leq |\{z|z \text{ is free in } \varepsilon\}|$

$$+ |\{z|z \text{ is free in } \eta\}$$

$$< \infty, \text{ by } G(i).$$

-But $|I| = \infty$ and so $|\{z|z \text{ is not free in } \varepsilon \text{ and } \eta\}| > 0$.

-Hence, there exists $z \in I$ not free in ε,η.

-We always had $[z/x]\varepsilon = [z/y]\eta$.

$\nmid$

1.1.3:REMARK:-

The above characterisation makes α_{i+1} easier to work with.

1.1.4:COR:-

(i) $(x,\top)\ \alpha_{i+1}\ (y,\top)$ and $(x,\bot)\ \alpha_{i+1}\ (y,\bot)$.

(ii) (y is not free in ε) $\Rightarrow$ $((x,\varepsilon)\ \alpha_{i+1}\ (y,[y/x]\varepsilon)$.

Proof:-

(i) -Let $z \in I$.

-Then $[z/x]\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\} = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\} = [z/y]\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by G(i).

-Hence, the result, by 1.1.2.

(ii) -Let $z \in I$.

-Then, $[z/x]\varepsilon = [z/y][y/x]\varepsilon$, by G(i).

-Hence, the result, by 1.1.2.

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1.1.5:REMARK:-

Part (ii) of the above corollary models the α-conversion rule of 0.3.6(i).

1.1.6:LEMMA:-

α_{i+1} is an equivalence relation on $I \times E_i$.

Proof:-

-Clear, using 1.1.2.

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1.1.7:NOTATION:-

We write $\lambda I.E_i$ for the set of equivalence classes,

$$(I \times E_i)/\alpha_{i+1}.$$

Also, we write $\lambda x.\varepsilon$ for the equivalence class, $[(x,\varepsilon)]$.

1.1.8:LEMMA:-

Let $x \neq y \in I$. Then,

$$\begin{aligned}\lambda x.\varepsilon = \lambda y.\eta &\Longleftrightarrow (x \text{ is not free in } \eta)\wedge(\varepsilon = [x/y]\eta)\\ &\Longleftrightarrow (y \text{ is not free in } \varepsilon)\wedge(\eta = [y/x]\varepsilon).\end{aligned}$$

Proof:-

($\Rightarrow$) -Suppose $\lambda x.\varepsilon = \lambda y.\eta$.

-Then, $[x/y]\eta = [x/x]\varepsilon$, by 1.1.2.

$\qquad\qquad = \varepsilon$, by G(i).

-But, y is not free in $[x/y]\eta$, by G(i).

-$\therefore$, y is not free in ε.

-Similarly, $[y/x]\varepsilon = \eta$ and x is not free in η.

($\Leftarrow$) -Let $z \in I$.

-Then, $[z/x]\varepsilon = [z/x][x/y]\eta = [z/y]\eta$, by G(i).

-$\therefore$, $\lambda x.\varepsilon = \lambda y.\eta$, by 1.1.2.

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1.1.9:DEF:- (*)

$\sqsubseteq$ is a relation on $\lambda I.E_i$ such that :-

$\lambda x.\varepsilon \sqsubseteq \lambda y.\eta$ if (there exists $z \in I$)(z is not free in ε,η)

$([z/x]\varepsilon \sqsubseteq [z/y]\eta)$.

1.1.10:LEMMA:-

$(\lambda x.\varepsilon \sqsubseteq \lambda y.\eta) \Longleftrightarrow (\forall z \in I)([z/x]\varepsilon \sqsubseteq [z/y]\eta)$.

Proof:-

-Same as 1.1.2, using monotonicity of $[z/z']$, by G(i).

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1.1.11:LEMMA:-

$\sqsubseteq$ is a well-defined partial ordering on $\lambda I.E_i$.

Proof:-

-Let $\lambda x.\varepsilon \sqsubseteq \lambda y.\eta$ and $\lambda x'.\varepsilon' = \lambda x.\varepsilon$ and $\lambda y'.\eta' = \lambda y.\eta$.

-Let $z \in I$. Then, $[z/x']\varepsilon' = [z/x]\varepsilon$, by 1.1.2.

$\qquad\qquad \sqsubseteq [z/y]\eta$, by 1.1.10.

$\qquad\qquad = [z/y']\eta'$, by 1.1.2.

-Hence, $\sqsubseteq$ is well-defined, by 1.1.10.

-Clearly, it is a partial ordering, by lemmas 1.1.10 and 1.1.2 (for anti-symmetry).

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(*) To prove $\sqsubseteq$ is well-defined directly, proceed as in 1.1.11 excepting that z must be chosen not free in ε, ε', η, η'.

1.1.12:LEMMA:-

(i) $\lambda x.\varepsilon = \lambda x.\eta \iff \varepsilon = \eta$.

(ii) $\lambda x.\varepsilon \sqsubseteq \lambda x.\eta \iff \varepsilon \sqsubseteq \eta$.

Proof:-

-Trivial, by 1.1.2,1.1.10 and $[z/z']$ is monotonic, by G(i).

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1.1.13:THEOREM:-

$\langle \lambda I.E_i, \sqsubseteq \rangle$ is a lattice, with :-

$$\lambda x.\varepsilon \sqcup \lambda y.\eta = \lambda z.([z/x]\varepsilon \sqcup [z/y]\eta)$$

and $\lambda x.\varepsilon \sqcap \lambda y.\eta = \lambda z.([z/x]\varepsilon \sqcap [z/y]\eta)$,

where $z \in I$ and is not free in $\varepsilon,\eta \in E_i$.

Proof:-

-Claim: the choice of z is immaterial :-

-Let z,z' be not free in ε,η and $z \neq z'$.

-Then, z' is not free in $[z/x]\varepsilon,[z/y]\eta$, by G(i).

-∴, $[z'/z]([z/x]\varepsilon \sqcup [z/y]\eta)$

$= [z'/z][z/x]\varepsilon \sqcup [z'/z][z/y]\eta$, by G(i).

$= [z'/x]\varepsilon \sqcup [z'/y]\eta$, by G(i).

-∴, $\lambda z.([z/x]\varepsilon \sqcup [z/y]\eta) = \lambda z'.([z'/x]\varepsilon \sqcup [z'/y]\eta)$, by 1.1.8, since z' is not free in $[z\ /x]\varepsilon \sqcup [z\ /y]\eta$, by G(i).

-Now, let $z' \in I$.

-Then, $[z'/x]\varepsilon,[z'/y]\eta \sqsubseteq [z'/x]\varepsilon \sqcup [z'/y]\eta$

$= [z'/z][z/x]\varepsilon \sqcup [z'/z][z/y]\eta$, by G(i).

$\sqsubseteq [z'/z]([z/x]\varepsilon \sqcup [z/y]\eta)$, by G(i).

-∴, $\lambda x.\varepsilon,\lambda y.\eta \sqsubseteq \lambda z.([z/x]\varepsilon \sqcup [z/y]\eta)$, by 1.1.10.

-Suppose $\lambda x.\varepsilon,\lambda y.\eta \sqsubseteq \lambda d.\delta$.

-By G(i), we can choose $z' \in I$ such that z' is not free in $\varepsilon,\eta,\delta \in E_i$ and $z' \neq z$.

-Then, z' is not free in $[z/x]\varepsilon,[z/y]\eta$, by G(i).

-But, $[z'/d]\delta \sqsupseteq [z'/x]\varepsilon,[z'/y]\eta$, by 1.1.10.

-So, $[z'/d]\delta \sqsupseteq [z'/x]\varepsilon \sqcup [z'/y]\eta$

$= [z'/z][z/x]\varepsilon \sqcup [z'/z][z/y]\eta$, by G(i).

$= [z'/z]([z/x]\varepsilon \sqcup [z/y]\eta)$, by G(i).

-But, also, we have z' not free in $[z/x]\varepsilon \sqcup [z/y]\eta$, by G(i).

-$\therefore$, $\lambda d.\delta \sqsupseteq \lambda z.([z/x]\varepsilon \sqcup [z/y]\eta)$, by definition 1.1.9.

-$\therefore$, $\lambda x.\varepsilon \sqcup \lambda y.\eta = \lambda z.([z/x]\varepsilon \sqcup [z/y]\eta)$.

-Similarly, $\lambda x.\varepsilon \sqcap \lambda y.\eta = \lambda z.([z/x]\varepsilon \sqcap [z/y]\eta)$.

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1.1.14:COR:-

(i) $\lambda x.\varepsilon \sqcup \lambda x.\eta = \lambda x.(\varepsilon \sqcup \eta)$

and $\lambda x.\varepsilon \sqcap \lambda x.\eta = \lambda x.(\varepsilon \sqcap \eta)$.

(ii) The $\top$ and $\bot$ of the lattice are given by $\lambda x.\top$ and $\lambda x.\bot$, respectively.

Proof:-

(i) -Let $z \in I$ and be not free in ε,η. Also, let $z \neq x$.

-Then, $\lambda x.\varepsilon \sqcup \lambda x.\eta = \lambda z.([z/x]\varepsilon \sqcup [z/x]\eta)$, by 1.1.13.

$= \lambda z.[z/x](\varepsilon \sqcup \eta)$, by G(i).

$= \lambda x.(\varepsilon \sqcup \eta)$, by 1.1.8, since z is not free in $\varepsilon \sqcup \eta$, by G(i).

-Similarly, $\lambda x.\varepsilon \sqcap \lambda x.\eta = \lambda x.(\varepsilon \sqcap \eta)$.

(ii) - Let $\lambda y.\eta \in \lambda I.E_i$ and $z \in I$.

-Then, $[z/x]\bot = \bot \sqsubseteq [z/y]\eta \sqsubseteq \top = [z/x]\top$, by G(i).

-So, $\lambda x.\bot \sqsubseteq \lambda y.\eta \sqsubseteq \lambda x.\top$, by 1.1.10.

-Hence, the result (N.B. $\lambda x.\bot = \lambda y.\bot$ and $\lambda x.\top = \lambda y.\top$, by 1.1.4(i)).

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1.1.15:LEMMA:-

The lattice, $\langle \lambda I.E_i, \sqsubseteq \rangle$, has finite depth.

Proof:-

-E_i has finite depth - say n - by G(i).

-Let $(\lambda x_0.\varepsilon_0 \sqsubseteq \lambda x_1.\varepsilon_1 \sqsubseteq \ldots\ldots \sqsubseteq \lambda x_n.\varepsilon_n)$ be a chain in the lattice.

-Now, $|\{z|z \text{ is free in } \varepsilon_0, \varepsilon_1, \ldots\ldots, \varepsilon_n\}|$

$$= |\bigcup_{i=0}^{n} \{z|z \text{ is free in } \varepsilon_i\}| < \infty, \text{ by } G(i).$$

-∴, $\{z|z \text{ is not free in } \varepsilon_0, \varepsilon_1, \ldots\ldots, \varepsilon_n\} \neq \emptyset$. Choose such a z.

-Then, by 1.1.10, $([z/x_0]\varepsilon_0 \sqsubseteq [z/x_1]\varepsilon_1 \sqsubseteq \ldots\ldots \sqsubseteq [z/x_n]\varepsilon_n)$ is a chain in E_i.

-So, $[z/x_j]\varepsilon_j = [z/x_{j+1}]\varepsilon_{j+1}$, for some $0 \leq j < n$.

-And, $\lambda x_j.\varepsilon_j = \lambda x_{j+1}.\varepsilon_{j+1}$, by definition 1.1.0.

-Thus, $\lambda I.E_i$ has finite depth.

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1.1.16:COR:-

$<\lambda I.E_i, \sqsubseteq>$ is a complete continuous countable lattice with all its elements isolated.

Proof:-

-$|\lambda I.E_i| \leq |I \times E_i| = \infty$, by $G(i)$.

-The rest is due to theorem 0.6.13.

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1.1.17:DEF:-

$$E_{i+1} := \lambda I.E_i + A_i.$$

1.1.18:LEMMA:-

E_{i+1} is a countable complete continuous lattice of finite depth with all its elements isolated.

Proof:-

-True about A_i, by $G(i)$.

-True about $\lambda I.E_i$, by 1.1.16.

-∴, true about their direct sum, E_{i+1}, by 0.6.18.

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1.1.19:REMARK:-

We have shown how to construct E_{i+1} given E_i and A_i. Next, we construct A_{i+1} using E_{i+1} and A_i. We could have used E_i but E_{i+1} makes the construction more efficient.

1.1.20:DEF:-

$\sim$ is a relation on $A_i \times E_{i+1}$ such that :-

$$(\alpha, \varepsilon) \sim (\alpha', \varepsilon') \text{ if } (\alpha = \top = \alpha')_{\vee}(\alpha = \bot = \alpha')$$
$$_{\vee}((\alpha = \alpha')_{\wedge}(\varepsilon = \varepsilon')).$$

1.1.21:LEMMA:-

$\sim$ is an equivalence relation on $A_i \times E_{i+1}$.

Proof:-

-Trivial.

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1.1.22:NOTATION:-

We write $A_i(E_{i+1})$ for the set of equivalence classes,

$$(A_i \times E_{i+1})/\sim.$$

Also, we write $\alpha(\varepsilon)$ for the equivalence class, $[(\alpha, \varepsilon)]$.

1.1.23:REMARK:-

The construction $A_i(E_{i+1})$ is going to be used to represent simple application of approximate normal forms. We factor out the equivalence, $\sim$, so that when the operator is "under-" or "over-defined", then so is the result.

1.1.24:DEF:-

$\sqsubseteq$ is a relation on $A_i(E_{i+1})$ such that :-

$$\alpha(\varepsilon) \sqsubseteq \alpha'(\varepsilon') \text{ if } (\alpha = \bot)_{\vee}(\alpha' = \top)_{\vee}((\alpha \sqsubseteq \alpha')_{\wedge}(\varepsilon \sqsubseteq \varepsilon')).$$

1.1.25:LEMMA:-

$\sqsubseteq$ is a well-defined partial ordering on $A_i(E_{i+1})$.

Proof:-

-Straightforward checking of all possible cases.

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1.1.26:THEOREM:-

$\langle A_i(E_{i+1}), \sqsubseteq \rangle$ is a lattice, with :-

$$\alpha(\varepsilon) \sqcup \alpha'(\varepsilon') = \left\{\begin{array}{l} \alpha'(\varepsilon'),\text{ if } \alpha = \bot \\ \alpha(\varepsilon),\text{ if } \alpha' = \bot \\ (\alpha \sqcup \alpha')(\varepsilon \sqcup \varepsilon'),\text{ otherwise} \end{array}\right\}$$

and,

$$\alpha(\varepsilon) \sqcap \alpha'(\varepsilon') = \left\{\begin{array}{l} \alpha'(\varepsilon'),\text{ if } \alpha = \top \\ \alpha(\varepsilon),\text{ if } \alpha' = \top \\ (\alpha \sqcap \alpha')(\varepsilon \sqcap \varepsilon'),\text{ otherwise} \end{array}\right\}.$$

<u>Proof:-</u>

-Consider $\alpha(\varepsilon)$ and $\alpha'(\varepsilon')$.

-Suppose $\alpha = \bot$:-

-Then, $\alpha(\varepsilon),\alpha'(\varepsilon') \sqsubseteq \alpha'(\varepsilon')$.

-If $\alpha(\varepsilon),\alpha'(\varepsilon') \sqsubseteq \beta(\eta)$, then $\alpha'(\varepsilon') \sqsubseteq \beta(\eta)$.

-$\therefore$, $\alpha(\varepsilon) \sqcup \alpha'(\varepsilon') = \alpha'(\varepsilon')$.

-Similarly, if $\alpha' = \bot$, then $\alpha(\varepsilon) \sqcup \alpha'(\varepsilon') = \alpha(\varepsilon)$.

-Suppose $\alpha \neq \bot \neq \alpha'$:-

-Then, $\alpha(\varepsilon),\alpha'(\varepsilon') \sqsubseteq (\alpha \sqcup \alpha')(\varepsilon \sqcup \varepsilon')$.

-If $\alpha(\varepsilon),\alpha'(\varepsilon') \sqsubseteq \beta(\eta)$, then <u>either</u> $\beta = \top$ - in which case, $(\alpha \sqcup \alpha')(\varepsilon \sqcup \varepsilon') \sqsubseteq \beta(\eta)$, <u>or</u> $\alpha,\alpha' \sqsubseteq \beta$ and $\varepsilon,\varepsilon' \sqsubseteq \eta$ - in which case, $\alpha \sqcup \alpha' \equiv \beta, \varepsilon \sqcup \varepsilon' \equiv \eta$ and so $(\alpha \sqcup \alpha')(\varepsilon \sqcup \varepsilon') \equiv \beta(\eta)$.

-$\therefore$, $\alpha(\varepsilon) \sqcup \alpha'(\varepsilon') = (\alpha \sqcup \alpha')(\varepsilon \sqcup \varepsilon')$.

-Similarly, for the greatest lower bounds.

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<u>1.1.27:LEMMA:-</u>

The lattice, $\langle A_i(E_{i+1}),\sqsubseteq\rangle$ has finite depth.

<u>Proof:-</u>

-A_i and E_{i+1} have finite depth - say n and m respectively - by $G(i)$ and 1.1.18.

-Let $(\alpha_0(\varepsilon_0) \sqsubseteq \alpha_1(\varepsilon_1) \sqsubseteq \ldots\ldots \sqsubseteq \alpha_{n+m-2}(\varepsilon_{n+m-2}))$ be a chain.

-Suppose the inequalities are all strict :-

-If $\alpha_0 = \bot$ then $\alpha_1 \neq \bot$. If $\alpha_{n+m-2} = \top$ then $\alpha_{n+m-3} \neq \top$.

-In either or neither or both of the above cases, we have :-

$$\alpha_0 \sqsubseteq \alpha_1 \sqsubseteq \ldots\ldots \sqsubseteq \alpha_{n+m-2} \text{ in } A_i$$

and $\varepsilon_0 \sqsubseteq \varepsilon_1 \sqsubseteq \ldots\ldots \sqsubseteq \varepsilon_{n+m-2}$ in E_{i+1}.

-Since A_i has finite depth of n, (m-1) of the α_i's must be the same.

-Similarly, (n-1) of the ε_i's are equal.

-But, (m-1) + (n-1) = m+n-2 > m+n-1, the number of elements in the original chain - i.e. it must overlap somewhere.

-So, there exists j such that $\alpha_j = \alpha_{j+1}$ and $\varepsilon_j = \varepsilon_{j+1}$.

-∴, there exists j such that $\alpha_j(\varepsilon_j) = \alpha_{j+1}(\varepsilon_{j+1})$ - ✗.

-Hence the lattice has finite depth (= n+m-2).

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1.1.28:COR:-

$\langle A_i(E_{i+1}), \sqsubseteq\rangle$ is a complete continuous countable lattice with all its elements isolated.

Proof:-

-$|A_i(E_{i+1})| \leq |A_i \times E_{i+1}| = \infty$, by $G(i)$ and 1.1.18.

-The rest is due to theorem 0.6.13.

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1.1.29:DEF:-

$A_{i+1} := I + A_i(E_{i+1})$.

1.1.30:LEMMA:-

A_{i+1} is a continuous countable complete lattice of finite depth with all its elements isolated.

Proof:-

-True about I, by 1.0.1.

-True about $A_i(E_{i+1})$, by 1.1.28.

-∴, true about their direct sum, A_{i+1}, by 0.6.18.

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1.2:Carrying through the Induction:-

1.2.0:REMARK:-

We have constructed E_{i+1} and A_{i+1} from E_i and A_i, but we still have to complete the induction hypothesis, $\mathcal{G}(i+1)$. For this, we must define the simple "change of variables" operator, $[x/y]$, and what it means to be "not free in" over E_{i+1} and A_{i+1}.

1.2.1:DEF:-

Let $x,y \in I$. Then,

$$[x/y] : E_{i+1} \longrightarrow E_{i+1}$$

$$\begin{Bmatrix} \lambda z.\,\varepsilon_i \\ \alpha_i \end{Bmatrix} \longmapsto \begin{Bmatrix} \lambda w.\,[x/y][w/z]\varepsilon_i \\ [x/y]\alpha_i \end{Bmatrix},$$

where $w \neq x$, $w \neq y$ and w is not free in ε_i. Also,

$$[x/y] : A_{i+1} \longrightarrow A_{i+1}$$

$$\begin{Bmatrix} z \\ \alpha_i(\varepsilon_{i+1}) \end{Bmatrix} \longmapsto \begin{Bmatrix} [x/y]z \\ [x/y]\alpha_i([x/y]\varepsilon_{i+1}) \end{Bmatrix}.$$

1.2.2:REMARK:-

We are using subscripts to indicate to which lattice an element belongs. Thus, $\alpha_i(\varepsilon_{i+1}) \in A_i(E_{i+1}) \subset A_{i+1}$, where $\alpha_i \in A_i$ and $\varepsilon_{i+1} \in E_{i+1}$. Also, strictly speaking, the functions $[x/y]$ should be subscripted to indicate on which lattice they operate. We have omitted them in the hope that the context makes things clear - e.g. :-

$$[x/y]_{E_{i+1}}\alpha_i = [x/y]_{A_i}\alpha_i \in A_i \subset E_{i+1},$$

$$[x/y]_{A_{i+1}}z = [x/y]_I z \in I \subset A_{i+1}$$

and $[x/y]_{A_{i+1}}\alpha_i(\varepsilon_{i+1}) = [x/y]_{A_i}\alpha_i([x/y]_{E_{i+1}}\varepsilon_{i+1}) \in A_i(E_{i+1}) \subset A_{i+1}$

1.2.3:LEMMA:-

The above maps are well-defined and doubly strict.

Proof:-

-The least obvious case is to show that $[x/y](\lambda z.\varepsilon_i)$ is well-define when $\top,\bot \neq \lambda z.\varepsilon_i \in \lambda I.E_i \subset E_{i+1}$.

-Claim: the choice of w in definition 1.2.1 does not matter :-

-Let $w,w' \neq x,y$ and w,w' be not free in ε_i.

-Also, suppose $w \neq w'$.

-Then, w is not free in $\varepsilon_i,[w'/z]\varepsilon_i,[x/y][w'/z]\varepsilon_i$, by G(i).

-But, $[w/w'][x/y][w'/z]\varepsilon_i = [x/y][w/w'][w'/z]\varepsilon_i$, by G(i).

$= [x/y][w/z]\varepsilon_i$, by G(i).

-$\therefore$, $\lambda w.[x/y][w/z]\varepsilon_i = \lambda w'.[x/y][w'/z]\varepsilon_i$, by 1.1.8.

-Claim: the choice of representative does not matter :-

-Let $\lambda z.\varepsilon_i = \lambda z'.\varepsilon'_i$ and $z \neq z'$.

-Then, z' is not free in ε_i and $[z'/z]\varepsilon_i = \varepsilon'_i$, by 1.1.8.

-Choose $w \neq x,y,z'$ and not free in ε_i.

-Then, w is not free in $[z'/z]\varepsilon_i$, by G(i).

-$\therefore$, $[x/y](\lambda z'.\varepsilon'_i) = [x/y](\lambda z'.[z'/z]\varepsilon_i)$

$= \lambda w.[x/y][w/z'][z'/z]\varepsilon_i$

$= \lambda w.[x/y][w/z]\varepsilon_i$, by G(i).

$= [x/y](\lambda z.\varepsilon_i)$.

-To complete the proof, check that $[x/y]$ is doubly strict on the various representations of $\top$ and $\bot$ in E_{i+1} and A_{i+1} - i.e. $\{\top_{A_i}, \lambda z.\top_{E_i}, \lambda z'.\top_{E_i}, \ldots\ldots\}, \{\bot_{A_i}, \lambda z.\bot_{E_i}, \lambda z'.\bot_{E_i}, \ldots\ldots\}, \{\top_{I'}, \top_{A_i}(\varepsilon_{i+1}), \top_{A_i}(\varepsilon'_{i+1}), \ldots\ldots\}$ and $\{\bot_{I'}, \bot_{A_i}(\varepsilon_{i+1}), \bot_{A_i}(\varepsilon'_{i+1}), \ldots\ldots\}$.

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1.2.4:LEMMA:-

(i) $[x/y](\lambda y.\varepsilon_i) = \lambda y.\varepsilon_i$.

(ii) $(z \neq x,y) \Rightarrow ([x/y](\lambda z.\varepsilon_i) = \lambda z.[x/y]\varepsilon_i)$.

Proof:-

(i) -Choose $w \neq x,y$ and not free in ε_i.

-Then, $[x/y](\lambda y.\varepsilon_i) = \lambda w.[x/y][w/y]\varepsilon_i$.

-But, y is not free in $[w/y]\varepsilon_i$, by G(i).

-So, $\lambda w.[x/y][w/y]\varepsilon_i = \lambda w.[w/y]\varepsilon_i$, by G(i).

$$= \lambda y.\varepsilon_i, \text{ by } 1.1.8.$$

(ii) -Choose $w \neq x,y$ and not free in ε_i, again.

-Then, $[x/y](\lambda z.\varepsilon_i) = \lambda w.[x/y][w/z]\varepsilon_i$

$= \lambda w.[w/z][x/y]\varepsilon_i$, by G(i).

$= \lambda z.[x/y]\varepsilon_i$, by 1.1.8, since w is not free in $[x/y]\varepsilon_i$, by G(i).

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1.2.5:LEMMA:-

$[x/x]$ is the identity function on E_{i+1}, A_{i+1}.

Proof:-

$$-[x/x]\left\{\begin{matrix}\lambda z.\varepsilon_i\\ \alpha_i\end{matrix}\right\} = \left\{\begin{matrix}\lambda w.[x/x][w/z]\varepsilon_i\\ [x/x]\alpha_i\end{matrix}\right\}$$

$$= \left\{\begin{matrix}\lambda w.[w/z]\varepsilon_i\\ \alpha_i\end{matrix}\right\}, \text{ by } G\text{(i).}$$

$$= \left\{\begin{matrix}\lambda z.\varepsilon_i, \text{ by } 1.1.8.\\ \alpha_i\end{matrix}\right\}.$$

$$-[x/x]\left\{\begin{matrix}z\\ \alpha_i(\varepsilon_{i+1})\end{matrix}\right\} = \left\{\begin{matrix}[x/x]z\\ [x/x]\alpha_i([x/x]\varepsilon_{i+1})\end{matrix}\right\}$$

$$= \left\{\begin{matrix}z, \text{ by } 1.0.3\text{(i).}\\ \alpha_i(\varepsilon_{i+1}), \text{ by } G\text{(i) and above.}\end{matrix}\right\}.$$

⊧

1.2.6:LEMMA:-

$[x/y]\eta = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\} \iff \eta = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, for $\eta \in E_{i+1}, A_{i+1}$.

Proof:-

$-[x/y](\lambda z.\varepsilon_i) = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\} \iff \lambda w.[x/y][w/z]\varepsilon_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$

$\iff [x/y][w/z]\varepsilon_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by 1.1.14(ii).

$\iff [w/z]\varepsilon_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by G(i).

$\iff \varepsilon_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by G(i).

$\iff \lambda z.\varepsilon_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by 1.1.14(ii).

$-[x/y]\alpha_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\} \iff \alpha_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by G(i).

$$-[x/y]z = \begin{Bmatrix}\top\\ \bot\end{Bmatrix} \Longleftrightarrow z = \begin{Bmatrix}\top\\ \bot\end{Bmatrix}\text{,by 1.0.3(ii).}$$

$$\begin{aligned}-[x/y]\alpha_i(\varepsilon_{i+1}) = \begin{Bmatrix}\top\\ \bot\end{Bmatrix} &\Longleftrightarrow [x/y]\alpha_i([x/y]\varepsilon_{i+1}) = \begin{Bmatrix}\top\\ \bot\end{Bmatrix}\\ &\Longleftrightarrow [x/y]\alpha_i = \begin{Bmatrix}\top\\ \bot\end{Bmatrix}\text{, by definition 1.1.24.}\\ &\Longleftrightarrow \alpha_i = \begin{Bmatrix}\top\\ \bot\end{Bmatrix}\text{, by } G(i)\text{.}\\ &\Longleftrightarrow \alpha_i(\varepsilon_{i+1}) = \begin{Bmatrix}\top\\ \bot\end{Bmatrix}\text{, by definition 1.1.24.}\end{aligned}$$

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1.2.7:LEMMA:-

Let $a,b,c,d \in I$ and $a \neq d$ and $b \neq c,d$. Then, $[a/b]$ and $[c/d]$ commute on E_{i+1}, A_{i+1}.

Proof:-

-Choose $w \neq a,b,c,d$ and not free in ε_i.

-Choose $w' \neq a,b,c,d,w$ and not free in ε_i.

-Then, w' is not free in $[c/d][w/z]\varepsilon_i, [a/b][w/z]\varepsilon_i$, by $G(i)$.

$$\begin{aligned}-\text{So, } [a/b][c/d]\begin{Bmatrix}\lambda z.\varepsilon_i\\ \alpha_i\end{Bmatrix} &= [a/b]\begin{Bmatrix}\lambda w.[c/d][w/z]\varepsilon_i\\ [c/d]\alpha_i\end{Bmatrix}\\ &= \begin{Bmatrix}\lambda w'.[a/b][w'/w][c/d][w/z]\varepsilon_i\\ [a/b][c/d]\alpha_i\end{Bmatrix}\\ &= \begin{Bmatrix}\lambda w'.[w'/w][a/b][c/d][w/z]\varepsilon_i\\ [c/d][a/b]\alpha_i\end{Bmatrix}\text{, by } G(i)\text{.}\\ &= \begin{Bmatrix}\lambda w'.[w'/w][c/d][a/b][w/z]\varepsilon_i\text{, by } G(i)\text{.}\\ [c/d][a/b]\alpha_i\end{Bmatrix}\\ &= \begin{Bmatrix}\lambda w'.[c/d][w'/w][a/b][w/z]\varepsilon_i\text{, by } G(i)\text{.}\\ [c/d][a/b]\alpha_i\end{Bmatrix}\\ &= [c/d]\begin{Bmatrix}\lambda w.[a/b][w/z]\varepsilon_i\\ [a/b]\alpha_i\end{Bmatrix}\\ &= [c/d][a/b]\begin{Bmatrix}\lambda z.\varepsilon_i\\ \alpha_i\end{Bmatrix}.\end{aligned}$$

$$-\text{Also, } [a/b][c/d]\begin{Bmatrix}z\\ \alpha_i(\varepsilon_{i+1})\end{Bmatrix} = [a/b]\begin{Bmatrix}[c/d]z\\ [c/d]\alpha_i([c/d]\varepsilon_{i+1})\end{Bmatrix}$$

$$= \left\{\begin{array}{l}[a/b][c/d]z \\ [a/b][c/d]\alpha_i([a/b][c/d]\varepsilon_{i+1})\end{array}\right\}$$

$$= \left\{\begin{array}{l}[c/d][a/b]z\text{, by 1.0.3(iii).} \\ [c/d][a/b]\alpha_i([c/d][a/b]\varepsilon_{i+1})\text{, by } G(i) \text{ and above.}\end{array}\right\}$$

$$= [c/d][a/b]\left\{\begin{array}{c}z \\ \alpha_i(\varepsilon_{i+1})\end{array}\right\}.$$

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<u>1.2.8:LEMMA:-</u>

$[x/y]$ is monotonic, and so, continuous, on E_{i+1}, A_{i+1}.

<u>Proof:-</u>

-Let $\lambda a.\varepsilon \sqsubseteq \lambda b.\eta$, where $a, b \in I$ and $\varepsilon, \eta \in E_i$.

-Choose $w \neq x, y$ and not free in ε, η.

-Then, $[w/a]\varepsilon \sqsubseteq [w/b]\eta$.

-So, $[x/y][w/a]\varepsilon \sqsubseteq [x/y][w/b]\eta$, since $[x/y]$ is monotonic, by $G(i)$.

-∴, $[x/y](\lambda a.\varepsilon) = \lambda w.[x/y][w/a]\varepsilon$

$\sqsubseteq \lambda w.[x/y][w/b]\varepsilon$, by 1.1.12(ii).

$= [x/y](\lambda b.\eta)$.

-If $\alpha_i \sqsubseteq \alpha_i'$, then $[x/y]\alpha_i \sqsubseteq [x/y]\alpha_i'$, by $G(i)$.

-If $z \sqsubseteq z'$, then $[x/y]z \sqsubseteq [x/y]z'$, by 1.0.3(iv).

-Let $\alpha_i(\varepsilon_{i+1}) \sqsubseteq \alpha_i'(\varepsilon_{i+1}')$.

-If $\alpha_i = \bot$ or $\alpha_i' = \top$, then the result is trivial, by 1.2.6.

-If not, then $\alpha_i \sqsubseteq \alpha_i'$ and $\varepsilon_{i+1} \sqsubseteq \varepsilon_{i+1}'$.

-Then, $[x/y]\alpha_i(\varepsilon_{i+1}) = [x/y]\alpha_i([x/y]\varepsilon_{i+1})$

$\sqsubseteq [x/y]\alpha_i'([x/y]\varepsilon_{i+1}')$, by $G(i)$ and above.

$= [x/y]\alpha_i'(\varepsilon_{i+1}')$.

-∴, $[x/y]$ is monotonic on E_{i+1}, A_{i+1} and so continuous, by 0.6.13, 1.1.18 and 1.1.30.

ǂ

1.2.9:DEF:-

$$x \text{ is } \underline{\text{NOT FREE IN}} \begin{Bmatrix} \lambda y.\varepsilon_i \\ \alpha_i \end{Bmatrix} \in E_{i+1} \text{ if :-}$$

$$\begin{Bmatrix} \underline{\text{either}}\ x = y\ \underline{\text{or}}\ (x \neq y) \wedge (x \text{ is not free in } \varepsilon_i \in E_i) \\ x \text{ is not free in } \alpha_i \in A_i \end{Bmatrix}.$$

$$x \text{ is } \underline{\text{NOT FREE IN}} \begin{Bmatrix} z \\ \alpha_i(\varepsilon_{i+1}) \end{Bmatrix} \in A_{i+1} \text{ if :-}$$

$$\begin{Bmatrix} x \text{ is not free in } z \in I' \\ \underline{\text{either}}\ (\alpha_i = \bot) \vee (\alpha_i = \top)\ \underline{\text{or}}\ (x \text{ is not free in } \alpha_i \in A_i \\ \wedge (x \text{ is not free in } \varepsilon_{i+1} \in E_{i+1}) \end{Bmatrix}.$$

1.2.10:LEMMA:-

The above definition is well-defined and x is not free in $\top, \bot \in E_{i+1}, A_{i+1}$.

Proof:-

-Clearly, x is not free in any of the representations of $\top$ and $\bot$ in E_{i+1} and A_{i+1} (see end of proof of 1.2.3 for list of these).

-The only problem is when x is not free in $\lambda y.\varepsilon_i$.

-Suppose $\lambda y.\varepsilon_i = \lambda z.\eta_i$, where $y \neq z$.

-Claim: x is not free in $\lambda z.\eta_i$:-

-If $x = z$, trivial.

-Suppose $x \neq z$. We must show x is not free in $\eta_i \in E_i$.

-Now, z is not free in ε_i and $\eta_i = [z/y]\varepsilon_i$, by 1.1.8.

-So, if $x = y$, then trivial, by $G(i)$.

-If $x \neq y$, then x is not free in ε_i and so, again, trivial, by $G(i)$.

-∴, the definition is well-defined.

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1.2.11:LEMMA:-

Let $x \neq y \in I$. Then, y is not free in $[x/y]\varepsilon_{i+1} \in E_{i+1}$ and y is not free in $[x/y]\alpha_{i+1} \in A_{i+1}$.

Proof:-

-Suppose $\varepsilon_{i+1} = \lambda z.\varepsilon_i$.

-Choose $w \neq x,y$ and not free in ε_i.

-Then, $[x/y](\lambda z.\varepsilon_i) = \lambda w.[x/y][w/z]\varepsilon_i$.

-But, y is not free in $[x/y][w/z]\varepsilon_i$, by $G(i)$, since $x \neq y$.

-∴, y is not free in $\lambda w.[x/y][w/z]\varepsilon_i$, since $y \neq w$.

-If $\varepsilon_{i+1} = \alpha_i$, then y is not free in $[x/y]\alpha_i$, by $G(i)$.

-If $\alpha_{i+1} = z$, then y is not free in $[x/y]z$, by 1.0.5(ii).

-Suppose $\alpha_{i+1} = \alpha_i(\varepsilon_{i+1})$.

-Now, y is not free in $[x/y]\alpha_i$, by $G(i)$.

-And, y is not free in $[x/y]\varepsilon_{i+1}$, by above.

-∴, y is not free in $[x/y]\alpha_i([x/y]\varepsilon_{i+1}) = [x/y]\alpha_i(\varepsilon_{i+1})$.

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1.2.12:LEMMA:-

$$(y \text{ is not free in } \left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}) \Rightarrow ([x/y]\left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\} = \left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}).$$

Proof:-

-Suppose $\varepsilon_{i+1} = \lambda z.\varepsilon_i$.

-If $y = z$, then $[x/y](\lambda y.\varepsilon_i) = \lambda y.\varepsilon_i)$, by 1.2.4(i).

-If $y \neq z$, then y is not free in $\varepsilon_i \in E_i$.

-Choose $w \neq x,y,z$ and not free in ε_i.

-Now, y is not free in $[w/z]\varepsilon_i$, by $G(i)$.

-So, $[x/y](\lambda z.\varepsilon_i) = \lambda w.[x/y][w/z]\varepsilon_i$

$= \lambda w.[w/z]\varepsilon_i$, by $G(i)$.

$= \lambda z.\varepsilon_i$, by 1.1.8.

-If $\varepsilon_{i+1} = \alpha_i$, then $[x/y]\alpha_i = \alpha_i$, by $G(i)$.

-If $\alpha_{i+1} = z$, then $[x/y]z = z$, by 1.0.5(iii).

-Suppose $\alpha_{i+1} = \alpha_i(\varepsilon_{i+1})$.

-If $\alpha_i = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, then $\alpha_{i+1} = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$ and so $x/y\ \alpha_{i+1} = \alpha_{i+1}$,by 1.2.6

-If not, then y is not free in $\alpha_i,\varepsilon_{i+1}$.

-So, $[x/y]\alpha_i(\varepsilon_{i+1}) = [x/y]\alpha_i([x/y]\varepsilon_{i+1})$

$= \alpha_i(\varepsilon_{i+1})$, by $G(i)$ and above.

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1.2.13:LEMMA:-

Let $x \neq a \in I$. Then,

$$(x \text{ is not free in} \left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}) \Rightarrow (x \text{ is not free in } [a/b]\left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}).$$

Proof:-

-Suppose $\varepsilon_{i+1} = \lambda y.\varepsilon_i$.

-Choose $w \neq a,b$ and not free in ε_i.

-Then, $\lambda y.\varepsilon_i = \lambda w.[w/y]\varepsilon_i$, by 1.1.4(ii).

-And, $[a/b](\lambda y.\varepsilon_i) = \lambda w.[a/b][w/y]\varepsilon_i$.

-Since x is not free in ε_{i+1}, either $x = w$ - in which case result is trivial - or $(x \neq w) \wedge (x$ is not free in $[w/y]\varepsilon_i)$ - in which case x is not free in $[a/b][w/y]\varepsilon_i$, by $G(i)$, since $x \neq a$, and so, again, we have the result.

-If $\varepsilon_{i+1} = \alpha_i$, we have the result by $G(i)$.

-If $\alpha_{i+1} = z$, we have the result by 1.0.5(iv).

-Suppose $\alpha_{i+i} = \alpha_i(\varepsilon_{i+i})$.

-If $\alpha_{i+1} = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, we have the result by 1.2.6 and 1.2.10.

-If not, then x is not free in $\alpha_i, \varepsilon_{i+1}$.

-Thus, x is not free in $[a/b]\alpha_i, [a/b]\varepsilon_{i+1}$, by $G(i)$ and above.

-So, x is not free in $[a/b]\alpha_i([a/b]\varepsilon_{i+1}) = [a/b]\alpha_i(\varepsilon_{i+1})$.

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1.2.14:LEMMA:-

$$(x \text{ is not free in } \left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}) \Rightarrow ([z/x][x/y]\left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\} = [z/y]\left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}).$$

Proof:-

-Assume $z \neq x \neq y$ - otherwise trivial, by 1.2.5.

-Suppose $\varepsilon_{i+1} = \lambda a.\varepsilon_i$.

-Assume, without loss of generality, that $x \neq a$ - since, if $x = a$, choose a' such that $x \neq a'$ and a' is not free in ε_i, and work with $\lambda a'.[a'/a]\varepsilon_i$, by 1.1.8. So, x is not free in ε_i.

-Choose $w, w' \in I$ such that $w \neq w'$ and $w, w' \neq x, y, z$ and w, w' are not free in ε_i.

-Then, $[z/x][x/y](\lambda a.\,\varepsilon_i) = [z/x](\lambda w.[x/y][w/a]\varepsilon_i)$

$= \lambda w'.[z/x][w'/w][x/y][w/a]\varepsilon_i$, since w' is not free in $[x/y][w/a]\varepsilon_i$, by G(i).

$= \lambda w'.[w'/w][z/x][x/y][w/a]\varepsilon_i$, by G(i)

$= \lambda w.[z/x][x/y][w/a]\varepsilon_i$, by 1.1.8, since w' is not free in $[z/x][x/y][w/a]\varepsilon_i$, by G(i).

$= \lambda w.[z/y][w/a]\varepsilon_i$, by G(i), since x is not free in $[w/a]\varepsilon_i$, by G(i).

$= [z/y](\lambda a.\,\varepsilon_i)$.

-If $\varepsilon_{i+1} = \alpha_i$, we have the result by G(i).

-If $\alpha_{i+1} = a$, we have the result by 1.0.5(v).

-Suppose $\alpha_{i+i} = \alpha_i(\varepsilon_{i+1})$.

-If $\alpha_i = \left\{\begin{matrix}\top \\ \bot\end{matrix}\right\}$, then the result is trivial, by 1.2.6.

-If not, then x is not free in $\alpha_i, \varepsilon_{i+1}$.

-So, $[z/x][x/y]\alpha_i(\varepsilon_{i+1}) = [z/x][x/y]\alpha_i([z/x][x/y]\varepsilon_{i+1})$

$= [z/y]\alpha_i([z/y]\varepsilon_{i+1})$, by G(i) and above.

$= [z/y]\alpha_i(\varepsilon_{i+i})$.

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1.2.15:LEMMA:-

$$(x \text{ is not free in } \left\{\begin{matrix}\varepsilon_{i+1}, \varepsilon'_{i+1} \\ \alpha_{i+1}, \alpha'_{i+1}\end{matrix}\right\}) \Rightarrow$$

$$(x \text{ is not free in } \left\{\begin{matrix}\varepsilon_{i+1} \sqcup \varepsilon'_{i+1}, \varepsilon_{i+1} \sqcap \varepsilon'_{i+1} \\ \alpha_{i+1} \sqcup \alpha'_{i+1}, \alpha_{i+1} \sqcap \alpha'_{i+1}\end{matrix}\right\}).$$

Proof:-

-If $\varepsilon_{i+1} \in \lambda I.E_i$ and $\varepsilon'_{i+1} \in A_i$, or vice-versa, then $\varepsilon_{i+1} \begin{Bmatrix}\sqcup \\ \sqcap\end{Bmatrix} \varepsilon'_{i+1} = \begin{Bmatrix}\top \\ \bot\end{Bmatrix}$, and in either case x is not free, by 1.2.10.

-Similarly, if $\alpha_{i+1},\alpha'_{i+1}$ are in different parts of A_{i+1}, the result is trivial.

-$\therefore$, suppose they are always in the same half of the lattice sum.

-Suppose $\varepsilon_{i+1} = \lambda a.\varepsilon$ and $\varepsilon'_{i+1} = \lambda b.\eta$.

-As in the proof of the last lemma, we may assume, w.l.o.g., that $x \neq a,b$ and, so, x is not free in ε,η.

-Then, $\lambda a.\varepsilon \sqcup \lambda b.\eta = \lambda x.([x/a]\varepsilon \sqcup [x/b]\eta)$, by theorem 1.1.13, and in which x is not free.

-Similarly, for the greatest lower bound.

-If $\varepsilon_{i+1} = \alpha_i$ and $\varepsilon'_{i+1} = \alpha'_i$, we have the result by *G*(i).

-If $\alpha_{i+1} = a$ and $\alpha_{i+1} = a'$, we have the result by 1.0.5(vi).

-Suppose $\alpha_{i+1} = \alpha(\beta)$ and $\alpha'_{i+1} = \gamma(\delta)$.

-If $\alpha = \bot$, then $\alpha(\beta) \sqcup \gamma(\delta) = \gamma(\delta)$, in which x is not free
and " $\sqcap$ " = $\bot$, " " x " " ".
-If $\alpha = \top$, then " $\sqcup$ " = $\top$, " " x " " "
and " $\sqcap$ " = $\gamma(\delta)$, " " x " " ".
-If $\gamma = \bot$, then " $\sqcup$ " = $\alpha(\beta)$, " " x " " "
and " $\sqcap$ " = $\bot$, " " x " " ".
-If $\gamma = \top$, then " $\sqcup$ " = $\top$, " " x " " "
and " $\sqcap$ " = $\alpha(\beta)$, " " x " " ".

-If none of these, then x is not free in $\alpha,\beta,\gamma,\delta$.

-Then, x is not free in $(\alpha \sqcup \gamma),(\alpha \sqcap \gamma),(\beta \sqcup \delta),(\beta \sqcap \delta)$, by *G*(i) and above.

-So, x is not free in $(\alpha \sqcup \gamma)(\beta \sqcup \delta),(\alpha \sqcap \gamma)(\beta \sqcap \delta)$

$$= \alpha(\beta) \sqcup \gamma(\delta), \alpha(\beta) \sqcap \gamma(\delta).$$

‡

1.2.16:LEMMA:-

$$(x \text{ is not free in } \left\{\begin{matrix}\varepsilon_{i+1}, \varepsilon'_{i+1}\\ \alpha_{i+i}, \alpha'_{i+1}\end{matrix}\right\}) \Rightarrow$$

$$([x/y]\left\{\begin{matrix}\varepsilon_{i+1} \sqcup \varepsilon'_{i+1}, \varepsilon_{i+1} \sqcap \varepsilon'_{i+1}\\ \alpha_{i+1} \sqcup \alpha'_{i+1}, \alpha_{i+1} \sqcap \alpha'_{i+1}\end{matrix}\right\}$$

$$= \left\{\begin{matrix}[x/y]\varepsilon_{i+1} \sqcup [x/y]\varepsilon'_{i+1}, [x/y]\varepsilon_{i+1} \sqcap [x/y]\varepsilon'_{i+1}\\ [x/y]\alpha_{i+1} \sqcup [x/y]\alpha'_{i+1}, [x/y]\alpha_{i+1} \sqcap [x/y]\alpha'_{i+1}\end{matrix}\right\}$$

Proof:-

-As in the proof of the last lemma, we may assume, w.l.o.g., that $\varepsilon_{i+1}, \varepsilon'_{i+1}, \alpha_{i+1}$ and α'_{i+1} are in the same halves of their respective lattice sums - otherwise we are operating on $\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$ and so the result is trivial, by 1.2.6.

-Suppose $\varepsilon_{i+1} = \lambda a.\varepsilon$ and $\varepsilon'_{i+1} = \lambda b.\eta$.

-Again, we may assume, w.l.o.g., that $a \neq x \neq b$, so that x is not free in ε, η.

-Choose $w \in I$ such that w is not free in ε, η and $w \neq x, y$. (N.B. it is always possible to make such a choice since I has an infinite number of elements and we are excluding only a finite number - the "free" variables of ε and η - by G(i).)

-Then, $[x/y](\lambda a.\varepsilon) \sqcup [x/y](\lambda b.\eta)$

$= (\lambda w.[x/y][w/a]\varepsilon) \sqcup (\lambda w.[x/y][w/b]\eta)$

$= \lambda w.([x/y][w/a]\varepsilon \sqcup [x/y][w/b]\eta)$, by 1.1.14(i).

$= \lambda w.([x/y]([w/a]\varepsilon \sqcup [w/b]\eta))$, by G(i), since x is not free in $[w/a]\varepsilon, [w/b]\eta$, by G(i).

$= [x/y]\lambda w.([w/a]\varepsilon \sqcup [w/b]\eta)$, by 1.2.4(ii).

$= [x/y](\lambda a.\varepsilon \sqcup \lambda b.\eta)$.

-Similarly, for the greatest lower bound.

-If $\varepsilon_{i+1} = \alpha_i$ and $\varepsilon'_{i+1} = \alpha'_i$, we have the result by G(i).

-If $\alpha_{i+1} = a$ and $\alpha_{i+1} = a'$, we have the result by 1.0.5(vii).

-Suppose $\alpha_{i+1} = \alpha(\beta)$ and $\alpha'_{i+1} = \gamma(\delta)$.

-It is trivial to check that the result holds when $\alpha, \gamma = \top, \bot$.

-Otherwise, x is not free in $\alpha, \beta, \gamma, \delta$.

-So, $[x/y](\alpha(\beta) \sqcup \gamma(\delta)) = [x/y](\alpha \sqcup \gamma)(\beta \sqcup \delta)$

$= [x/y](\alpha \sqcup \gamma)([x/y](\beta \sqcup \delta))$

$= ([x/y]\alpha \sqcup [x/y]\gamma)([x/y]\beta \sqcup [x/y]\delta)$,

by $G(i)$ and above.

$= [x/y]\alpha([x/y]\beta) \sqcup [x/y]\gamma([x/y]\delta)$,

by 1.1.26, since $[x/y]\alpha \neq \bot \neq [x/y]\gamma$, by 1.2.6.

$= [x/y]\alpha(\beta) \sqcup [x/y]\gamma(\delta)$.

-Similarly, for the greatest lower bound.

-N.B. if x were free originally, we would have the inequalities of $G(i+1)$ as a result of the monotonicity of $[x/y]$, proved in 1.2.8.

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__1.2.17:LEMMA:-__

$$|\{x \in I | x \text{ is free in } \left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\}\}| < \infty.$$

__Proof:-__

-$|\{x | x \text{ is free in } \lambda a.\varepsilon_i\}| \leq |\{x | x \text{ is free in } \varepsilon_i\}| < \infty$, by $G(i)$.

-If $\varepsilon_{i+1} = \alpha_i$, we have the result by $G(i)$.

-If $\varepsilon_{i+1} = a$, we have the result by 1.0.5(viii).

-$|\{x | x \text{ is free in } \alpha_i(\varepsilon_{i+1})\}| \leq |\{x | x \text{ is free in } \alpha_i\}| + |\{x | x \text{ is free in } \varepsilon_{i+1}\}|$, since we have equality if $\alpha_i \neq \top, \bot$ and 0 otherwise.

$< \infty$, by $G(i)$ and above.

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__1.2.18:THEOREM:-__

$(\forall i \geq 0)G(i)$.

__Proof:-__

-$G(0)$, by 1.0.9.

-For $i \geq 0$, $G(i) \Rightarrow G(i+1)$, by 1.1.18,1.1.30,1.2.5,1.2.6,1.2.7, 1.2.8,1.2.10,1.2.11,1.2.12,1.2.13,1.2.14,1.2.15,1.2.16 and 1.2.17.

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2:PROJECTIONS AND THE INVERSE LIMITS E_∞, A_∞.

2.0:Initial Projections:-

2.0.0:DEF:-

$$\phi_{1,0} : E_1 \longrightarrow E_0$$

$$\left\{\begin{matrix}\lambda x.\varepsilon_0 \\ \alpha_0\end{matrix}\right\} \longmapsto \left\{\begin{matrix}\bot, \text{ if } \varepsilon_0 \neq \top. \\ \alpha_0\end{matrix}\right\},$$

$$\phi_{0,1} : E_0 \longrightarrow E_1$$

$$\varepsilon_0 \longmapsto \varepsilon_0,$$

$$\theta_{1,0} : A_1 \longrightarrow A_0$$

$$\left\{\begin{matrix}x \\ \alpha_0(\varepsilon_1)\end{matrix}\right\} \longmapsto \left\{\begin{matrix}x \\ \bot, \text{ if } \alpha_0 \neq \top.\end{matrix}\right\}$$

and $\theta_{0,1} : A_0 \longrightarrow A_1$

$$\alpha_0 \longmapsto \alpha_0.$$

2.0.1:LEMMA:-

The above maps are well-defined and doubly strict.

Proof:-

-By definition, they are doubly strict.

-The only equivalence classes not $\top$ or $\bot$ are in $\lambda I.E_0$.

-If $\lambda x.\varepsilon_0 = \lambda y.\eta_0 \neq \top$, then $\phi_{1,0}(\lambda x.\varepsilon_0) = \bot = \phi_{1,0}(\lambda y.\eta_0)$.

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2.0.2:LEMMA:-

The following diagrams commute :-

$$\begin{tikzcd}
E_0 \arrow[r, shift left, "\phi_{0,1}"] \arrow[d, shift left] & E_1 \arrow[l, shift left, "\phi_{1,0}"] \arrow[d, shift left, "{[x/y]}"] \\
E_0 \arrow[u, shift left, "{[x/y]}"] \arrow[r, shift left, "\phi_{0,1}"] & E_1 \arrow[l, shift left, "\phi_{1,0}"] \arrow[u, shift left]
\end{tikzcd}$$

and,

$$\begin{tikzcd}
A_0 \arrow[r, shift left, "\theta_{0,1}"] \arrow[d, shift left] & A_1 \arrow[l, shift left, "\theta_{1,0}"] \arrow[d, shift left, "{[x/y]}"] \\
A_0 \arrow[u, shift left, "{[x/y]}"] \arrow[r, shift left, "\theta_{0,1}"] & A_1 \arrow[l, shift left, "\theta_{1,0}"] \arrow[u, shift left]
\end{tikzcd}$$

Proof:-

-Straightforward.

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2.0.3:LEMMA:-

$\phi_{1,0}, \phi_{0,1}, \theta_{1,0}, \theta_{0,1}$ are monotonic and, so, continuous.

Proof:-

-Monotonicity is straightforward.

-Continuity follows from 0.6.13, since the lattices have finite depth by 1.2.18, and hence the ACC by 0.6.16.

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2.0.4:LEMMA:-

$E_0 \lhd E_1$ and $A_0 \lhd A_1$.

Proof:-

$$-\phi_{0,1} \circ \phi_{1,0} \left\{\begin{matrix} \lambda x.\, \varepsilon_0 \quad \neq \top \\ \alpha_0 \end{matrix}\right\} = \phi_{0,1} \left\{\begin{matrix} \bot \\ \alpha_0 \end{matrix}\right\} = \left\{\begin{matrix} \bot \\ \alpha_0 \end{matrix}\right\} \sqsubseteq \left\{\begin{matrix} \lambda x.\, \varepsilon_0 \\ \alpha_0 \end{matrix}\right\}.$$

$$-\phi_{1,0} \circ \phi_{0,1}(\varepsilon_0) = \phi_{1,0}(\varepsilon_0) = \varepsilon_0.$$

$$-\theta_{0,1} \circ \theta_{1,0} \left\{\begin{matrix} x \\ \alpha_0(\varepsilon_1) \quad \neq \top \end{matrix}\right\} = \theta_{0,1} \left\{\begin{matrix} x \\ \bot \end{matrix}\right\} = \left\{\begin{matrix} x \\ \bot \end{matrix}\right\} \sqsubseteq \left\{\begin{matrix} x \\ \alpha_0(\varepsilon_1) \end{matrix}\right\}.$$

$$-\theta_{1,0} \circ \theta_{0,1}(\alpha_0) = \theta_{1,0}(\alpha_0) = \alpha_0.$$

ǂ

2.0.5:LEMMA:-

(i) x is not free in $\left\{\begin{matrix} \varepsilon_0 \\ \alpha_0 \end{matrix}\right\}$ => x is not free in $\left\{\begin{matrix} \phi_{0,1}(\varepsilon_0) \\ \theta_{0,1}(\alpha_0) \end{matrix}\right\}$.

(ii) x is not free in $\left\{\begin{matrix} \varepsilon_1 \\ \alpha_1 \end{matrix}\right\}$ => x is not free in $\left\{\begin{matrix} \phi_{1,0}(\varepsilon_1) \\ \theta_{1,0}(\alpha_1) \end{matrix}\right\}$.

Proof:-

-Trivial.

ǂ

2.0.6:DEF:- (Induction Hypothesis - H)

$H(i) \equiv (\phi_{i,i-1}, \phi_{i-1,i}, \theta_{i,i-1}, \theta_{i-1,i}$ are well-defined, doubly strict and continuous)

& (the following diagrams commute :-

$$\begin{tikzcd}
E_{i-1} \arrow[r, shift left, "\phi_{i-1,i}"] \arrow[d, shift left] & E_i \arrow[l, shift left, "\phi_{i,i-1}"] \arrow[d, shift left, "{[x/y]}"] \\
E_{i-1} \arrow[r, shift left, "\phi_{i-1,i}"] \arrow[u, shift left, "{[x/y]}"] & E_i \arrow[l, shift left, "\phi_{i,i-1}"] \arrow[u, shift left]
\end{tikzcd}$$

and,

$$\begin{tikzcd}
A_{i-1} \arrow[r, shift left, "\theta_{i-1,i}"] \arrow[d, shift left] & A_i \arrow[l, shift left, "\theta_{i,i-1}"] \arrow[d, shift left, "{[x/y]}"] \\
A_{i-1} \arrow[r, shift left, "\theta_{i-1,i}"] \arrow[u, shift left, "{[x/y]}"] & A_i \arrow[l, shift left, "\theta_{i,i-1}"] \arrow[u, shift left]
\end{tikzcd}$$)

& (x is not free in $\left\{\begin{matrix}\varepsilon_{i-1}\\ \alpha_{i-1}\end{matrix}\right\} \Rightarrow$ x is not free in $\left\{\begin{matrix}\phi_{i-1,i}(\varepsilon_{i-1})\\ \theta_{i-1,i}(\alpha_{i-1})\end{matrix}\right\}$)

& (x is not free in $\left\{\begin{matrix}\varepsilon_{i}\\ \alpha_{i}\end{matrix}\right\} \Rightarrow$ x is not free in $\left\{\begin{matrix}\phi_{i,i-1}(\varepsilon_{i})\\ \theta_{i,i-1}(\alpha_{i})\end{matrix}\right\}$)

& ($E_{i-1} \lhd E_i$ and $A_{i-1} \lhd A_i$).

2.0.7:LEMMA:-

$H(1)$.

Proof:-

-By lemmas 2.0.1,2.0.2,2.0.3,2.0.4 and 2.0.5.

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2.1:The Other Projections:-

2.1.0:DEF:-

Suppose $H(i)$, for some $i \geq 1$. Then,

$$\phi_{i+1,i} : E_{i+1} \longrightarrow E_i$$
$$\left\{\begin{array}{c}\lambda x.\,\varepsilon_i \\ \alpha_i\end{array}\right\} \longmapsto \left\{\begin{array}{l}\lambda x.\,\phi_{i,i-1}(\varepsilon_i) \\ \theta_{i,i-1}(\alpha_i)\end{array}\right\},$$

$$\phi_{i,i+1} : E_i \longrightarrow E_{i+1}$$
$$\left\{\begin{array}{c}\lambda x.\,\varepsilon_{i-1} \\ \alpha_{i-1}\end{array}\right\} \longmapsto \left\{\begin{array}{l}\lambda x.\,\phi_{i-1,i}(\varepsilon_{i-1}) \\ \theta_{i-1,i}(\alpha_{i-1})\end{array}\right\},$$

$$\theta_{i+1,i} : A_{i+1} \longrightarrow A_i$$
$$\left\{\begin{array}{c}x \\ \alpha_i(\varepsilon_{i+1})\end{array}\right\} \longmapsto \left\{\begin{array}{l}x \\ \theta_{i,i-1}(\alpha_i)(\phi_{i+1,i}(\varepsilon_{i+1}))\end{array}\right\},$$

$$\theta_{i,i+1} : A_i \longrightarrow A_{i+1}$$
$$\left\{\begin{array}{c}x \\ \alpha_{i-1}(\varepsilon_i)\end{array}\right\} \longmapsto \left\{\begin{array}{l}x \\ \theta_{i-1,i}(\alpha_{i-1})(\phi_{i,i+1}(\varepsilon_i))\end{array}\right\},$$

2.1.1:LEMMA:-

The above maps are well-defined and doubly strict.

Proof:-

-First, check well-definedness and double strictness on the various representations of $\top$ and $\bot$ in E_{i+1}, A_{i+1}, E_i and A_i.

-Then, we must deal with the equivalence classes of $\lambda I.E_i, \lambda I.E_{i-1}$.

-Suppose $\lambda a.\varepsilon_i = \lambda b.\eta_i \in E_{i+1}$.

-Let $x \in I$.

-Now, $[x/a]\phi_{i,i-1}(\varepsilon_i) = \phi_{i,i-1}([x/a]\varepsilon_i)$, by $H(i)$.

$= \phi_{i,i-1}([x/b]\eta_i)$, by 1.1.2.

$= [x/b]\phi_{i,i-1}(\eta_i)$, by $H(i)$.

-Hence, $\lambda a.\phi_{i,i-1}(\varepsilon_i) = \lambda b.\phi_{i,i-1}(\eta_i)$, by 1.1.2.

-i.e. $\phi_{i+1,i}(\lambda a.\varepsilon_i) = \phi_{i+1,i}(\lambda b.\eta_i)$.

-Similarly, $\phi_{i,i+1}(\lambda a.\varepsilon_{i-1}) = \phi_{i,i+1}(\lambda b.\eta_{i-1})$ if $\lambda a.\varepsilon_{i-1} = \lambda b.\eta_{i-1}$.

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2.1.2:LEMMA:-

$$[x/y]\circ\phi_{i+1,i} = \phi_{i+1,i}\circ[x/y],$$
$$[x/y]\circ\phi_{i,i+1} = \phi_{i,i+1}\circ[x/y],$$
$$[x/y]\circ\theta_{i+1,i} = \theta_{i+1,i}\circ[x/y],$$
$$[x/y]\circ\theta_{i,i+1} = \theta_{i,i+1}\circ[x/y].$$

Proof:-

-$\phi_{i+1,i}\circ[x/y](\lambda a.\varepsilon_i) = \phi_{i+1,i}(\lambda w.[x/y][w/a]\varepsilon_i)$, where $w \neq x,y$ and is not free in ε_i.

$= \lambda w.\phi_{i,i-1}([x/y][w/a]\varepsilon_i)$

$= \lambda w.[x/y][w/a]\phi_{i,i-1}(\varepsilon_i)$, by $H(i)$.

$= [x/y](\lambda a.\phi_{i,i-1}(\varepsilon_i))$, since w is not free in $\phi_{i,i-1}(\varepsilon_i)$, by $H(i)$.

$= [x/y]\circ\phi_{i+1,i}(\lambda a.\varepsilon_i)$.

-$\phi_{i+1,i}\circ[x/y]\alpha_i = \theta_{i,i-1}\circ[x/y]\alpha_i = [x/y]\circ\theta_{i,i-1}(\alpha_i)$, by $H(i)$.

$= [x/y]\circ\phi_{i+1,i}(\alpha_i)$.

-Similarly, $[x/y]\circ\phi_{i,i+1} = \phi_{i,i+1}\circ[x/y]$.

-$\theta_{i+1,i}\circ[x/y]a = [x/y]a = [x/y]\circ\theta_{i+1,i}(a)$.

-$\theta_{i+1,i}\circ[x/y]\alpha_i(\varepsilon_{i+1}) = \theta_{i+1,i}([x/y]\alpha_i([x/y]\varepsilon_{i+1}))$

$= \theta_{i,i-1}\circ[x/y]\alpha_i(\phi_{i+1,i}\circ[x/y]\varepsilon_{i+1})$

$= [x/y]\circ\theta_{i,i-1}(\alpha_i)([x/y]\circ\phi_{i+1,i}(\varepsilon_{i+1}))$, by $H(i)$ and above.

$= [x/y](\theta_{i,i-1}(\alpha_i)(\phi_{i+1,i}(\varepsilon_{i+1})))$

$= [x/y]\circ\theta_{i+1,i}(\alpha_i(\varepsilon_{i+1}))$.

-Similarly, $[x/y]\circ\theta_{i,i+1} = \theta_{i,i+1}\circ[x/y]$.

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2.1.3:LEMMA:-

(i) x is not free in $\left\{\begin{matrix}\varepsilon_{i+1}\\ \alpha_{i+1}\end{matrix}\right\} \Rightarrow$ x is not free in $\left\{\begin{matrix}\phi_{i+1,i}(\varepsilon_{i+1})\\ \theta_{i+1,i}(\alpha_{i+1})\end{matrix}\right\}$

(ii) x is not free in $\left\{\begin{matrix}\varepsilon_{i}\\ \alpha_{i}\end{matrix}\right\} \Rightarrow$ x is not free in $\left\{\begin{matrix}\phi_{i,i+1}(\varepsilon_{i})\\ \theta_{i,i+1}(\alpha_{i})\end{matrix}\right\}$.

Proof:-

(i) -Suppose $\varepsilon_{i+1} = \lambda y.\varepsilon_i$.

-As in the proof of 1.2.14, we may assume, w.l.o.g., that $x \neq y$ and, so, x is not free in ε_i.

-Then, x is not free in $\phi_{i,i-1}(\varepsilon_i)$, by $H(i)$.

-Thus, x is not free in $\lambda y.\phi_{i,i-1}(\varepsilon_i) = \phi_{i+1,i}(\lambda y.\varepsilon_i)$.

-If $\varepsilon_{i+1} = \alpha_i$, then x is not free in $\alpha_i \in A_i \Rightarrow x$ is not free in $\theta_{i,i-1}(\alpha_i) \in A_{i-1}$, by $H(i)$, $\Rightarrow x$ is not free in $\phi_{i+1,i}(\alpha_i) \in E_i$.

-If $\alpha_{i+1} = a$, then x is not free in $a \in I \Rightarrow x$ is not free in $a \in$
$\Rightarrow x$ is not free in $\theta_{i+1,i}(a) \in A_i$.

-Suppose $\alpha_{i+1} = \alpha_i(\varepsilon_{i+1})$.

-If $\alpha_i = \begin{Bmatrix} \top \\ \bot \end{Bmatrix}$, then $\theta_{i+1,i}(\alpha_{i+1}) = \begin{Bmatrix} \top \\ \bot \end{Bmatrix}$, by 2.1.1.

-If not, then x is not free in $\alpha_i \in A_i$ and x is not free in $\varepsilon_{i+1} \in E_{i+1}$.

-Then, x is not free in $\theta_{i,i-1}(\alpha_i) \in A_{i-1}$ and x is not free in $\phi_{i+1,i}(\varepsilon_{i+1}) \in E_i$, by $H(i)$ and above.

-So, x is not free in $\theta_{i,i-1}(\alpha_i)(\phi_{i+1,i}(\varepsilon_{i+1})) \in A_i$.

-i.e. x is not free in $\theta_{i+1,i}(\alpha_i(\varepsilon_{i+1})) \in A_i$.

(ii) -Similar to part (i).

‡

2.1.4:LEMMA:-

The maps $\phi_{i+1,i}, \phi_{i,i+1}, \theta_{i+1,i}$ and $\theta_{i,i+1}$ are monotonic and, hence, continuous.

Proof:-

-Let $\varepsilon_{i+1} \sqsubseteq \varepsilon'_{i+1}$.

-If $\varepsilon'_{i+1} = \top$, then trivial, by 2.1.1.

-Suppose $\varepsilon_{i+1} = \lambda a.\varepsilon_i \sqsubseteq \lambda b.\eta_i = \varepsilon'_{i+1}$.

-Let $x \in I$. Then, $[x/a]\circ\phi_{i,i-1}(\varepsilon_i) = \phi_{i,i-1}\circ[x/a]\varepsilon_i$, by $H(i)$.

$\sqsubseteq \phi_{i,i-1}\circ[x/b]\eta_i$, by $H(i)$ and 1.1.10.

$= [x/b]\circ\phi_{i,i-1}(\eta_i)$.

-So, $\lambda a.\phi_{i,i-1}(\varepsilon_i) \sqsubseteq \lambda b.\phi_{i,i-1}(\eta_i)$, by 1.1.10.

-i.e. $\phi_{i+1,i}(\lambda a.\varepsilon_i) \sqsubseteq \phi_{i+1,i}(\lambda b.\eta_i)$.

-If $\varepsilon_{i+1} = \alpha_i \sqsubseteq \alpha_i' = \varepsilon_{i+1}'$, then trivial, by $H(i)$.

-∴, $\phi_{i+1,i}$ is monotonic.

-Similarly, $\phi_{i,i+1}$ is monotonic.

-Let $\alpha_{i+1} \sqsubseteq \alpha_{i+1}'$.

-If $\alpha_{i+1} = a \sqsubseteq a' = \alpha_{i+1}'$, then trivial, since $\theta_{i+1,i}$ is the identity map.

-Suppose $\alpha_{i+1} = \alpha_i(\varepsilon_{i+1}) \sqsubseteq \alpha_i'(\varepsilon_{i+1}') = \alpha_{i+1}'$.

-If $\alpha_i = \bot$ or $\alpha_i' = \top$, then trivial, by 2.1.1.

-If not, then $\alpha_i \sqsubseteq \alpha_i'$ and $\varepsilon_{i+1} \sqsubseteq \varepsilon_{i+1}'$.

-Then, $\theta_{i,i-1}(\alpha_i) \sqsubseteq \theta_{i,i-1}(\alpha_i')$ and $\phi_{i+1,i}(\varepsilon_{i+1}) \sqsubseteq \phi_{i+1,i}(\varepsilon_{i+1}')$ by $H(i)$ and above.

$$
\begin{aligned}
\text{-So, } \theta_{i+1,i}(\alpha_i(\varepsilon_{i+1})) &= \theta_{i,i-1}(\alpha_i)(\phi_{i+1,i}(\varepsilon_{i+1}))\\
&\sqsubseteq \theta_{i,i-1}(\alpha_i')(\phi_{i+1,i}(\varepsilon_{i+1}'))\\
&= \theta_{i+1,i}(\alpha_i'(\varepsilon_{i+1}')).
\end{aligned}
$$

-∴, $\theta_{i+1,i}$ is monotonic.

-Similarly, $\theta_{i,i+1}$ is monotonic.

-Hence, the maps are continuous, by 0.6.13,0.6.16 and 1.2.18.

†

2.1.5:LEMMA:-

$E_i \lhd E_{i+1}$ and $A_i \lhd A_{i+1}$.

Proof:-

$$-\phi_{i,i+1}\circ\phi_{i+1,i}\begin{Bmatrix}\lambda x.\varepsilon_i\\ \alpha_i\end{Bmatrix} = \begin{Bmatrix}\lambda x.\phi_{i-1,i}\circ\phi_{i,i-1}(\varepsilon_i)\\ \theta_{i-1,i}\circ\theta_{i,i-1}(\alpha_i)\end{Bmatrix} \sqsubseteq \begin{Bmatrix}\lambda x.\varepsilon_i\\ \alpha_i\end{Bmatrix}, \text{ by } H(i)$$

and 1.1.12.

$$-\phi_{i+1,i}\circ\phi_{i,i+1}\begin{Bmatrix}\lambda x.\varepsilon_{i-1}\\ \alpha_{i-1}\end{Bmatrix} = \begin{Bmatrix}\lambda x.\phi_{i,i-1}\circ\phi_{i-1,i}(\varepsilon_{i-1})\\ \theta_{i,i-1}\circ\theta_{i-1,i}(\alpha_{i-1})\end{Bmatrix} = \begin{Bmatrix}\lambda x.\varepsilon_{i-1}\\ \alpha_{i-1}\end{Bmatrix},$$

by $H(i)$ and 1.1.12.

$$-\theta_{i,i+1}\circ\theta_{i+1,i}\left\{\begin{matrix} a \\ \alpha_i(\varepsilon_{i+1})\end{matrix}\right\} = \left\{\begin{matrix} a \\ \theta_{i-1,i}\circ\theta_{i,i-1}(\alpha_i)(\phi_{i,i+1}\circ\phi_{i+1,i}(\varepsilon_{i+1}))\end{matrix}\right\}$$

$$\sqsubseteq\left\{\begin{matrix} a \\ \alpha_i(\varepsilon_{i+1})\end{matrix}\right\},\text{ by } H(i) \text{ and def. 1.1.24.}$$

$$-\theta_{i+1,i}\circ\theta_{i,i+1}\left\{\begin{matrix} a \\ \alpha_{i-1}(\varepsilon_i)\end{matrix}\right\} = \left\{\begin{matrix} a \\ \theta_{i,i-1}\circ\theta_{i-1,i}(\alpha_{i-1})(\phi_{i+1,i}\circ\phi_{i,i+1}(\varepsilon_i))\end{matrix}\right\}$$

$$=\left\{\begin{matrix} a \\ \alpha_{i-1}(\varepsilon_i)\end{matrix}\right\},\text{ by } H(i) \text{ and above.}$$

ǂ

2.1.6:THEOREM:-

$(\forall i \geq 1)H(i)$.

Proof:-

-$H(1)$, by 2.0.7.

-For $i \geq 1$, $H(i) \Rightarrow H(i+1)$, by 2.1.1,2.1.2,2.1.3,2.1.4 and 2.1.5.

-∴, by induction, the theorem.

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2.2:Inverse Limits:-

2.2.0:DEF:-

E_∞ := Inverse limit of $\langle E_i,\phi_{i+1,i}\rangle_{i=0}^{\infty}$

$= \{\langle\varepsilon_0,\varepsilon_1,........\rangle \mid \varepsilon_i \in E_i$ and $\varepsilon_i = \phi_{i+1,i}(\varepsilon_{i+1})\}$.

A_∞ := Inverse limit of $\langle A_i,\theta_{i+1,i}\rangle_{i=0}^{\infty}$

$= \{\langle\alpha_0,\alpha_1,........\rangle \mid \alpha_i \in A_i$ and $\alpha_i = \theta_{i+1,i}(\alpha_{i+1})\}$.

2.2.1:THEOREM:-

E_∞ and A_∞ are complete continuous lattices.

Proof:-

-By theorems 1.2.18,2.1.6 and 0.6.22.

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2.2.2:DEF:-

Define $\phi_{i,j}, \theta_{i,j}, \phi_{i,\infty}, \theta_{i,\infty}, \phi_{\infty,i}$ and $\theta_{\infty,i}$ as in 0.6.21.

2.2.3:LEMMA:-

(i) All the above maps are doubly strict and continuous.

(ii) The following diagrams commute :-

$$\begin{tikzcd}
E_i \arrow[r, shift left, "\phi_{i,j}"] \arrow[d, shift left] & E_j \arrow[l, shift left, "\phi_{j,i}"] \arrow[d, shift left, "{[x/y]}"] \\
E_i \arrow[r, shift left, "\phi_{i,j}"] \arrow[u, shift left, "{[x/y]}"] & E_j \arrow[l, shift left, "\phi_{j,i}"] \arrow[u, shift left]
\end{tikzcd}$$

and

$$\begin{tikzcd}
A_i \arrow[r, shift left, "\theta_{i,j}"] \arrow[d, shift left] & A_j \arrow[l, shift left, "\theta_{j,i}"] \arrow[d, shift left, "{[x/y]}"] \\
A_i \arrow[r, shift left, "\theta_{i,j}"] \arrow[u, shift left, "{[x/y]}"] & A_j \arrow[l, shift left, "\theta_{j,i}"] \arrow[u, shift left]
\end{tikzcd} .$$

(iii) x is not free in $\left\{\begin{matrix}\varepsilon_i \\ \alpha_i\end{matrix}\right\} \Rightarrow$ x is not free in $\left\{\begin{matrix}\phi_{i,j}(\varepsilon_i) \\ \theta_{i,j}(\alpha_i)\end{matrix}\right\}$.

(iv) For $i \le j$, $E_i \lhd E_j$ and $A_i \lhd A_j$. Also, $E_i \lhd E_\infty$ and $A_i \lhd A_\infty$

<u>Proof:-</u>

-Either trivial or by 0.6.22.

ǂ

2.2.4:DEF:-

Let $x,y \in I$. Then,

$$\begin{array}{rl} [x/y] : E_\infty & \longrightarrow E_\infty \\ \langle\varepsilon_i\rangle_{i=0}^{\infty} & \longmapsto \langle[x/y]\varepsilon_i\rangle_{i=0}^{\infty} \end{array}$$

and,

$$\begin{array}{rl} [x/y] : A_\infty & \longrightarrow A_\infty \\ \langle\alpha_i\rangle_{i=0}^{\infty} & \longmapsto \langle[x/y]\alpha_i\rangle_{i=0}^{\infty} . \end{array}$$

x is <u>NOT FREE IN $\langle\varepsilon_i\rangle_{i=0}^{\infty} \in E_\infty$</u> if $(\forall i \ge 0)$(x is not free in $\varepsilon_i \in E_i$)

x is <u>NOT FREE IN $\langle\alpha_i\rangle_{i=0}^{\infty} \in A_\infty$</u> if $(\forall i \ge 0)$(x is not free in $\alpha_i \in A_i$)

2.2.5:LEMMA:-

(i) The following diagrams commute :-

$$\begin{tikzcd}
E_i \arrow[r, shift left, "\phi_{i,\infty}"] \arrow[d, shift left] & E_\infty \arrow[l, shift left, "\phi_{\infty,i}"] \arrow[d, shift left] \arrow[d, phantom, shift left=4, "{[x/y]}"] \\
E_i \arrow[u, shift left] \arrow[u, phantom, shift left=4, "{[x/y]}"] \arrow[r, shift left, "\phi_{i,\infty}"] & E_\infty \arrow[l, shift left, "\phi_{\infty,i}"] \arrow[u, shift left]
\end{tikzcd}$$

and

$$\begin{tikzcd}
A_i \arrow[r, shift left, "\theta_{i,\infty}"] \arrow[d, shift left] & A_\infty \arrow[l, shift left, "\theta_{\infty,i}"] \arrow[d, shift left] \arrow[d, phantom, shift left=4, "{[x/y]}"] \\
A_i \arrow[u, shift left] \arrow[u, phantom, shift left=4, "{[x/y]}"] \arrow[r, shift left, "\theta_{i,\infty}"] & A_\infty \arrow[l, shift left, "\theta_{\infty,i}"] \arrow[u, shift left]
\end{tikzcd} .$$

(ii) x is not free in $\left\{\begin{matrix}\varepsilon_i \\ \alpha_i\end{matrix}\right\}$ ⟹ x is not free in $\left\{\begin{matrix}\phi_{i,\infty}(\varepsilon_i) \\ \theta_{i,\infty}(\alpha_i)\end{matrix}\right\}$.

(iii) x is not free in $\left\{\begin{matrix}\varepsilon \\ \alpha\end{matrix}\right\}$ ⟹ x is not free in $\left\{\begin{matrix}\phi_{\infty,i}(\varepsilon) \\ \theta_{\infty,i}(\alpha)\end{matrix}\right\}$.

Proof:-

-Trivial, using 2.2.3(ii) and (iii).

$\dagger$

2.2.6:NOTATION:-

When we use symbols like ε, α' for elements of E_∞, A_∞, we will assume they are of the form $<\varepsilon_i>_{i=0}^{\infty}$ and $<\alpha_i'>_{i=0}^{\infty}$, respectively.

2.2.7:RECALL:-

Let $\varepsilon, \varepsilon' \in E_\infty$ and $\alpha, \alpha' \in A_\infty$. Then, by 0.6.22,

$$\varepsilon \sqcup \varepsilon' = <\bigsqcup_{j=0}^{\infty}\{\phi_{j,i}(\varepsilon_j \sqcup \varepsilon_j')\}>_{i=0}^{\infty}$$

and $\alpha \sqcup \alpha' = <\bigsqcup_{j=0}^{\infty}\{\theta_{j,i}(\alpha_j \sqcup \alpha_j')\}>_{i=0}^{\infty}$, where the sets form directed chains.*

2.2.8:THEOREM:-

(i) $[x/y]$ is a continuous map on E_∞, A_∞.

(ii) $[x/x]$ is the identity map on E_∞, A_∞.

(iii) $(a \neq d) \wedge (b \neq c,d)$ ⟹ ($[a/b]$ and $[c/d]$ commute).

(* these reults are reflected for [illegible]'s

(iv) $[x/y]\eta = \begin{Bmatrix} \top \\ \bot \end{Bmatrix} \Longleftrightarrow \eta = \begin{Bmatrix} \top \\ \bot \end{Bmatrix}, \forall \eta \in E_\infty, A_\infty$.

(v) x is not free in $\top$ or $\bot$ in E_∞, A_∞.

(vi) $(x \neq y) \Rightarrow$ (y is not free in $[x/y]\eta$, $\forall \eta \in E_\infty, A_\infty$).

(vii) (y is not free in $\eta \in E_\infty, A_\infty$) $\Rightarrow$ $([x/y]\eta = \eta)$.

(viii) $(x \neq a) \wedge$ (x is not free in $\eta \in E_\infty, A_\infty$) $\Rightarrow$ (x is not free in $[a/b]\eta$).

(ix) (y is not free in $\eta \in E_\infty, A_\infty$) $\Rightarrow$ $([x/y][y/z]\eta = [x/z]\eta)$.

(x) (x is not free in $\eta, \eta' \in E_\infty, A_\infty$) $\Rightarrow$ (x is not free in $\eta \sqcup \eta', \eta \sqcap \eta' \in E_\infty, A_\infty$).

(xi) (x is not free in $\eta, \eta' \in E_\infty, A_\infty$) $\Rightarrow$

$$\left([x/a]\begin{Bmatrix} \eta \sqcup \eta' \\ \eta \sqcap \eta' \end{Bmatrix} = \begin{Bmatrix} [x/a]\eta \sqcup [x/a]\eta' \\ [x/a]\eta \sqcap [x/a]\eta' \end{Bmatrix}\right).$$

If x were free, then we would have the usual inequalities.

Proof:-

(i) -By 0.6.22(v), $f: L \to E_\infty$ is continuous iff $\phi_{\infty,i} \circ f: L \to E_i$ is continuous for all $i \geq 0$.

-But, $\phi_{\infty,i} \circ [x/y] = [x/y] \circ \phi_{\infty,i}$, by 2.2.5(i), which, being the composition of continuous functions, is continuous.

-Hence, $[x/y]$ is continuous on E_∞ and, similarly, on A_∞.

(ii),(iii),(iv),(v),(vi),(vii),(viii) and (ix) -Trivial.

(x) -Let x be not free in $\eta, \eta' \in E_\infty$.

-Then, $(\forall j \geq 0)$(x is not free in $\eta_j, \eta_j' \in E_j$).

-So, $(\forall j \geq 0)$(x is not free in $\eta_j \sqcup \eta_j' \in E_j$), by 1.2.18.

-Thus, $(\forall i, j \geq 0)$(x is not free in $\phi_{j,i}(\eta_j \sqcup \eta_j') \in E_i$), by 2.2.3.

-But, $\eta \sqcup \eta' = \langle \bigsqcup_{j=0}^{\infty} \{\phi_{j,i}(\eta_j \sqcup \eta_j')\} \rangle_{i=0}^{\infty}$, by 2.2.7.

-Since E_i has finite depth, and by 0.6.13, $\bigsqcup_{j=0}^{\infty} \{\phi_{j,i}(\eta_j \sqcup \eta_j')\} = \phi_{k,i}(\eta_k \sqcup \eta_k')$, for some $k \geq 0$.*

-Thus, $(\forall i \geq 0)$(x is not free in $\bigsqcup_{j=0}^{\infty} \{\phi_{j,i}(\eta_j \sqcup \eta_j')\}$).

-i.e. x is not free in $\eta \sqcup \eta' \in E_\infty$.

-Similarly, x is not free in $\eta \sqcup \eta' \in A_\infty$ and $\eta \sqcup \eta' \in E_\infty, A_\infty$.

(* this result is reflected for $\sqcap$'s)

(xi) -In general, we have the inequalities, by part (i).

-Let x be not free in $\eta, \eta' \in E_\infty$.

-Then, $[x/a](\eta \sqcup \eta') = \langle [x/a] \bigsqcup_{j=0}^{\infty} \{\phi_{j,i}(\eta_j \sqcup \eta_j')\} \rangle_{i=0}^{\infty}$, by 2.2.7.

$$= \langle \bigsqcup_{j=0}^{\infty} \{[x/a] \circ \phi_{j,i}(\eta_j \sqcup \eta_j')\} \rangle_{i=0}^{\infty}$$

$$= \langle \bigsqcup_{j=0}^{\infty} \{\phi_{j,i} \circ [x/a](\eta_j \sqcup \eta_j')\} \rangle_{i=0}^{\infty}, \text{ by 2.2.3(ii).}$$

$$= \langle \bigsqcup_{j=0}^{\infty} \{\phi_{j,i}([x/a]\eta_j \sqcup [x/a]\eta_j')\} \rangle_{i=0}^{\infty}, \text{ by 1.2.18.}$$

$$= [x/a]\eta \sqcup [x/a]\eta', \text{ by 2.2.7.}$$

-Similarly, for the other three cases.

‡

2.2.9:REMARK:-

We have now constructed an E_∞ which contains elements like those exampled in 0.7.27. If $\varepsilon \in E_\infty$, we can see that it is nearly always of the form,

$$\langle \bot, \lambda x.\varepsilon_0', \lambda x.\varepsilon_1', \lambda x.\varepsilon_2', \ldots\ldots\ldots \rangle,$$

where $\varepsilon' \in E_\infty$ or,

$$\langle \alpha_0', \alpha_0', \alpha_1', \alpha_2', \alpha_3', \alpha_4', \ldots\ldots\ldots\ldots \rangle,$$

where $\alpha' \in A_\infty$.

Further, if $\alpha \in A_\infty$, it is always either of the form,

$$\langle a,a,a,a,a,a,a,a,a,a,\ldots\ldots\ldots \rangle,$$

where $a \in I$ or,

$$\langle \bot, \alpha_0'(\varepsilon_1'), \alpha_1'(\varepsilon_2'), \alpha_2'(\varepsilon_3'), \ldots\ldots \rangle,$$

where $\alpha' \in A_\infty$ and $\varepsilon' \in E_\infty$.

This uncertainty in the decomposition of E_∞ arises because it is possible for elements of E_∞ to have infinitely many free variables. For example, let $\langle x_0, x_1, x_2, \ldots\ldots \rangle$ be an enumeration of the elements of I. Consider,

$$\varepsilon = \langle \bot, \bot, x_0(\bot), x_0(x_1(\bot)), x_0(x_1(x_2(\bot))), \ldots\ldots\ldots \rangle.$$

Then, x is not free in $\varepsilon \iff$ x is not free in ε_i, $\forall i \geq 0 \iff x \neq x_i$

$\forall i \geq 0$ - $\not X$, since the x_i's were an ennumeration. Thus, all elements of I are free in ε.

Suppose we want to construct an element like $\lambda ?.?(\varepsilon)$. What can we choose for "?"? There is nothing left! However, such an element exists quite happily in E_∞ :-

$<\bot,\bot,\bot,\lambda x_0.x_0(\bot),\lambda x_1.x_1(x_0(\bot)),\lambda x_2.x_2(x_0(x_1(\bot))),........>$.

There is no way to express this element in the form :-

$<\bot,\lambda x.\varepsilon_0,\lambda x.\varepsilon_1,\lambda x.\varepsilon_2,........>$.

This awkwardness comes about because we use the same set for both the free variables and the not free variables. The not free variables are really only for use as pointers and they do not have the actual physical significance of the free ones.

Technically, there may be several ways of getting out of this (e.g. reserving a special symbol, $\overline{x}$, for just such an eventuality and not allowing it to occur free - using different sets, I and $\overline{I}$, for free and not free variables - a suggestion of Reynolds - [55] - altering all the definitions accordingly and checking them!), but we do not think it worth all the trouble because :-

(a) it makes the constructions clumsy and inelegant,

(b) we don't mind E_∞ catering for more than the pure λ-calculus with terms like $\lambda ?.?(\varepsilon)$ above,

(c) when we deal with the λ-definable elements of E_∞, the problem does not occur

and (d) E_∞ decomposes very nicely anyhow, provided we aren't so naive about it (2.2.20).

<u>2.2.10:DEF:-</u>

Let $\phi\phi_{i+1,i}$ be $\phi_{i+2,i+1}$ restricted to $\lambda I.E_{i+1}$,

$\phi\phi_{i,i+1}$ " $\phi_{i+1,i+2}$ " " $\lambda I.E_i$,

$\theta\theta_{i+1,i}$ " $\theta_{i+2,i+1}$ " " $A_{i+1}(E_{i+2})$ and

$\theta\theta_{i,i+1}$ " $\theta_{i+1,i+2}$ " " $A_i(E_{i+1})$.

2.2.11:LEMMA:-

$(\forall i \geq 2)(\mathcal{H}(i)$ restricted to $\lambda I.E_{i-1}$ and $A_{i-1}(E_i))$.

Proof:-

-Trivial.

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2.2.12:DEF:-

$\lambda I.E_\infty$:= Inverse limit of $\langle \lambda I.E_i, \phi\phi_{i+1,i}\rangle_{i=0}^{\infty}$.

$A_\infty(E_\infty)$:= Inverse limit of $\langle A_i(E_{i+1}), \theta\theta_{i+1,i}\rangle_{i=0}^{\infty}$.

Also, $\phi\phi_{i,j}, \theta\theta_{i,j}, \phi\phi_{i,\infty}, \theta\theta_{i,\infty}, \phi\phi_{\infty,i}$ and $\theta\theta_{\infty,i}$ as in 0.6.21. Also, $[x/y]$ on $\lambda I.E_\infty$ and $A_\infty(E_\infty)$ and NOT FREE IN are defined coordinate-wise, as in 2.2.4.

2.2.13:LEMMA:-

Suitable adaptions of 2.2.1,2.2.3,2.2.5 and 2.2.8.

Proof:-

-Straightforward.

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2.2.14:LEMMA:-

Let $\varepsilon \in E_\infty$ and $\varepsilon \neq \top,\bot$. Then, either :-

$(\forall i \geq 1)$(there exists $x_i \in I$ and $\varepsilon'_{i-1} \in E_{i-1})(\varepsilon_i = \lambda x_i.\varepsilon'_{i-1})$,

or (exclusively) :-

$(\forall i \geq 1)$(there exists $\alpha_{i-1} \in A_{i-1})(\varepsilon_i = \alpha_{i-1})$.

Proof:-

-Since $\varepsilon \neq \top,\bot$, there exists $j \geq 1$ such that $\varepsilon_j \neq \top,\bot$.

-Either $\varepsilon_j = \lambda x_j.\varepsilon'_{j-1}$ where $\varepsilon'_{j-1} \neq \top,\bot$:-

-Clearly, $\forall 1 \leq i \leq j$, $\varepsilon_i = \lambda x_j.\varepsilon'_{i-1}$, where $\varepsilon'_{i-1} = \phi_{j-1,i-1}(\varepsilon'_{j-1})$.

-Suppose $j < i$. Consider ε_i. If $\varepsilon_i = \alpha_{i-1}$, then $\varepsilon_j = \alpha_{j-1} \in A_{j-1} \subset E_j - \chi$. So, $\varepsilon_i \neq \alpha_{i-1}$ - i.e. there exists $x_i \in I$ and $\varepsilon'_{i-1} \in E_{i-1}$ such that $\varepsilon_i = \lambda x_i.\varepsilon'_{i-1}$.

-Or $\varepsilon_j = \alpha_{j-1} \neq \top,\bot$:-

-Clearly, $\forall 1 \leq i \leq j$, $\varepsilon_i = \alpha_{i-1}$, where $\alpha_{i-1} = \theta_{j-1,i-1}(\alpha_{j-1})$.

-Suppose $j < i$. Consider ε_i. If $\varepsilon_i = \lambda x_i . \varepsilon'_{i-1}$, then $\varepsilon_j = \lambda x_i . \varepsilon'_{j-1} \in \lambda I.E_{j-1} \subset E_j - \chi$. So, $\varepsilon_i = \alpha_{i-1}$.

ǂ

2.2.15:DEF:-

$$\pi_1 : E_\infty \longrightarrow \lambda I.E_\infty$$

$$\varepsilon \longmapsto \left\{ \begin{array}{l} \top, \text{ if } \varepsilon = \top. \\ \langle \lambda x_i . \varepsilon'_{i-1} \rangle_{i=1}^{\infty} \text{ , if \underline{either}-clause above.} \\ \bot, \text{ if \underline{or}-clause above.} \\ \bot, \text{ if } \varepsilon = \bot. \end{array} \right\}$$

and

$$\pi_2 : E_\infty \longrightarrow A_\infty$$

$$\varepsilon \longmapsto \left\{ \begin{array}{l} \top, \text{ if } \varepsilon = \top. \\ \bot, \text{ if \underline{either}-clause above.} \\ \langle \alpha_{i-1} \rangle_{i=1}^{\infty}, \text{ if \underline{or}-clause above.} \\ \bot, \text{ if } \varepsilon = \bot. \end{array} \right\},$$

where $x_{i-1}, \varepsilon'_{i-1}$ and α_{i-1} are as given by 2.2.14. Also,

$$\pi_1^{-1} : \lambda I.E_\infty \longrightarrow E_\infty$$

$$\langle \lambda x_i . \varepsilon_i \rangle_{i=0}^{\infty} \longmapsto \langle \bot, \langle \lambda x_i . \varepsilon_i \rangle_{i=0}^{\infty} \rangle$$

and

$$\pi_2^{-1} : A_\infty \longrightarrow E_\infty$$

$$\langle \alpha_i \rangle_{i=0}^{\infty} \longmapsto \langle \alpha_0, \langle \alpha_i \rangle_{i=0}^{\infty} \rangle .$$

2.2.16:LEMMA:-

π_1, π_2, π_1^{-1} and π_2^{-1} are well-defined and continuous.

Proof:-

-Trivial, by 2.2.13,2.2.14 and 0.6.22.

ǂ

2.2.17:LEMMA:-

Let $\alpha \in A_\infty$ and $\alpha \neq \top, \bot$. Then, <u>either</u> :-

$(\forall i \geq 0)$(there exists $a \in I$)$(\alpha_i = a)$, or (exclusively) :-

(∀i ≥ 1)(there exists $\alpha'_{i-1} \in A_{i-1}$ and $\varepsilon_i \in E_i$)($\alpha_i = \alpha'_{i-1}(\varepsilon_i)$).

Proof:-

-Trivial, similar to 2.2.14.

ǂ

2.2.18:DEF:-

$$\psi_1 : A_\infty \longrightarrow I'$$

$$\alpha \longmapsto \left\{\begin{array}{l} \top, \text{ if } \alpha = \top. \\ a, \text{ if } \underline{\text{either}}\text{-clause above.} \\ \bot, \text{ if } \underline{\text{or}}\text{-clause above.} \\ \bot, \text{ if } \alpha = \bot. \end{array}\right\}$$

and

$$\psi_2 : A_\infty \longrightarrow A_\infty(E_\infty)$$

$$\alpha \longmapsto \left\{\begin{array}{l} \top, \text{ if } \alpha = \top. \\ \bot, \text{ if } \underline{\text{either}}\text{-clause above.} \\ \langle \alpha'_{i-1}(\varepsilon_i)\rangle_{i=1}^{\infty}, \text{ if } \underline{\text{or}}\text{-clause above.} \\ \bot, \text{ if } \alpha = \bot. \end{array}\right\},$$

where a, α'_{i-1} and ε_i are as given by 2.2.17. Also,

$$\psi_1^{-1} : I' \longrightarrow A_\infty$$

$$a \longmapsto \langle a\rangle_{i=0}^{\infty}$$

and

$$\psi_2^{-1} : A_\infty(E_\infty) \longrightarrow A_\infty$$

$$\langle \alpha_{i-1}(\varepsilon_i)\rangle_{i=1}^{\infty} \longmapsto \langle \bot, \langle \alpha_{i-1}(\varepsilon_i)\rangle_{i=1}^{\infty}\rangle.$$

2.2.19:LEMMA:-

$\psi_1, \psi_2, \psi_1^{-1}$ and ψ_2^{-1} are well-defined and continuous.

Proof:-

-Trivial, as in 2.2.16.

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2.2.20:THEOREM:-

Under the maps defined in 2.2.15 and 2.2.18,

$$\lambda I.E_\infty, A_\infty \lhd E_\infty$$

$$\text{and } I', A_\infty(E_\infty) \lhd A_\infty.$$

Further, under the obvious direct sum maps, $\pi_1 + \pi_2$, $\pi_1^{-1} + \pi_2^{-1}$, $\psi_1 + \psi_2$ and $\psi_1^{-1} + \psi_2^{-1}$, we have :-

$$E_\infty \triangleq \lambda I.E_\infty + A_\infty$$

$$\text{and } A_\infty \triangleq I' + A_\infty(E_\infty).$$

Proof:-

-Trivial.

$\nmid$

3:APPLICATION AND SUBSTITUTION IN E_∞.

3.0:Finite Applications and Substitution:-

3.0.0:REMARK:-

In any model of the λ-calculus we ought to define an application so as to capture the notion of β-reduction,

$$Ap : E_\infty \times E_\infty \to E_\infty$$

This should extend the partial application which already exists, by 2.2.20, where $\varepsilon = \alpha \in A_\infty, \subseteq E_\infty$, and $\delta \in E_\infty \Rightarrow Ap(\varepsilon, \delta) = Ap(\alpha, \delta) = \alpha(\delta) \in A_\infty(E_\infty), \subseteq A_\infty \subseteq E_\infty$.

We now only have to define Ap for when $\varepsilon = \lambda x.\varepsilon' \in \lambda I.E_\infty \subseteq E_\infty$.

We will define Ap coordinate-wise following the definition of β-reduction, 0.3.6, as closely as possible. Thus, we will also have to define a substitution operator,

$$[\varepsilon /x] : E_\infty \to E_\infty,$$

similarly coordinate-wise. This will, of course, have to extend the operator we already have when $\varepsilon = a \in I, \subseteq A_\infty \subseteq E_\infty$.

3.0.1:DEF.:-

Let $x \in I$ and $\varepsilon_i \in E_i$. Then,

$$[\varepsilon_i/x]_I : I' \longrightarrow E_i$$
$$y \longmapsto \begin{Bmatrix} y, \text{ if } x \neq y \\ \varepsilon_i, \text{ if } x = y \end{Bmatrix}$$

3.0.2:LEMMA:-

$[\varepsilon_i/x]_I$ is monotonic, and so continuous. It is doubly strict. Also, $\varepsilon_i \sqsubseteq \varepsilon_i' \Rightarrow [\varepsilon_i/x]_I \sqsubseteq [\varepsilon_i'/x]_I$ (in $[I' \to E_i]$).

Proof:-

-Trivial.

3.0.3:DEF:-

$$[\varepsilon_1/x]_{E_0} := [\varepsilon_1/x]_I \in [I'\to E_1], = [E_0\to E_1]$$

$$[\varepsilon_2/x]_{A_0} := [\varepsilon_2/x]_I \in [I'\to E_2], = [A_0\to E_2]$$

For $i \geq 2$,

$$[\varepsilon'_i/x]_{E_{i-1}} : E_{i-1} \longrightarrow E_i$$

$$\left\{\begin{matrix} \lambda y.\, \varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \lambda z.[\varepsilon'_{i-1}/x]_{E_{i-2}}[z/y]\varepsilon_{i-2}\,, \\ [\varepsilon'_i/x]_{A_{i-2}}\alpha_{i-2} \end{matrix}\right\}$$

where $z \neq x$ and z is not free in $\varepsilon'_{i-1}, \varepsilon_{i-2}$.

For $i \geq 3$,

$$[\varepsilon'_i/x]_{A_{i-2}} : A_{i-2} \longrightarrow E_i$$

$$\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} [\varepsilon'_i/x]_I y \\ Ap_{i-1}([\varepsilon'_{i-1}/x]_{A_{i-3}}\alpha_{i-3}, \\ \qquad [\varepsilon'_{i-1}/x]_{E_{i-2}}\varepsilon_{i-2}) \end{matrix}\right\}$$

For $i \geq 1$,

$$Ap_i : E_i \times E_i \longrightarrow E_{i+1}$$

$$\left\{\begin{matrix} (\alpha_{i-1}, \varepsilon'_i) \\ (\lambda x.\, \varepsilon_{i-1}, \varepsilon'_i) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \alpha_{i-1}(\varepsilon'_i) \\ \phi_{i,i+1}([\varepsilon'_i/x]_{E_{i-1}}\varepsilon_{i-1}) \end{matrix}\right\}$$

3.0.4:REMARK:-

In the above definition, where there appear expressions like $\varepsilon'_i, \varepsilon'_{i-1}$ and $\alpha_{i-2}, \alpha_{i-3}$, we imply that :-

$$\phi_{i,i-1}(\varepsilon'_i) = \varepsilon'_{i-1} \text{ and } \theta_{i-2,i-3}(\alpha_{i-2}) = \alpha_{i-3}.$$

The definition seems rather complicated! There are many different ways the types of the functions could have been arranged - e.g. $Ap_i \in [E_i \times E_i \to E_i]$ - but we still have to use various projections and inclusions to make them fit: the above seems to be the cleanest way to do it. All the ways would be equivalent, however, in that they lead to the same function Ap, when it is eventually defined as $\bigsqcup_{i=0}^{\infty} Ap_i$.

As before, see 1.2.2, we will omit most of the subscripts on the substitution operators, hoping the context indicates on which lattice they are operating in each instance.

The definition is inductively defined and so, most of the theorems on it will require proofs by induction. The following "induction schema" is designed to handle such proofs.

3.0.5:LEMMA:- ($SPQR$-induction)

$$\begin{array}{l}
\vdash (S(i),\forall i\geq 0) \wedge Q(1) \wedge R(2) \\
\vdash (P(i-1) \wedge Q(i-1) \wedge R(i-1) \wedge S(i) \Rightarrow R(i)),\forall i\geq 3 \\
\vdash (Q(i-1) \wedge R(i) \Rightarrow Q(i)),\forall i\geq 2 \\
\vdash (Q(i) \Rightarrow P(i)),\forall i\geq 1 \\
\hline
\quad (P(i) \wedge Q(i)),\forall i\geq 1 \\
\text{and} \quad R(i),\forall i\geq 2.
\end{array}$$

Proof:-

-$Q(1)$ gives us $P(1)$.

-$Q(1) \wedge R(2)$ gives us $Q(2)$, which gives us $P(2)$.

-$\therefore$we have $P(2) \wedge Q(2) \wedge R(2)$.

-But,$P(i) \wedge Q(i) \wedge R(i) \Rightarrow R(i+1)$, since $S(i),\forall i\geq 0$.

-And,$Q(i) \wedge R(i+1) \Rightarrow Q(i+1)$.

-And,$Q(i+1) \Rightarrow P(i+1)$.

-$\therefore$,$\forall i\geq 2, P(i) \wedge Q(i) \wedge R(i) \Rightarrow P(i+1) \wedge Q(i+1) \wedge R(i+1)$.

-$\therefore$,by ordinary induction, we have,

$\forall i\geq 2,\ P(i) \wedge Q(i) \wedge R(i)$.

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3.0.6:REMARK:-

We will use the above scheme as follows :-

$S(i) \leftrightarrow$ sentences about $[\varepsilon_i/x]_I$
$P(i) \leftrightarrow$ " " Ap_i
$Q(i) \leftrightarrow$ " " $[\varepsilon_i/x]_{E_{i-1}}$
$R(i) \leftrightarrow$ " " $[\varepsilon_i/x]_{A_{i-2}}$.

3.0.7:COR:-

Ap_i and $[\varepsilon_i'/x]_{E_{i-1}}$ are defined, $\forall i \geq 1$. Also, $[\varepsilon_i'/x]_{A_{i-2}}$ is defined, $\forall i \geq 2$.

Proof:-

-Let $S(i) \equiv [\varepsilon_i/x]_I$ is defined.
- " $P(i) \equiv Ap_i$ " " .
- " $Q(i) \equiv [\varepsilon_i/x]_{E_{i-1}}$ " " .
- " $R(i) \equiv [\varepsilon_i/x]_{A_{i-2}}$ " " .

-Then, $\forall i \geq 0$, $S(i)$, by defn. 3.0.1.

-And, we have $Q(1) \wedge R(2)$, since $Q(1) \equiv S(1)$ and $R(2) \equiv S(2)$, by 3.0.3.

-Clearly, $\forall i \geq 2$, $Q(i-1) \wedge R(i) \Rightarrow Q(i)$, by defn. 3.0.3.

- " , $\forall i \geq 3$, $P(i-1) \wedge Q(i-1) \wedge R(i-1) \wedge S(i) \Rightarrow R(i)$, by 3.0.3.

- " , $\forall i \geq 1$, $Q(i) \Rightarrow P(i)$, by 3.0.3.

- , by lemma 3.0.5, $P(i) \wedge Q(i)$, $\forall i \geq 1$, and $R(i)$, $\forall i \geq 2$.

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3.1:Basic Properties of $[\varepsilon_i/x]$ and Ap_i:-

3.1.0:DEF:-

Let $\left\{\begin{matrix} S \\ Q \\ R \end{matrix}\right\}^{(*)}(i) \equiv ([\varepsilon_i/x]$ is well-defined and doubly strict)

& $([z/z'][[z/z']\varepsilon_i/x] = [z/z'][\varepsilon_i/x])$

& ($x \neq z, z'$ and z' is not free in $\varepsilon_i \Rightarrow [\varepsilon_i/x][z/z'] = [z/z'][\varepsilon_i/x]$)

& (x is not free in $\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} \Rightarrow [\varepsilon_i/x][x/y]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} = [\varepsilon_i/y]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}$)

(*) In this, and similar, definitions, we are defining three hypotheses simultaenously - choose one from the $\left\{\begin{matrix} \text{top} \\ \text{middle} \\ \text{lower} \end{matrix}\right\}$ alternatives consistently.

3.0.7:COR:-

Ap_i and $[\varepsilon_i'/x]_{E_{i-1}}$ are defined, $\forall i \geq 1$. Also, $[\varepsilon_i'/x]_{A_{i-2}}$ is defined, $\forall i \geq 2$.

Proof:-

-Let $S(i) \equiv [\varepsilon_i/x]_I$ is defined.

- " $P(i) \equiv Ap_i$ " " .

- " $Q(i) \equiv [\varepsilon_i/x]_{E_{i-1}}$ " " .

- " $R(i) \equiv [\varepsilon_i/x]_{A_{i-2}}$ " " .

-Then, $\forall i \geq 0$, $S(i)$, by defn. 3.0.1.

-And, we have $Q(1) \wedge R(2)$, since $Q(1) \equiv S(1)$ and $R(2) \equiv S(2)$, by 3.0.3.

-Clearly, $\forall i \geq 2$, $Q(i-1) \wedge R(i) \Rightarrow Q(i)$, by defn. 3.0.3.

- " , $\forall i \geq 3$, $P(i-1) \wedge Q(i-1) \wedge R(i-1) \wedge S(i) \Rightarrow R(i)$, by 3.0.3.

- " , $\forall i \geq 1$, $Q(i) \Rightarrow P(i)$, by 3.0.3.

- , by lemma 3.0.5, $P(i) \wedge Q(i)$, $\forall i \geq 1$, and $R(i)$, $\forall i \geq 2$.

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3.1:Basic Properties of $[\varepsilon_i/x]$ and Ap_i:-

3.1.0:DEF:-

Let $\left\{\begin{matrix} S \\ Q \\ R \end{matrix}\right\}(i) \equiv$ ($[\varepsilon_i/x]$ is well-defined and doubly strict)

& $([z/z'][[z/z']\varepsilon_i/x] = [z/z'][\varepsilon_i/x])$

& ($x \neq z, z'$ and z' is not free in $\varepsilon_i \Rightarrow [\varepsilon_i/x][z/z'] = [z/z'][\varepsilon_i/x]$)

& (x is not free in $\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} \Rightarrow [\varepsilon_i/x][x/y]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} = [\varepsilon_i/y]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}$)

& (a is not free in $\varepsilon_i', \left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} \Rightarrow$

a is not free in $[\varepsilon_i'/x]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}$)

& (a is not free in $\varepsilon_i' \Rightarrow$ a is not free in $[\varepsilon_i'/a]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}$).

Let $P(i) \equiv (Ap_i$ is well-defined)

& $(Ap_i(\left\{\begin{matrix} \top \\ \bot \end{matrix}\right\}, \varepsilon_i') = \left\{\begin{matrix} \top \\ \bot \end{matrix}\right\})$

& $([z/z']Ap_i(\varepsilon_i, \varepsilon_i') = Ap_i([z/z']\varepsilon_i, [z/z']\varepsilon_i')$

& (a is not free in $\varepsilon_i, \varepsilon_i' \Rightarrow$ a is not free in $Ap_i(\varepsilon_i, \varepsilon_i')$).

3.1.1:LEMMA:-

$(\forall i \geq 0) S(i)$.

Proof:-

-Straightforeward examination of all possibilities.

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3.1.2:LEMMA:-

Suppose $P(i-1) \wedge R(i-1)$, for some $i \geq 3$. Then, $[\varepsilon_i'/x]_{A_{i-2}}$ is well-defined and doubly strict.

Proof:-

$$-[\varepsilon_i'/x](\top_{A_{i-2}}) = [\varepsilon_i'/x]\left\{\begin{matrix} \top_I \\ \top_{A_{i-3}}(\varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} [\varepsilon_i'/x]\top_I \\ Ap_{i-1}([\varepsilon_{i-1}'/x]\top_{A_{i-3}}, [\varepsilon_{i-1}'/x]\varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \top_I, \text{ by 3.1.1} \\ Ap_{i-1}(\top_{E_{i-1}}, [\varepsilon_{i-1}'/x]\varepsilon_{i-2}), \text{ by } R(i-1) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \top_I \\ \top_{E_i}, \text{ by } P(i-1) \end{matrix}\right\} = \top_{E_i}.$$

-Similarly, $[\varepsilon_i'/x](\bot_{A_{i-2}}) = \bot_{E_i}$.

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3.1.3:LEMMA:-

Suppose $P(i-1)\wedge Q(i-1)\wedge R(i-1)$, for some $i\geq 3$. Let $z,z'\in I'$.

Then, $[z/z'][[z/z']\varepsilon'_i/x]_{A_{i-2}} = [z/z'][\varepsilon'_i/x]_{A_{i-2}}$.

Proof:-

$$-[z/z'][[z/z']\varepsilon'_i/x]\begin{Bmatrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{Bmatrix}$$

$$= [z/z']\begin{Bmatrix} [[z/z']\varepsilon'_i/x]y \\ Ap_{i-1}([[z/z']\varepsilon'_{i-1}/x]\alpha_{i-3},[[z/z']\varepsilon'_{i-1}/x]\varepsilon_{i-2}) \end{Bmatrix},$$

by 2.1.6, since $\phi_{i,i-1}\cdot[z/z']\varepsilon'_i = [z/z']\cdot\phi_{i,i-1}(\varepsilon'_i) = [z/z']\varepsilon'_{i-1}$.

$$=\begin{Bmatrix} [z/z'][[z/z']\varepsilon'_i/x]y \\ Ap_{i-1}([z/z'][[z/z']\varepsilon'_{i-1}/x]\alpha_{i-3},[z/z'][[z/z']\varepsilon'_{i-1}/x]\varepsilon_{i-2}) \end{Bmatrix},$$

by $P(i-1)$.

$$=\begin{Bmatrix} [z/z'][\varepsilon'_i/x]y, \text{ by 3.1.1.} \\ Ap_{i-1}([z/z'][\varepsilon'_{i-1}/x]\alpha_{i-3},[z/z'][\varepsilon'_{i-1}/x]\varepsilon_{i-2}), \text{ by } Q(i-1) \end{Bmatrix}{}_{\wedge R(i-1)}$$

$$= [z/z']\begin{Bmatrix} [\varepsilon'_i/x]y \\ Ap_{i-1}([\varepsilon'_{i-1}/x]\alpha_{i-3},[\varepsilon'_{i-1}/x]\varepsilon_{i-2}), \text{ by } P(i-1) \end{Bmatrix}$$

$$= [z/z'][\varepsilon'_i/x]\begin{Bmatrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{Bmatrix}.$$

ǂ

3.1.4:LEMMA:-

Suppose $P(i-1)\wedge Q(i-1)\wedge R(i-1)$, for some $i\geq 3$. Let $x,z,z'\in I$ such that $x\neq z,z'$ and z' is not free in ε'_i. Then,

$[\varepsilon'_i/x][z/z'] = [z/z'][\varepsilon'_i/x]$.

Proof:-

$$-[\varepsilon'_i/x][z/z']\begin{Bmatrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{Bmatrix}$$

$$=\begin{Bmatrix} [\varepsilon'_i/x][z/z']y \\ Ap_{i-1}([\varepsilon'_{i-1}/x][z/z']\alpha_{i-3},[\varepsilon'_{i-1}/x][z/z']\varepsilon_{i-2}) \end{Bmatrix}$$

$$=\left\{\begin{matrix}[z/z'][\varepsilon_i'/x]y, \text{ by 3.1.1.}\\ Ap_{i-1}([z/z'][\varepsilon_{i-1}'/x]\alpha_{i-3},[z/z'][\varepsilon_{i-1}'/x]\varepsilon_{i-2}), \text{ by } Q(i-1)\end{matrix}\right\}$$

$\wedge R(i-1)$, since z' is not free in $\varepsilon_{i-1}' = \phi_{i,i-1}(\varepsilon_i')$, by 2.1.6.

$$= [z/z']\left\{\begin{matrix}[\varepsilon_i'/x]y\\ Ap_{i-1}([\varepsilon_{i-1}'/x]\alpha_{i-3},[\varepsilon_{i-1}'/x]\varepsilon_{i-2}), \text{ by } P(i-1).\end{matrix}\right\}$$

$$= [z/z'][\varepsilon_i'/x]\left\{\begin{matrix}y\\ \alpha_{i-3}(\varepsilon_{i-2})\end{matrix}\right\}.$$

ǂ

3.1.5:LEMMA:-

Suppose $P(i-1)\wedge Q(i-1)\wedge R(i-1)$, for some $i\geq 3$. Let x be not free in $\alpha_{i-2}\in A_{i-2}$. Then,

$$[\varepsilon_i'/x][x/y](\alpha_{i-2}) = [\varepsilon_i'/y](\alpha_{i-2}).$$

Proof:-

$$-[\varepsilon_i'/x][x/y]\left\{\begin{matrix}z\\ \alpha_{i-3}(\varepsilon_{i-2})\end{matrix}\right\}$$

$$=\left\{\begin{matrix}[\varepsilon_i'/x][x/y]z\\ Ap_{i-1}([\varepsilon_{i-1}'/x][x/y]\alpha_{i-3},[\varepsilon_{i-1}'/x][x/y]\varepsilon_{i-2})\end{matrix}\right\}$$

$$=\left\{\begin{matrix}[\varepsilon_i'/y]z, \text{ by 3.1.1.}\\ \left\{\begin{matrix}Ap_{i-1}([\varepsilon_{i-1}'/y]\alpha_{i-3},[\varepsilon_{i-1}'/y]\varepsilon_{i-2}), \text{ if (*), by } Q(i-1)\wedge R(i-1)\\ Ap_{i-1}(\begin{smallmatrix}\top\\ \bot\end{smallmatrix},[\varepsilon_{i-1}'/x][x/y]\varepsilon_{i-2}), \text{ if (**), by } R(i-1).\end{matrix}\right\}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}[\varepsilon_i'/y]\left\{\begin{matrix}z\\ \alpha_{i-3}(\varepsilon_{i-2})\end{matrix}\right\}, \text{ if } \alpha_{i-3}\neq\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}.\\ \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}, \text{ if } \alpha_{i-3}=\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}, = [\varepsilon_i'/y]\alpha_{i-3}(\varepsilon_{i-2}).\end{matrix}\right\}.$$

ǂ

(*) $\alpha_{i-3}\neq\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$ and x is not free in $\alpha_{i-3},\varepsilon_{i-2}$.

(**) $\alpha_{i-3}=\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$.

3.1.6:LEMMA:-

Suppose $P(i-1) \wedge Q(i-1) \wedge R(i-1)$, for some $i \geq 3$. Let a be not free in $\varepsilon_i', \alpha_{i-2}$. Then, a is not free in $[\varepsilon_i'/x]\alpha_{i-2}, \in E_i$.

Proof:-

-If $\alpha_{i-2} = y \in I$, then a is not free in $[\varepsilon_i'/x]y$, by 3.1.1.

-If $\alpha_{i-2} = \alpha_{i-3}(\varepsilon_{i-2})$, then either $\alpha_{i-3} = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, in which case $[\varepsilon_i'/x]\alpha_{i-2} = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$, by 3.1.2, and a is not free in it; or a is not free in $\alpha_{i-3}, \varepsilon_{i-2}$ and then, since a is not free in ε_{i-1}', $= \phi_{i,i-1}(\varepsilon_i')$, by 2.1.6, we have a not free in $[\varepsilon_{i-1}'/x]\alpha_{i-3}$, $[\varepsilon_{i-1}'/x]\varepsilon_{i-2}$, by $Q(i-1) \wedge R(i-1)$.

-∴, a is not free in $Ap_{i-1}([\varepsilon_{i-1}'/x]\alpha_{i-3}, [\varepsilon_{i-1}'/x]\varepsilon_{i-2})$, by $P(i-1)$, $= [\varepsilon_i'/x]\alpha_{i-2}$.

ǂ

3.1.7:LEMMA:-

Suppose $P(i-1) \wedge Q(i-1) \wedge R(i-1)$, for some $i \geq 3$. Let a be not free in ε_i'. Then, a is not free in $[\varepsilon_i'/a]\alpha_{i-2}$.

Proof:-

-If $\alpha_{i-2} = y$, then a is not free in $[\varepsilon_i'/a]y$, by 3.1.1.

-If $\alpha_{i-2} = \alpha_{i-3}(\varepsilon_{i-2})$, then, since a is not free in ε_{i-1}', by 2.1.6, a is not free in $[\varepsilon_{i-1}'/a]\alpha_{i-3}, [\varepsilon_{i-1}'/a]\varepsilon_{i-2}$, by $Q(i-1) \wedge R(i-1)$.

-∴, a is not free in $Ap_{i-1}([\varepsilon_{i-1}'/a]\alpha_{i-3}, [\varepsilon_{i-1}'/a]\varepsilon_{i-2})$, by $P(i-1)$, $= [\varepsilon_i'/a]\alpha_{i-2}$.

ǂ

3.1.8:LEMMA:-

$P(i-1) \wedge Q(i-1) \wedge R(i-1) \Rightarrow R(i)$, $\forall i \geq 3$.

Proof:-

-By lemmas 3.1.2,3.1.3,3.1.4,3.1.5,3.1.6 and 3.1.7.

ǂ

3.1.9:LEMMA:-

Suppose $\mathcal{Q}(i-1)\wedge\mathcal{R}(i)$, for some $i\geq 2$. Then, $[\varepsilon'_i/x]_{E_{i-1}}$ is well-defined and doubly strict.

Proof:-

-First of all, check that the choice of z in defn. 3.0.3 does not matter.

-Let $x,z,z'\in I$. Let $x\neq z,z'$. Let $z\neq z'$.

-Let z,z' be not free in $\varepsilon'_{i-1},\varepsilon_{i-2}$.

-Claim: $\lambda z.[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2} = \lambda z'.[\varepsilon'_{i-1}/x][z'/y]\varepsilon_{i-2}$:-

-Now, z' is not free in $[z/y]\varepsilon_{i-2}$, by 1.2.18.

-$\therefore$, z' is not free in $[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}$, by $\mathcal{Q}(i-1)$.

-Also, $[z'/z][\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}$

$= [\varepsilon'_{i-1}/x][z'/z][z/y]\varepsilon_{i-2}$, by $\mathcal{Q}(i-1)$.

$= [\varepsilon'_{i-1}/x][z'/y]\varepsilon_{i-2}$, by 1.2.18.

-Hence, by lemma 1.1.8, the claim is established.

-Next, suppose $\lambda y.\varepsilon_{i-2} = \lambda w.[w/y]\varepsilon_{i-2}$, where w is not free in ε_{i-2} and $w\neq y$. Then,

$[\varepsilon'_i/x](\lambda w.[w/y]\varepsilon_{i-2}) = \lambda z.([\varepsilon'_{i-1}/x][z/w][w/y]\varepsilon_{i-2})$,

where $z\neq x,w$ and is not free in ε_{i-2} ($\Rightarrow$ z is not free in $[w/y]\varepsilon_{i-2}$, by 1.2.18).

$= \lambda z.([\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2})$, by 1.2.18.

$= [\varepsilon'_i/x](\lambda y.\varepsilon_{i-2})$.

-Finally, check for strictness. Well,

$$[\varepsilon'_i/x]\top_{E_{i-1}} = [\varepsilon'_i/x]\begin{Bmatrix}\lambda y.\top_{E_{i-2}}\\ \top_{A_{i-2}}\end{Bmatrix}$$

$$= \begin{Bmatrix}\lambda z.[\varepsilon'_{i-1}/x][z/y]\top_{E_{i-2}}\\ [\varepsilon'_i/x]\top_{A_{i-2}}\end{Bmatrix} = \begin{Bmatrix}\lambda z.\top_{E_{i-1}}, \text{ by } \mathcal{Q}(i-1).\\ \top_{E_i}, \text{ by } \mathcal{R}(i).\end{Bmatrix} = \top_{E_i}.$$

-Similarly, $[\varepsilon'_i/x]\bot_{E_{i-1}} = \bot_{E_i}$.

⊧

3.1.10:LEMMA:-

Suppose $Q(i-1)\wedge R(i)$, for some $i\geq 2$. Let $z,z'\in I$. Then, $[z/z'][[z/z']\varepsilon'_i/x] = [z/z'][\varepsilon'_i/x]$.

Proof:-

$$-[z/z'][[z/z']\varepsilon'_i/x]\left\{\begin{array}{l}\lambda y.\varepsilon_{i-2}\\ \alpha_{i-2}\end{array}\right.$$

$$= [z/z']\left\{\begin{array}{l}\lambda w.[[z/z']\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}\\ [[z/z']\varepsilon'_i/x]\alpha_{i-2}\end{array}\right. \text{, where } w\neq x,z,z' \text{ and}\left.\vphantom{\begin{array}{l}a\\a\end{array}}\right\}$$

is not free in $\varepsilon'_i,\varepsilon_{i-2}$, since $\phi_{i,i-1}\cdot[z/z']\varepsilon'_i = [z/z']\varepsilon'_{i-1}$, by 1.2.18.

$$= \left\{\begin{array}{l}\lambda w.[z/z'][[z/z']\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}\text{, by 1.2.4(ii).}\\ [z/z'][[z/z']\varepsilon'_i/x]\alpha_{i-2}\end{array}\right\}$$

$$= \left\{\begin{array}{l}\lambda w.[z/z'][\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}\text{, by } Q(i-1).\\ [z/z'][\varepsilon'_i/x]\alpha_{i-2}\text{, by } R(i).\end{array}\right\}$$

$$= [z/z']\left\{\begin{array}{l}\lambda w.[\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}\text{, by 1.2.4(ii).}\\ [\varepsilon'_i/x]\alpha_{i-2}\end{array}\right\}$$

$$= [z/z'][\varepsilon'_i/x]\left\{\begin{array}{l}\lambda y.\varepsilon_{i-2}\\ \alpha_{i-2}\end{array}\right\}.$$

ǂ

3.1.11:LEMMA:-

Suppose $Q(i-1)\wedge R(i)$, for some $i\geq 2$. Let $x\neq z,z'$ and z' be not free in $\varepsilon'_i\in E_i$. Then,

$[\varepsilon'_i/x][z/z'] = [z/z'][\varepsilon'_i/x]$.

Proof:-

-If $z=z'$, trivial by 1.2.18. So, suppose $z\neq z'$.

-Choose $w,w'\in I$ such that $w,w'\neq z',x$ and $w\neq z,w'$ and w,w' are not free in $\varepsilon'_{i-1},\varepsilon_{i-2}$.

-Then,

$$[z/z'][\varepsilon'_i/x]\left\{\begin{array}{l}\lambda y.\varepsilon_{i-2}\\ \alpha_{i-2}\end{array}\right\} = [z/z]\left\{\begin{array}{l}\lambda w.[\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}\\ [\varepsilon'_{i-1}/x]\alpha_{i-2}\end{array}\right\}$$

$$= \left\{ \begin{matrix} \lambda w'.[z/z'][w'/w][\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2} \\ [z/z'][\varepsilon'_i/x]\alpha_{i-2} \end{matrix} \right. \text{, since } w' \text{ is not free} \Bigg\}$$

in $[\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}$, by $Q(i-1)$ and 1.2.18.

$$= \left\{ \begin{matrix} \lambda w'.[w'/w][z/z'][\varepsilon'_{i-1}/x][w/y]\varepsilon_{i-2}, \text{ by } 1.2.18. \\ [z/z'][\varepsilon'_i/x]\alpha_{i-2} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} \lambda w'.[\varepsilon'_{i-1}/x][w'/w][z/z'][w/y]\varepsilon_{i-2}, \text{ by } Q(i-1). \\ [\varepsilon'_i/x][z/z']\alpha_{i-2}, \text{ by } R(i). \end{matrix} \right\}$$

$$= [\varepsilon'_i/x] \left\{ \begin{matrix} \lambda w.[z/z'][w/y]\varepsilon_{i-2} \\ [z/z']\alpha_{i-2} \end{matrix} \right. \text{, since } w' \text{ is not free in } [z/z'][w/y]\varepsilon_{i-2}. \Bigg\}$$

$$= [\varepsilon'_i/x][z/z'] \left\{ \begin{matrix} \lambda y.\varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix} \right\}.$$

ǂ

3.1.12:LEMMA:-

Suppose $Q(i-1)\wedge R(i)$, for some $i\geq 2$. Let x be not free in ε_{i-1}. Then,

$$[\varepsilon'_i/x][x/y]\varepsilon_{i-1} = [\varepsilon'_i/y]\varepsilon_{i-1}.$$

Proof:-

-Choose w,w′ I such that $w,w'\neq x,y$ and w,w′ are not free in ε_{i-1} and w′ is not free in ε'_i and $w\neq w'$. (N.B. we can always make such choices since there are only finitely many free variables in $\varepsilon_{i-1}, \varepsilon'_i$, by 1.2.18, and I is infinite.)

-Then, w,w′ are not free in $\phi_{i-1,i-2}(\varepsilon_{i-1}), = \varepsilon_{i-2}$, and $\phi_{i,i-1}(\varepsilon'_i), = \varepsilon'_{i-1}$, by 2.1.6.

-∴, w is not free in $[x/y][w/z]\varepsilon_{i-2} \in E_{i-2}$, by 1.2.18.

$$\text{-So, } [\varepsilon'_i/x][x/y] \left\{ \begin{matrix} \lambda z.\varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix} \right\} = [\varepsilon'_i/x] \left\{ \begin{matrix} \lambda w.[x/y][w/z]\varepsilon_{i-2} \\ [x/y]\alpha_{i-2} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} \lambda w'.[\varepsilon'_{i-1}/x][w'/w][x/y][w/z]\varepsilon_{i-2} \\ [\varepsilon'_i/x][x/y]\alpha_{i-2} \end{matrix} \right\}$$

$$= \left\{ \begin{array}{l} \lambda w'.[\varepsilon'_{i-1}/x][x/y][w'/z]\varepsilon_{i-2}, \text{ by 1.2.14 (twice).} \\ [\varepsilon'_i/x][x/y]\alpha_{i-2} \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} \lambda w'.[\varepsilon'_{i-1}/y][w'/z]\varepsilon_{i-2}, \text{ by } Q(i-1), & \text{since x is not free in} \\ [\varepsilon'_i/y]\alpha_{i-2}, \text{ by } R(i). & [w'/z]\varepsilon_{i-2}, \text{ by 2.1.6 and} \end{array} \right\}$$

$$\text{1.2.14.}$$

$$= [\varepsilon'_i/y] \left\{ \begin{array}{l} \lambda z.\varepsilon_{i-2} \\ \alpha_{i-2} \end{array} \right\}.$$

ǂ

3.1.13:LEMMA:-

Suppose $Q(i-1) \wedge R(i)$, for some $i \geq 2$. Let a be not free in $\varepsilon'_i, \varepsilon_{i-1}$. Then, a is not free in $[\varepsilon'_i/x]\varepsilon_{i-1}$.

Proof:-

-Suppose, w.l.o.g., $\varepsilon_{i-1} = \lambda y.\varepsilon_{i-2}$, with $y \neq a$ - by 1.1.8.

-Then, a is not free in ε_{i-2}.

-But, a is not free in ε'_{i-1}, by 2.1.6.

-Choose $z \in I$ such that $z \neq x,a$ and z is not free in $\varepsilon'_{i-1}, \varepsilon_{i-2}$.

-Then, a is not free in $[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}$, by 1.2.14 and $Q(i-1)$.

-∴, a is not free in $\lambda z.[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}, = [\varepsilon'_i/x]\varepsilon_{i-1}$.

-On the other hand, suppose $\varepsilon_{i-1} = \alpha_{i-2}$.

-Then, we have the result by $R(i)$.

ǂ

3.1.14:LEMMA:-

Suppose $Q(i-1) \wedge R(i)$, for some $i \geq 2$. Let a be not free in ε'_i. Then, a is not free in $[\varepsilon'_i/a]\varepsilon_{i-1}$.

Proof:-

-Suppose $\varepsilon_{i-1} = \lambda y.\varepsilon_{i-2}$. Now, a is not free in ε'_{i-1}, by 2.1.6.

-Choose $z \neq a$ and not free in $\varepsilon'_{i-1}, \varepsilon_{i-2}$.

-Then, a is not free in $[\varepsilon'_{i-1}/a][z/y]\varepsilon_{i-2}$, by $Q(i-1)$.

-∴, a is not free in $\lambda z.[\varepsilon'_{i-1}/a][z/y]\varepsilon_{i-2}, = [\varepsilon'_i/a](\lambda y.\varepsilon_{i-2})$.

-If $\varepsilon_{i-1} = \alpha_{i-2}$, we have the result by $R(i)$.

ǂ

3.1.15:LEMMA:-

$\mathcal{Q}(i-1) \wedge \mathcal{R}(i) \Rightarrow \mathcal{Q}(i)$, $\forall i \geq 2$.

Proof:-

-By lemmas 3.1.9, 3.1.10, 3.1.11, 3.1.12, 3.1.13 and 3.1.14.

ǂ

3.1.16:LEMMA:-

Suppose $\mathcal{Q}(i)$, for some $i \geq 1$. Then, Ap_i is well-defined and,
$$Ap_i\left(\left\{\begin{matrix} {}^{\top}E_i \\ {}^{\perp}E_i \end{matrix}\right\}, \varepsilon_i'\right) = \left\{\begin{matrix} {}^{\top}E_{i+1} \\ {}^{\perp}E_{i+1} \end{matrix}\right\}.$$

Proof:-

-Clearly, Ap_i is well-defined on its second argument.

-Suppose $\lambda x.\varepsilon_{i-1} = \lambda y.[y/x]\varepsilon_{i-1}$, where $y \neq x$ and y is not free in ε_{i-1}.

-Then,
$$\begin{aligned} Ap_i(\lambda y.[y/x]\varepsilon_{i-1}, \varepsilon_i') &= \phi_{i,i+1}([\varepsilon_i'/y][y/x]\varepsilon_{i-1}) \\ &= \phi_{i,i+1}([\varepsilon_i'/x]\varepsilon_{i-1}), \text{ by } \mathcal{Q}(i). \\ &= Ap_i(\lambda x.\varepsilon_{i-1}, \varepsilon_i'). \end{aligned}$$

-Also,
$$\begin{aligned} Ap_i\left(\left\{\begin{matrix} \lambda x.{}^{\top}E_{i-1} \\ {}^{\top}A_{i-1} \end{matrix}\right\}, \varepsilon_i'\right) &= \left\{\begin{matrix} \phi_{i,i+1}([\varepsilon_i'/x]{}^{\top}E_{i-1}) \\ {}^{\top}A_{i-1}(\varepsilon_i') \end{matrix}\right\} \\ &= \left\{\begin{matrix} \phi_{i,i+1}({}^{\top}E_i), \text{ by } \mathcal{Q}(i). \\ {}^{\top}E_{i+1} \end{matrix}\right\} \\ &= {}^{\top}E_{i+1}, \text{ by 2.1.6.} \end{aligned}$$

-Similarly, $Ap_i({}^{\perp}E_i, \varepsilon_i') = {}^{\perp}E_{i+1}$.

ǂ

3.1.17:LEMMA:-

Suppose $\mathcal{Q}(i)$, for some $i \geq 1$. Let $z, z' \in I$. Then,
$$[z/z']Ap_i(\varepsilon_i, \varepsilon_i') = Ap_i([z/z]\varepsilon_i, [z/z']\varepsilon_i').$$

Proof:-

-Suppose $z \neq z'$. Otherwise trivial, by 1.2.18.

-Then, $Ap_i([z/z']\left\{\begin{matrix}\lambda x.\varepsilon_{i-1} \\ \alpha_{i-1}\end{matrix}\right. ,[z/z']\varepsilon_i')\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\}$

$= Ap_i(\left\{\begin{matrix}\lambda w.[z/z'][w/x]\varepsilon_{i-1} \\ [z/z']\alpha_{i-1}\end{matrix}\right\},[z/z']\varepsilon_i')$, where $w \neq z,z'$ and is not free in ε_{i-1}.

$= \left\{\begin{matrix}\phi_{i,i+1}([[z/z']\varepsilon_i'/w][z/z'][w/x]\varepsilon_{i-1}) \\ [z/z']\alpha_{i-1}([z/z']\varepsilon_i')\end{matrix}\right\}$

$= \left\{\begin{matrix}\phi_{i,i+1}([z/z'][[z/z']\varepsilon_i'/w][w/x]\varepsilon_{i-1}) \\ [z/z']\alpha_{i-1}(\varepsilon_i')\end{matrix}\right\}$, by $\mathcal{Q}(i)$, since $w \neq z,z'$ and z' is not free in $[z/z']\varepsilon_i'$, by 1.2.18.

$= \left\{\begin{matrix}\phi_{i,i+1}([z/z'][\varepsilon_i'/w][w/x]\varepsilon_{i-1}) \\ [z/z']\alpha_{i-1}(\varepsilon_i')\end{matrix}\right\}$, by $\mathcal{Q}(i)$.

$= \left\{\begin{matrix}\phi_{i,i+1}([z/z'][\varepsilon_i'/x]\varepsilon_{i-1}) \\ [z/z']\alpha_{i-1}(\varepsilon_i')\end{matrix}\right\}$, by $\mathcal{Q}(i)$.

$= \left\{\begin{matrix}[z/z']\cdot\phi_{i,i+1}([\varepsilon_i'/x]\varepsilon_{i-1}), \text{ by 2.1.6.} \\ [z/z']\alpha_{i-1}(\varepsilon_i')\end{matrix}\right\}$

$= [z/z']\cdot Ap_i(\left\{\begin{matrix}\lambda x.\varepsilon_{i-1} \\ \alpha_{i-1}\end{matrix}\right\},\varepsilon_i')$.

ǂ

<u>3.1.18:LEMMA:-</u>

Suppose $\mathcal{Q}(i)$, for some $i \geq 1$. Let a be not free in $\varepsilon_i,\varepsilon_i'$. Then, a is not free in $Ap_i(\varepsilon_i,\varepsilon_i')$.

<u>Proof:-</u>

-Suppose $\varepsilon_i = \lambda x.\varepsilon_{i-1}$. Assume, w.l.o.g., $x \neq a$, by 1.1.8.

-Then, a is not free in ε_{i-1}.

-∴, a is not free in $[\varepsilon_i'/x]\varepsilon_{i-1}$, by $\mathcal{Q}(i)$.

- , a is not free in $\phi_{i,i+1}([\varepsilon_i'/x]\varepsilon_{i-1})$, by 2.1.6,

$= Ap_i(\lambda x.\varepsilon_{i-1},\varepsilon_i')$.

-If $\varepsilon_i = \alpha_{i-1}$, then the result is trivial.

ǂ

3.1.19:LEMMA:-

$Q(i) \Rightarrow P(i)$, $\forall i \geq 1$.

Proof:-

-By lemmas 3.1.16,3.1.17 and 3.1.18.

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3.1.20:THEOREM:-

$(\forall i \geq 1)(P(i) \wedge Q(i))$ and $(\forall i \geq 2) R(i)$.

Proof:-

-We have $Q(1)$ and $R(2)$, since $Q(1) \equiv S(1)$ and $R(2) \equiv S(2)$.

-∴, by lemmas 3.1.1,3.1.8,3.1.15,3.1.19 and 3.0.5, we have the result.

ǂ

3.2:Continuity of Ap_i and $[\varepsilon_i/x]$:-

3.2.0:DEF:-

$$\text{Let } \left\{\begin{matrix} S \\ Q \\ R \end{matrix}\right\}(i) \equiv ([\varepsilon_i/x] \text{ is monotone})$$

$$\& \ (\varepsilon_i \sqsubseteq \eta_i \Rightarrow [\varepsilon_i/x] \sqsubseteq [\eta_i/x]).$$

Let $P(i) \equiv (Ap_i$ is monotone).

3.2.1:REMARK:-

We have redefined the S,P,Q and R of 3.1.0, whose scope is for section 3.1 only. The present definition is valid for section 3.2 only. When we refer to these sections, we must be careful to make sure what S,P,Q and R represent.

3.2.2:LEMMA:-

$S(i)$, $\forall i \geq 0$.

Proof:-

-By 3.0.2.

3.2.3:LEMMA:-

$\mathcal{P}(i-1)\wedge\mathcal{Q}(i-1)\wedge\mathcal{R}(i-1)\Rightarrow\mathcal{R}(i)$, $\forall i\geq 3$.

Proof:-

-Suppose $\mathcal{P}(i-1)\wedge\mathcal{Q}(i-1)\wedge\mathcal{R}(i-1)$, for some $i\geq 3$.

-Let $\alpha_{i-2} \sqsubseteq \eta_{i-2} \in A_{i-2}$. Claim: $[\varepsilon_i'/x]\alpha_{i-2} \sqsubseteq [\varepsilon_i'/x]\eta_{i-2}$:-

-If $\alpha_{i-2} = \perp$ or $\eta_{i-2} = \top$, then trivial, since $[\varepsilon_i'/x]$ is doubly strict, by 3.1.20.

-∴, suppose $\alpha_{i-2} \neq \perp$ and $\eta_{i-2} \neq \top$.

-∴, α_{i-2} and η_{i-2} are in the same sub-lattice of A_{i-2}, that is either I′ or $A_{i-3}(E_{i-2})$.

-If $\alpha_{i-2} = y$, then $\eta_{i-2} = y$ also, and we are home.

-Suppose $\alpha_{i-2} = \alpha_{i-3}(\varepsilon_{i-2})$ and $\eta_{i-2} = \eta_{i-3}(\delta_{i-2})$.

-Then, $\alpha_{i-3} \sqsubseteq \eta_{i-3}$ and $\varepsilon_{i-2} \sqsubseteq \delta_{i-2}$, since $\alpha_{i-2} \neq \perp$ and $\eta_{i-2} \neq \top$. Thus, $[\varepsilon_i'/x]\alpha_{i-2} = [\varepsilon_i'/x]\alpha_{i-3}(\varepsilon_{i-2})$

$= Ap_{i-1}([\varepsilon_{i-1}'/x]\alpha_{i-3},[\varepsilon_{i-1}'/x]\varepsilon_{i-2})$

$\sqsubseteq Ap_{i-1}([\varepsilon_{i-1}'/x]\eta_{i-3},[\varepsilon_{i-1}'/x]\delta_{i-2})$, by $\mathcal{P}(i-1)\wedge\mathcal{Q}(i-1)\wedge\mathcal{R}(i-1)$.

$= [\varepsilon_i'/x]\eta_{i-3}(\delta_{i-2})$

$= [\varepsilon_i'/x]\eta_{i-2}$, and the claim is established.

-∴, $[\varepsilon_i'/x]$ is monotonic on A_{i-2}.

-Suppose $\varepsilon_i' \sqsubseteq \eta_i'$. Then,

$$[\varepsilon_i'/x]\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\} = \left\{\begin{matrix} [\varepsilon_i'/x]y \\ Ap_{i-1}([\varepsilon_{i-1}'/x]\alpha_{i-3},[\varepsilon_{i-1}'/x]\varepsilon_{i-2}) \end{matrix}\right\}$$

$$\sqsubseteq \left\{\begin{matrix} [\eta_i'/x]y\text{, by 3.0.2.} \\ Ap_{i-1}([\eta_{i-1}'/x]\alpha_{i-3},[\eta_{i-1}'/x]\varepsilon_{i-2})\text{, by } \mathcal{P}(i-1)\wedge\mathcal{Q}(i-1)\wedge\mathcal{R}(i-1) \text{ and} \end{matrix}\right\}$$

since $\phi_{i,i-1}$ is monotone, by 2.1.6.

$$= [\eta_i'/x]\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\}.$$

‡

3.2.4:LEMMA:-

$\mathcal{Q}(i-1)\wedge\mathcal{R}(i)\Rightarrow\mathcal{Q}(i)$, $\forall i\geq 2$.

Proof:-

-Suppose $\mathcal{Q}(i-1)\wedge\mathcal{R}(i)$, for some $i\geq 2$.

-Suppose $\varepsilon_{i-1}\sqsubseteq\eta_{i-1}\in A_{i-1}$.

-Claim: $[\varepsilon'_i/x]\varepsilon_{i-1}\sqsubseteq[\varepsilon'_i/x]\eta_{i-1}$:-

-As in the last proof,we may assume that $\varepsilon_{i-1}\neq\bot$, $\eta_{i-1}\neq\top$ and they are either both in $\lambda I.E_{i-2}$ or both in A_{i-2}.

-Suppose $\varepsilon_{i-1}=\lambda y.\varepsilon_{i-2}$ and $\eta_{i-1}=\lambda w.\delta_{i-2}$.

-Choose $z\in I$ such that $z\neq x$ and is not free in ε'_{i-1}, ε_{i-2} and δ_{i-2}. Then,

$[\varepsilon'_i/x]\varepsilon_{i-1}=[\varepsilon'_i/x](\lambda y.\varepsilon_{i-2})$

$=\lambda z.[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}$

$\sqsubseteq\lambda z.[\varepsilon'_{i-1}/x][z/w]\delta_{i-2}$, by $\mathcal{Q}(i-1)$ and 1.1.12, since $[z/y]\varepsilon_{i-2}\sqsubseteq[z/w]\delta_{i-2}$, by defn. 1.1.9.

$=[\varepsilon'_i/x](\lambda w.\delta_{i-2})$

$=[\varepsilon'_i/x]\eta_{i-1}$.

-If $\varepsilon_{i-1}=\alpha_{i-2}$ and $\eta_{i-1}=\beta_{i-2}\in A_{i-2}$, we have the claim by $\mathcal{R}(i)$. Hence, the claim is established.

-$\therefore$, $[\varepsilon'_i/x]$ is monotonic on E_{i-1}.

-Suppose $\varepsilon'_i\sqsubseteq\eta'_i$. Then,

$$[\varepsilon'_i/x]\left\{\begin{array}{l}\lambda y.\varepsilon_{i-2}\\ \alpha_{i-2}\end{array}\right.=\left\{\begin{array}{l}\lambda z.[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2}\\ {}[\varepsilon'_i/x]\alpha_{i-2}\end{array}\right\},$$ where $z\neq x$ and is not free in ε'_{i-1},η'_{i-1} and ε_{i-2}.

$$\sqsubseteq\left\{\begin{array}{l}\lambda z.[\eta'_{i-1}/x][z/y]\varepsilon_{i-2},\text{ by }\mathcal{Q}(i-1)\\ {}[\eta'_i/x]\alpha_{i-2},\text{ by }\mathcal{R}(i)\end{array}\right\},$$ since $\varepsilon'_{i-1}\sqsubseteq\eta'_{i-1}$, by 2.1.6, and by 1.1.12.

$$=[\eta'_i/x]\left\{\begin{array}{l}\lambda y.\varepsilon_{i-2}\\ \alpha_{i-2}\end{array}\right\}.$$

ǂ

3.2.5:LEMMA:-

$Q(i) \Rightarrow P(i)$, $\forall i \geq 1$.

Proof:-

-Suppose $Q(i)$, for some $i \geq 1$.

-Let $(\varepsilon_i, \varepsilon'_i) \sqsubseteq (\eta_i, \eta'_i) \in E_i \times E_i$.

-We may assume that $\varepsilon_i \neq \bot$ and $\eta_i \neq \top$, as before, since $Ap_i(\left\{\begin{matrix}\top\\ \bot\end{matrix}\right., \varepsilon'_i) = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right.$, by 3.1.20, and so monotonicity would be clear.

-We are trying to establish $Ap_i(\varepsilon_i, \varepsilon'_i) \sqsubseteq Ap_i(\eta_i, \eta'_i)$.

-Now, ε_i and η_i are in the same sublattice of E_i.

-Suppose $\varepsilon_i = \alpha_{i-1} \sqsubseteq \beta_{i-1} = \eta_i$. Then,

$$\begin{aligned} Ap_i(\alpha_{i-1}, \varepsilon'_i) &= \alpha_{i-1}(\varepsilon'_i) \\ &\sqsubseteq \beta_{i-1}(\varepsilon'_i) = Ap_i(\beta_{i-1}, \varepsilon'_i). \end{aligned}$$

-Suppose $\varepsilon_i = \lambda x.\varepsilon_{i-1}$ and $\eta_i = \lambda y.\delta_{i-1}$.

-Let z be not free in $\varepsilon_{i-1}, \delta_{i-1}$. Then,

$$\begin{aligned} Ap_i(\lambda x.\varepsilon_{i-1}, \varepsilon'_i) &= \phi_{i,i+1}([\varepsilon'_i/x]\varepsilon_{i-1}) \\ &= \phi_{i,i+1}([\varepsilon'_i/z][z/x]\varepsilon_{i-1}), \text{ by 3.1.20.} \\ &\sqsubseteq \phi_{i,i+1}([\varepsilon'_i/z][z/y]\delta_{i-1}), \text{ by } Q(i) \text{ and 2.1.6.} \\ &= \phi_{i,i+1}([\varepsilon'_i/y]\delta_{i-1}), \text{ by 3.1.20.} \\ &= Ap_i(\lambda y.\delta_{i-1}, \varepsilon'_i). \end{aligned}$$

-∴, $Ap_i(\varepsilon_i, \varepsilon'_i) \sqsubseteq Ap_i(\eta_i, \eta'_i)$

$$\begin{aligned} &= Ap_i(\left\{\begin{matrix}\lambda y.\delta_{i-1}\\ \beta_{i-1}\end{matrix}\right\}, \varepsilon'_i) \\ &= \left\{\begin{matrix}\phi_{i,i+1}([\varepsilon'_i/y]\delta_{i-1})\\ \beta_{i-1}(\varepsilon'_i)\end{matrix}\right\} \\ &\sqsubseteq \left\{\begin{matrix}\phi_{i,i+1}([\eta'_i/y]\delta_{i-1}), \text{ by } Q(i) \text{ and 2.1.6.}\\ \beta_{i-1}(\eta'_i)\end{matrix}\right\} \\ &= Ap_i(\left\{\begin{matrix}\lambda y.\delta_{i-1}\\ \beta_{i-1}\end{matrix}\right\}, \eta'_i) = Ap_i(\eta_i, \eta'_i). \end{aligned}$$

ǂ

3.2.6:THEOREM:-

$(\forall i\geq 1)((P(i)_{\wedge}Q(i))$ and $(\forall i\geq 2)R(i)$.

Proof:-

-$Q(1)\equiv S(1)$ and $R(2)\equiv S(2)$.

-∴, $Q(1)_{\wedge}R(2)$, by 3.2.2.

-∴, by lemmas 3.2.2, 3.2.3, 3.2.4, 3.2.5 and 3.0.5, we are home.

ǂ

3.2.7:COR:-

Ap_i is continuous on $E_i \times E_i$ and $[\varepsilon_i/x]$ is continuous on I', E_{i-1} and A_{i-2}.

Proof:-

-By 1.2.18, I', E_{i-1}, A_{i-2} and E_i have finite depth.

-∴, $E_i \times E_i$ has finite depth.

-∴, the functions are continuous, by 3.2.6 and 0.6.13.

ǂ

3.3:More Properties of Ap_i and $[\varepsilon_i/x]$:-

3.3.0:LEMMA:-

Let $\begin{Bmatrix} S \\ Q \\ R \end{Bmatrix}(i) \equiv$ (x is not free in $\begin{Bmatrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix} \Rightarrow$

$[\varepsilon'_i/x]\begin{Bmatrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix} = \begin{Bmatrix} z \\ \phi_{i-1,i}(\varepsilon_{i-1}) \\ \theta_{i-2,i-1}(\alpha_{i-2}) \end{Bmatrix}$).

Then, $(\forall i\geq 0)S(i)$ and $(\forall i\geq 1)Q(i)$ and $(\forall i\geq 2)R(i)$.

Proof:-

-Claim: $S(i)$, $\forall i\geq 0$:-

-x is not free in $z \Rightarrow x \not\equiv z \Rightarrow [\varepsilon_i/x]z = z$, by 3.0.1.

-<u>Claim:</u> $\mathcal{Q}(i-1)\wedge\mathcal{R}(i-1)\Rightarrow\mathcal{R}(i)$, $\forall i\geq 3$:-

-Let x be not free in α_{i-2}.

-Assume $\alpha_{i-2}\neq\top,\bot$, since otherwise result is trivial, by 3.1.20 and 2.1.6. Then,

$$[\varepsilon_i'/x]\alpha_{i-2} = [\varepsilon_i'/x]\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} [\varepsilon_i'/x]y \\ Ap_i([\varepsilon_{i-1}'/x]\alpha_{i-3},[\varepsilon_{i-1}'/x]\varepsilon_{i-2} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} y, \text{ by (i), which we have above.} \\ Ap_i(\theta_{i-3,i-2}(\alpha_{i-3}),\phi_{i-2,i-1}(\varepsilon_{i-2})) \end{matrix}\right\}, \text{ by } \mathcal{Q}(i-1)$$

$\wedge\mathcal{R}(i-1)$, since x is not free in $\alpha_{i-3}, \varepsilon_{i-2}$.

$$= \left\{\begin{matrix} y \\ \theta_{i-3,i-2}(\alpha_{i-3})(\phi_{i-2,i-1}(\varepsilon_{i-2})) \end{matrix}\right\}$$

$$= \theta_{i-2,i-1}\left(\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\}\right) = \theta_{i-2,i-1}(\alpha_{i-2}).$$

-<u>Claim:</u> $\mathcal{Q}(i-1)\wedge\mathcal{R}(i)\Rightarrow\mathcal{Q}(i)$, $\forall i\geq 2$:-

-Let x be not free in ε_{i-1}. Then,

$$[\varepsilon'/x]\varepsilon_{i-1} = [\varepsilon_i'/x]\left\{\begin{matrix} \lambda y.\varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \lambda z.[\varepsilon_{i-1}'/x][z/y]\varepsilon_{i-2} \\ [\varepsilon_i'/x]\alpha_{i-2} \end{matrix}\right\}, \text{ where } z\neq x \text{ and is not free in } \varepsilon_{i-1}', \varepsilon_{i-2}.$$

$$= \left\{\begin{matrix} \lambda z.\phi_{i-2,i-1}([z/y]\varepsilon_{i-2}) \\ \theta_{i-2,i-1}(\alpha_{i-2}) \end{matrix}\right\}, \text{ by } \mathcal{Q}(i-1)\wedge\mathcal{R}(i),$$

since x is not free in $[z/y]\varepsilon_{i-2}$, by 1.2.18.

$$= \phi_{i-1,i}\left(\left\{\begin{matrix} \lambda z.[z/y]\varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix}\right\}\right)$$

$$= \phi_{i-1,i}\left(\left\{\begin{matrix} \lambda y.\varepsilon_{i-2}, \text{ by 1.1.8.} \\ \alpha_{i-2} \end{matrix}\right\}\right) = \phi_{i-1,i}(\varepsilon_{i-1}).$$

-Let $\mathcal{P}(i)\equiv$ <u>true</u>. Then, since $\mathcal{Q}(1)\equiv\mathcal{S}(1)$ and $\mathcal{R}(2)\equiv\mathcal{S}(2)$, we have the result by 3.0.5.

$\perp$

3.3.1:COR:-

Let x be not free in ε_{i-1}. Then, $\forall i \geq 1$,

$$\mathrm{Ap}_i(\lambda x.\varepsilon_{i-1}, \varepsilon_i') = \phi_{i-1,i+1}(\varepsilon_{i-1}).$$

Proof:-

$$-\mathrm{Ap}_i(\lambda x.\varepsilon_{i-1}, \varepsilon_i') = \phi_{i,i+1}([\varepsilon_i'/x]\varepsilon_{i-1})$$

$$= \phi_{i,i+1}\cdot\phi_{i-1,i}(\varepsilon_{i-1}), \text{ by 3.3.0.}$$

$$= \phi_{i-1,i+1}(\varepsilon_{i-1}).$$

ǂ

3.3.2:COR:-

(i) $[\varepsilon_i'/x](\lambda x.\varepsilon_{i-2}) = \phi_{i-1,i}(\lambda x.\varepsilon_{i-2})$, $\forall i \geq 1$.

(ii) $(y \neq x)\wedge(y$ is not free in $\varepsilon_i') \Rightarrow$

$$([\varepsilon_i'/x](\lambda y.\varepsilon_{i-2}) = \lambda y.[\varepsilon_{i-1}'/x]\varepsilon_{i-2}).$$

Proof:-

(i) -By defn. 1.2.9, x is not free in $\lambda x.\varepsilon_{i-2}$.

-∴, by 3.3.0, we have the result.

(ii) -Choose $z \neq x$ and not free in $\varepsilon_{i-1}', \varepsilon_{i-2}$.

-Then, $[\varepsilon_i'/x](\lambda y.\varepsilon_{i-2}) = \lambda z.[\varepsilon_{i-1}'/x][z/y]\varepsilon_{i-2}$

$= \lambda z.[z/y][\varepsilon_{i-1}'/x]\varepsilon_{i-2}$, by 3.1.20.

$= y.[\varepsilon_{i-1}'/x]\varepsilon_{i-2}$, since z is not free in $[\varepsilon_{i-1}'/x]\varepsilon_{i-2}$, by 3.1.20.

ǂ

3.3.3:REMARK:-

Theorems 3.1.20 and 3.2.7 gave us the basic properties of Ap_i and $[\varepsilon_i/x]$. Results 3.3.0, 3.3.1 and 3.3.2 show properties which are useful for manipulating these operators.

The following lemma relates the operator $[\varepsilon_i/x]$ with the old variable swap operator $[y/x]$. Be careful not to confuse $[y/x] \in [E_{i-1} \rightarrow E_i]$, an instance of $[\varepsilon_i/x]$ when $\varepsilon_i = y$, with $[y/x] \in [E_i \rightarrow E_i]$.

3.3.4:LEMMA:-

$$\text{Let } \left\{\begin{matrix} S \\ Q \\ R \end{matrix}\right\}(i) \equiv \left([w/x]_{\left\{\begin{matrix} I \\ E_{i-1} \\ A_{i-2} \end{matrix}\right\}} \left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}\right.$$

$$= \left\{\begin{matrix} [w/x]z \quad \text{(of defn. 1.0.2.)} \\ \phi_{i-1,i}([w/x]\varepsilon_{i-1}) \quad \text{(of defn. 1.2.1.)} \quad). \\ \theta_{i-2,i-1}([w/x]\alpha_{i-2}) \quad \text{(of defn. 1.2.1.)} \end{matrix}\right\}$$

Then, $(\forall i \geq 0) S(i)$ and $(\forall i \geq 1) Q(i)$ and $(\forall \geq 2 \} R(i)$.

Proof:-

-Clearly, $S(i)$, $\forall i \geq 0$, by defn. 3.0.1.

-Claim: $Q(i-1) \wedge R(i-1) \Rightarrow R(i)$, $\forall i \geq 3$:-

$$-[w/x]_{A_{i-2}} \alpha_{i-2} = [w/x]_{A_{i-2}} \left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} [w/x]_I y \\ Ap_{i-1}([w/x]_{A_{i-3}} \alpha_{i-3}, [w/x]_{E_{i-2}} \varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} [w/x]y, \text{ by } S(i). \\ Ap_{i-1}(\theta_{i-3,i-2} \circ [w/x]\alpha_{i-3}, \phi_{i-2,i-1} \circ [w/x]\varepsilon_{i-2}), \text{ by } Q(i-1) \wedge R(i-1). \end{matrix}\right\}$$

$$= \left\{\begin{matrix} [w/x]y \\ \theta_{i-3,i-2} \circ [w/x]\alpha_{i-3}(\phi_{i-2,i-1} \circ [w/x]\varepsilon_{i-2}) \end{matrix}\right\}$$

$$= \theta_{i-2,i-1}\left(\left\{\begin{matrix} [w/x]y \\ [w/x]\alpha_{i-3}([w/x]\varepsilon_{i-2}) \end{matrix}\right\}\right)$$

$$= \theta_{i-2,i-1}\left([w/x]\left\{\begin{matrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{matrix}\right\}\right) = \theta_{i-2,i-1}([w/x]\alpha_{i-2}).$$

-Claim: $Q(i-1) \wedge R(i) \Rightarrow Q(i)$, $\forall i \geq 2$:-

$$-[w/x]_{E_{i-1}}(\varepsilon_{i-1}) = [w/x]_{E_{i-1}} \left\{\begin{matrix} \lambda y . \varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \lambda z . [w/x]_{E_{i-2}} [z/y] \varepsilon_{i-2} \\ [w/x]_{A_{i-2}} (\alpha_{i-2}) \end{matrix}\right\}, \text{ where } z \neq x \text{ and } z \text{ is not free in } w, \varepsilon_{i-2}.$$

$$= \left\{\begin{matrix} \lambda z . \phi_{i-2,i-1}([w/x][z/y]\varepsilon_{i-2}), \text{ by } Q(i-1). \\ \theta_{i-2,i-1}([w/x]\alpha_{i-2}), \text{ by } R(i). \end{matrix}\right\}$$

$$= \phi_{i-1,i}(\left\{\begin{matrix}\lambda z.[w/x][z/y]\,\varepsilon_{i-2}\\ [w/x]\alpha_{i-2}\end{matrix}\right\})$$

$$= \phi_{i-1,i}([w/x]\left\{\begin{matrix}\lambda y.\,\varepsilon_{i-2}\\ \alpha_{i-2}\end{matrix}\right\}) = \phi_{i-1,i}([w/x]\,\varepsilon_{i-1}).$$

-But, $R(2) \equiv S(2)$ and $Q(1) \equiv S(1)$.

-∴, we have the result, by 3.0.5, letting $p(i) \equiv$ <u>true</u>.

$\ddagger$

3.3.5:LEMMA:-

$$\text{Let } \left\{\begin{matrix}S\\ Q\\ R\end{matrix}\right\}(i) \equiv (\phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\left\{\begin{matrix}z\\ \varepsilon_i\\ \alpha_{i-1}\end{matrix}\right\} \equiv [\varepsilon'_i/x]\left\{\begin{matrix}z\\ \varepsilon_{i-1}\\ \alpha_{i-2}\end{matrix}\right\}).$$

Let $P(i) \equiv (\phi_{i+2,i+1}\circ Ap_{i+1}(\varepsilon_{i+1},\varepsilon'_{i+1}) \equiv Ap_i(\varepsilon_i,\varepsilon'_i))$.

Then, $(\forall i \geq 0)\,S(i)$ and $(\forall \geq 1)(P(i) \wedge Q(i))$ and $(\forall i \geq 2)\,R(i)$.

Proof:-

-Clearly, $\phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]z = [\varepsilon'_i/x]z$, $\forall i \geq 0$.

-∴, $S(i)$, $\forall i \geq 0$.

-Claim: $R(2)$:-

-If $\alpha_1 = z$, we have result by $S(2)$.

-If $\alpha_1 = \alpha'_0(\varepsilon'_1)$, then $\alpha_0 = \perp$ and result is trivial, since $[\varepsilon'_2/x]\perp = \perp$, by 3.1.20.

-Claim: $Q(1)$:-

-If $\varepsilon_1 = z \in E_0', = I'$ then we have the result by $S(1)$.

-If $\varepsilon_1 = \lambda y.\,\varepsilon''_0$, then $\varepsilon_0 = \perp$ and, as above, we are home.

-Claim: $P(i-1) \wedge Q(i-1) \wedge R(i-1) \Rightarrow R(i)$, $\forall i \geq 3$:-

$$-\phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\alpha'_{i-1} = \phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\left\{\begin{matrix}y\\ \alpha_{i-2}(\varepsilon_{i-1})\end{matrix}\right\}$$

$$= \phi_{i+1,i}(\left\{\begin{matrix}[\varepsilon'_{i+1}/x]y\\ Ap_i([\varepsilon'_i/x]\alpha_{i-2},[\varepsilon'_i/x]\varepsilon_{i-1})\end{matrix}\right\})$$

$$\equiv \left\{\begin{matrix}[\varepsilon'_i/x]y, \text{ by } S(i).\\ Ap_{i-1}(\phi_{i,i-1}\circ[\varepsilon'_i/x]\alpha_{i-2},\phi_{i,i-1}\circ[\varepsilon'_i/x]\varepsilon_{i-1}), \text{ by } P(i-1).\end{matrix}\right\}$$

$$\sqsupseteq \left\{\begin{matrix}[\varepsilon'_i/x]y \\ Ap_{i-1}([\varepsilon'_{i-1}/x]\alpha_{i-3},[\varepsilon'_{i-1}/x]\varepsilon_{i-2}),\ \text{by } Q(i-1)\wedge R(i-1) \text{ and } 3.2.6.\end{matrix}\right\}$$

$$= [\varepsilon'_i/x]\left\{\begin{matrix}y \\ \alpha_{i-3}(\varepsilon_{i-2})\end{matrix}\right\} = [\varepsilon'_i/x]\alpha'_{i-2}.$$

-<u>Claim:</u> $Q(i-1)\wedge R(i)\Rightarrow Q(i),\ \forall i\geq 2$:-

$$-\phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\varepsilon''_i = \phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\left\{\begin{matrix}\lambda y.\varepsilon_{i-1} \\ \alpha_{i-1}\end{matrix}\right\}$$

$$= \phi_{i+1,i}(\left\{\begin{matrix}\lambda z.[\varepsilon'_i/x][z/y]\varepsilon_{i-1} \\ [\varepsilon'_{i+1}/x]\alpha_{i-1}\end{matrix}\right\}),\ \text{where } z \neq x \text{ and not free in } \varepsilon'_i,\varepsilon_{i-1}.$$

$$= \left\{\begin{matrix}\lambda z.\phi_{i,i-1}\circ[\varepsilon'_i/x][z/y]\varepsilon_{i-1} \\ \phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\alpha_{i-1}\end{matrix}\right\}$$

$$\sqsupseteq \left\{\begin{matrix}\lambda z.[\varepsilon'_{i-1}/x][z/y]\varepsilon_{i-2},\ \text{by } Q(i-1),2.1.6 \text{ and } 1.1.12. \\ [\varepsilon'_i/x]\alpha_{i-2},\ \text{by } R(i).\end{matrix}\right\}$$

$$= [\varepsilon'_i/x]\left\{\begin{matrix}\lambda y.\varepsilon_{i-2},\ \text{since z is not free in } \varepsilon'_{i-1},\varepsilon_{i-2},\ \text{by } 2.1.6. \\ \alpha_{i-2}\end{matrix}\right\}$$

$$= [\varepsilon'_i/x]\varepsilon''_{i-1}.$$

-<u>Claim:</u> $Q(i)\Rightarrow P(i),\ \forall i\geq 1$:-

$$-\phi_{i+2,i+1}\circ Ap_{i+1}(\left\{\begin{matrix}\lambda x.\varepsilon_i \\ \alpha_i\end{matrix}\right\},\varepsilon'_{i+1})$$

$$= \phi_{i+2,i+1}(\left\{\begin{matrix}\phi_{i+1,i+2}\circ[\varepsilon'_{i+1}/x]\varepsilon_i \\ \alpha_i(\varepsilon'_{i+1})\end{matrix}\right\}) = \left\{\begin{matrix}[\varepsilon'_{i+1}/x]\varepsilon_i,\ \text{by } 2.1.6. \\ \alpha_{i-1}(\varepsilon'_i)\end{matrix}\right\}$$

$$\sqsupseteq \left\{\begin{matrix}\phi_{i,i+1}\circ\phi_{i+1,i}\circ[\varepsilon'_{i+1}/x]\varepsilon_i,\ \text{by } 2.1.6. \\ \alpha_{i-1}(\varepsilon'_i)\end{matrix}\right\}$$

$$\sqsupseteq \left\{\begin{matrix}\phi_{i,i+1}\circ[\varepsilon'_i/x]\varepsilon_{i-1},\ \text{by } Q(i) \\ \alpha_{i-1}(\varepsilon'_i)\end{matrix}\right\} = Ap_i(\left\{\begin{matrix}\lambda x.\varepsilon_{i-1} \\ \alpha_{i-1}\end{matrix}\right\},\varepsilon'_i).$$

-∧, by 3.0.5, we have the result.

ǂ

<u>3.3.6:LEMMA:-</u>

In lemma 3.3.5, we cannot have equality.

<u>Proof:-</u>

-Let $y \neq a,b$ and $\bot \neq a,b$.

-Then, $\phi_{3,2}\circ[\lambda x.\varepsilon_2/y]a(b) = \phi_{3,2}(a(b)) = a(b)$.

-But, $[\lambda x.\varepsilon_1/y]\bot = \bot$. Contradiction to equality -$\times\!\!\!\!\times$.

ǂ

3.3.7:REMARK:-

The inequality in lemma 3.3.5 is highly regrettable, since otherwise it would be very easy to prove later that E_∞ modelled β-reduction. However, it does seem to be inherent in the system. It arises because of the inclusion, $\phi_{i,i+1}$, in definition 3.0.3 and, as was mentioned in 3.0.4, there seems to be no way to define the operators avoiding an inclusion.

There is an analogous situation in Scott's D_∞ model, where the application, also defined coordinatewise, fails to provide an exact sequence of resulting coordinates, and so necessitates the taking of a least upper bound.

Thus, we maintain that this inexactness is a feature of the λ-calculus itself and may be a reason why it can be so exceedingly difficult to prove theorems about it.

3.3.8:COR:-

Let $i \geq k$. Then, $\forall k \geq \begin{Bmatrix}0\\1\\2\end{Bmatrix}$, $\phi_{i,k}\circ[\varepsilon_i'/x]\begin{Bmatrix}z\\ \varepsilon_{i-1}\\ \alpha_{i-2}\end{Bmatrix} \equiv [\varepsilon_k'/x]\begin{Bmatrix}z\\ \varepsilon_{k-1}\\ \alpha_{k-2}\end{Bmatrix}$.

Also, $\forall k \geq 1$, $\phi_{i+2,k+2}\circ Ap_{i+1}(\varepsilon_{i+1},\varepsilon_{i+1}') \equiv Ap_{k+1}(\varepsilon_{k+1},\varepsilon_{k+1}')$.

Proof:-

-Trivial, by repeated applications of 3.3.5.

ǂ

3.3.9:COR:-

The sets $\left\{\phi_{i+1,\infty}\circ Ap_i(\varepsilon_i,\varepsilon_i') \mid i\geq 1\right\}$, $\left\{\phi_{i,\infty}\circ[\varepsilon_i'/x]\varepsilon_{i-1} \mid i\geq 1\right\}$ and $\left\{\phi_{i,\infty}\circ[\varepsilon_i'/x]\alpha_{i-2} \mid i\geq 2\right\}$ are directed in E_∞.

Proof:-

-$\forall i\geq 1$, $\phi_{i+2,\infty}\circ Ap_{i+1}(\varepsilon_{i+1},\varepsilon_{i+1}')$

$$\sqsupseteq \phi_{i+1,\infty}\circ\phi_{\infty,i+1}\circ\phi_{i+2,\infty}\circ Ap_{i+1}(\varepsilon_{i+1},\varepsilon'_{i+1}),\ \text{by } 2.2.3(\text{iv}).$$

$$= \phi_{i+1,\infty}\circ\phi_{i+2,i+1}\circ Ap_{i+1}(\varepsilon_{i+1},\varepsilon'_{i+1}),\ \text{by } 2.2.3(\text{iv}).$$

$$\sqsupseteq \phi_{i+1,\infty}\circ Ap_{i}(\varepsilon_{i},\varepsilon'_{i}),\ \text{by } 3.3.5.$$

-Similarly, $\forall i\geq 1,\ \phi_{i+1,\infty}\circ[\varepsilon'_{i+1}/x]\varepsilon_i \sqsupseteq \phi_{i,\infty}\circ[\varepsilon'_i/x]\varepsilon_{i-1}$.

- " , $\forall i\geq 2,\ \phi_{i+1,\infty}\circ[\varepsilon'_{i+1}/x]\alpha_{i-1} \sqsupseteq \phi_{i,\infty}\circ[\varepsilon'_i/x]\alpha_{i-2}$.

-∴, the sets form, in fact, directed chains in E_∞.

⊧

3.4:Infinite Application and Substitution:-

3.4.0:DEF:-

$$\begin{aligned} Ap:\ E_\infty\times E_\infty &\longrightarrow E_\infty \\ (\varepsilon,\varepsilon') &\longmapsto \bigsqcup_{i=1}^{\infty}\left\{\phi_{i+1,\infty}\circ Ap_i(\varepsilon_i,\varepsilon'_i)\right\} \end{aligned}$$

$$\begin{aligned} [\varepsilon'/x]_E:\ E_\infty &\longrightarrow E_\infty \\ \varepsilon &\longmapsto \bigsqcup_{i=1}^{\infty}\left\{\phi_{i,\infty}\circ[\varepsilon'_i/x]\varepsilon_{i-1}\right\} \end{aligned}$$

$$\begin{aligned} [\varepsilon'/x]_A:\ A_\infty &\longrightarrow E_\infty \\ \alpha &\longmapsto \bigsqcup_{i=2}^{\infty}\left\{\phi_{i,\infty}\circ[\varepsilon'_i/x]\alpha_{i-2}\right\} \end{aligned}$$

3.4.1:REMARK:-

As before, we will omit subscripts from the substitution operators when safe to do so.

The following lemma gives a characterisation of the operators which is sometimes easier to use.

3.4.2:LEMMA:-

$$Ap(\varepsilon,\varepsilon') = \left\langle \bigsqcup_{i=1}^{\infty}\left\{\phi_{i+1,j}\circ Ap_i(\varepsilon_i,\varepsilon'_i)\right\}\right\rangle_{j=0}^{\infty},$$

$$[\varepsilon'/x]\varepsilon = \left\langle \bigsqcup_{i=1}^{\infty}\left\{\phi_{i,j}\circ[\varepsilon'_i/x]\varepsilon_{i-1}\right\}\right\rangle_{j=0}^{\infty} \text{ and}$$

$$[\varepsilon'/x]\varepsilon = \left\langle \bigsqcup_{i=2}^{\infty}\left\{\phi_{i,j}\circ[\varepsilon'_i/x]\alpha_{i-2}\right\}\right\rangle_{j=0}^{\infty},$$

where the above sets are directed.

Proof:-

$-\phi_{\infty,j}\circ Ap(\varepsilon,\varepsilon') = \phi_{\infty,j}(\bigsqcup_{i=1}^{\infty}\{\phi_{i+1,\infty}\circ Ap_i(\varepsilon_i,\varepsilon_i')\})$

$= \bigsqcup_{i=1}^{\infty}\{\phi_{\infty,j}\circ\phi_{i+1,\infty}\circ Ap_i(\varepsilon_i,\varepsilon_i')\}$, since $\phi_{\infty,j}$ is continuous, by 2.2.3(i), and the set is directed, by 3.3.9, which implies the resulting set is also directed.

$= \bigsqcup_{i=1}^{\infty}\{\phi_{i+1,j}\circ Ap_i(\varepsilon_i,\varepsilon_i')\}$, by 2.2.3(iv).

-Similarly for the others.

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3.4.3:REMARK:-

Notice that in the above characterisation of Ap and $[\varepsilon/x]$, since the sets are subsets of various E_j's, which have finite depth by 1.2.18, the actual j'th coordinate must be one of the $\phi_{i+1,j}\circ Ap_i(\varepsilon_i,\varepsilon_i')$ etc..., for some finite i. This is useful in some of the later proofs.

3.4.4:THEOREM:-

(i) $[\varepsilon'/x]$ is doubly strict over E_∞ and A_∞.

(ii) $Ap(\{^{\top}_{\perp}\},\varepsilon') = \{^{\top}_{\perp}\}$.

(iii) $[z/z'][[z/z']\varepsilon'/x] = [z/z'][\varepsilon'/x]$.

(iv) $[z/z']Ap(\varepsilon,\varepsilon') = Ap([z/z']\varepsilon,[z/z']\varepsilon')$.

(v) $(x \neq z,z')\wedge(z'$ is not free in $\varepsilon') \Rightarrow$

$([\varepsilon'/x][z/z'] = [z/z'][\varepsilon'/x])$.

(vi) (x is not free in $\varepsilon,\alpha) \Rightarrow$

$([\varepsilon'/x][x/y]\{^{\varepsilon}_{\alpha}\} = [\varepsilon'/y]\{^{\varepsilon}_{\alpha}\})$.

(vii) (a is not free in $\varepsilon',\{^{\varepsilon}_{\alpha}\}) \Rightarrow$

(a is not free in $[\varepsilon'/x]\{^{\varepsilon}_{\alpha}\}$).

(viii) (a is not free in ε') $\Rightarrow$

(a is not free in $[\varepsilon'/a]\{^{\varepsilon}_{\alpha}\}$).

(ix) (a is not free in ε,ε') => (a is not free in $Ap(\varepsilon,\varepsilon')$).

(x) (x is not free in $\{^{\varepsilon}_{\alpha}\}$) => ($[\varepsilon'/x]\{^{\varepsilon}_{\alpha}\} = \{^{\varepsilon}_{\alpha}\}$).

(xi) $[\varepsilon'/x]$ and Ap are continuous maps.

(xii) $[w/x]_E = [w/x]$ and $\pi_2 \circ [w/x]_A = [w/x]$.

<u>Proof:-</u>

-(i),(ii),(iii),(iv),(v),(vi) and (x) are trivial, by 2.1.6,2.2.8 and 3.1.20.

-(vii),(viii) and (ix) are trivial, by 3.4.2,1.2.18 and 2.1.6, noting remark 3.4.3.

-(xi) is trivial, by 3.2.7, since we note that,

$$Ap = \bigsqcup_{i=1}^{\infty}\{\phi_{i+1,\infty} \circ Ap_i \circ (\phi_{\infty,i},\phi_{\infty,i})\},$$

where $(\phi_{\infty,i},\phi_{\infty,i})$ is the clearly continuous projection on the cross-product defined in the obvious way. Thus, Ap is the least upper bound of continuous maps and is therefore continuous itself. Similarly for $[\varepsilon'/x]$.

-(xii) Well, by 3.4.2 :-

$$\begin{aligned}[w/x]_E\varepsilon &= <\bigsqcup_{i=1}^{\infty}\{\phi_{i,j} \circ [w/x]_{E_{i-1}} \varepsilon_{i-1}\}>_{j=0}^{\infty} \\ &= <\bigsqcup_{i=1}^{\infty}\{\phi_{i,j} \circ \phi_{i-1,i} \circ [w/x]\varepsilon_{i-1}\}>_{j=0}^{\infty}, \text{ by 3.3.4.} \\ &= <\bigsqcup_{i=0}^{\infty}\{\phi_{i,j} \circ [w/x]\varepsilon_i\}>_{j=0}^{\infty}, \text{ by 2.2.3(iv).} \\ &= <[w/x]\varepsilon_j>_{j=0}^{\infty} = [w/x]\varepsilon.\end{aligned}$$

Similarly for $[w/x]_A$ - the π_2 is a trivial detail to get the types of the functions compatible.

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3.4.5:NOTATION:-

Let $\varepsilon \in E_\infty$ and $x \in I$. Then, write x for $<x>_{i=0}^{\infty} \in E_\infty$, and $\lambda x.\varepsilon$ for $<\iota,<\lambda x.\varepsilon_i>_{i=0}^{\infty}> \in E_\infty$.

3.4.6:LEMMA:-

(i) x is not free in y $\Leftrightarrow$ $x \neq y$.

(ii) x is not free in $\lambda y.\varepsilon$ $\Leftrightarrow$ $x = y$ or x is not free in ε.

(iii) x is not free in $Ap(\varepsilon,\eta)$ $\Leftarrow$ x is not free in ε,η.

Proof:-

-(i) and (ii) are trivial, by 3.4.5.

-(iii) is the same as 3.4.4(ix).

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3.4.7:LEMMA:-

Let y be not free in ε. Then, $\lambda x.\varepsilon = \lambda y.[y/x]\varepsilon$.

Proof:-

$$-\lambda x.\varepsilon = \langle \iota, \langle \lambda x.\varepsilon_i \rangle_{i=0}^{\infty} \rangle$$

$$= \langle \iota, \langle \lambda y.[y/x]\varepsilon_i \rangle_{i=0}^{\infty} \rangle, \text{ by 1.1.8.}$$

$$= \lambda y.[y/x]\varepsilon, \text{ by 2.1.6.}$$

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3.4.8:LEMMA:-

$Ap(\lambda x.\varepsilon,\varepsilon') = [\varepsilon'/x]\varepsilon$.

Proof:-

$$-Ap(\lambda x.\varepsilon,\varepsilon') = \bigsqcup_{i=1}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\lambda x.\varepsilon_{i-1},\varepsilon'_i)$$

$$= \bigsqcup_{i=1}^{\infty} \phi_{i,\infty} \circ [\varepsilon'_i/x]\varepsilon_{i-1}$$

$$= [\varepsilon'/x]\varepsilon.$$

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3.4.9:COR:-

Let x be not free in $\varepsilon \in E_\infty$. Then,

$$Ap(\lambda x.\varepsilon,\varepsilon') = \varepsilon.$$

Proof:-

$$-Ap(\lambda x.\varepsilon,\varepsilon') = [\varepsilon'/x]\varepsilon, \text{ by 3.4.8}$$

$$= \varepsilon, \text{ by 3.4.4(x).}$$

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3.4.10:LEMMA:-

(i) $$[\varepsilon/x]y = \begin{Bmatrix} y, \text{ if } x \neq y \in I \\ \varepsilon, \text{ if } x = y \in I \end{Bmatrix}.$$

(ii) $[\varepsilon'/x](\lambda x.\varepsilon) = \lambda x.\varepsilon.$

(iii) $(y \neq x) \wedge (y \text{ is not free in } \varepsilon') \Rightarrow$

$$([\varepsilon'/x](\lambda y.\varepsilon) = \lambda y.[\varepsilon'/x]\varepsilon).$$

(iv) $(z \neq x) \wedge (z \text{ is not free in } \varepsilon',\varepsilon) \Rightarrow$

$$([\varepsilon'/x](\lambda y.\varepsilon) = \lambda z.[\varepsilon'/x][z/y]\varepsilon).$$

Proof:-

-(i) $$[\varepsilon/x]y = <\bigsqcup_{i=1}^{\infty}\phi_{i,j}\circ[\varepsilon_i/x]y>_{j=0}^{\infty}$$

$$= \begin{Bmatrix} <\bigsqcup_{i=1}^{\infty}\phi_{i,j}(y)>_{j=0}^{\infty}, \text{ if } x \neq y. \\ <\bigsqcup_{i=1}^{\infty}\phi_{i,j}(\varepsilon_i)>_{j=0}^{\infty}, \text{ if } x = y. \end{Bmatrix}$$

$$= \begin{Bmatrix} <y>_{j=0}^{\infty}, \text{ if } x \neq y. \\ <\varepsilon_j>_{j=0}^{\infty}, \text{ if } x = y. \end{Bmatrix} = \begin{Bmatrix} y, \text{ if } x \neq y \\ \varepsilon, \text{ if } x = y \end{Bmatrix}.$$

-(ii) x is not free in $\lambda x.\varepsilon$, by 3.4.6(ii). Thus, we have the result by 3.4.4(x).

-(iii) $$[\varepsilon'/x](\lambda y.\varepsilon) = <\bigsqcup_{i=2}^{\infty}\phi_{i,j}\circ[\varepsilon'_i/x](\lambda y.\varepsilon_{i-2})>_{j=0}^{\infty}$$

$$= <\bigsqcup_{i=2}^{\infty}\phi_{i,j}(\lambda y.[\varepsilon'_{i-1}/x]\varepsilon_{i-2})>_{j=0}^{\infty}, \text{ by 3.3.2(ii)}$$

$$= <\bot,<\bigsqcup_{i=2}^{\infty}\lambda y.\phi_{i-1,j-1}\circ[\varepsilon'_{i-1}/x]\varepsilon_{i-2}>_{j=1}^{\infty}>$$

$$= <\bot,<\lambda y.\bigsqcup_{i=1}^{\infty}\phi_{i,j-1}\circ[\varepsilon'_i/x]\varepsilon_{i-1}>_{j=1}^{\infty} \text{ , by 1.1.14(i).}$$

$$= <\bot,<\lambda y.\phi_{\infty,j-1}\circ[\varepsilon'/x]\varepsilon>_{j=1}^{\infty}>, \text{ by 3.4.2.}$$

$$= <\bot,<\phi_{\infty,j}(\lambda y.[\varepsilon'/x]\varepsilon)>_{j=1}^{\infty}> = \lambda y.[\varepsilon'/x]\varepsilon.$$

-(iv) $[\varepsilon'/x](\lambda y.\varepsilon) = [\varepsilon'/x](\lambda z.[z/y]\varepsilon)$, by 3.4.7.

$= \lambda z.[\varepsilon'/x][z/y]\varepsilon$, by part (iii) above.

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3.4.11: REMARK:-

We have shown that $[\varepsilon/x]$ extends $[y/x]$, by 3.4.4(xii), as was required in remark 3.0.0. We see that the concept of "not free in" in E_∞ is very similar to the notion of λ-calculus, by 3.4.6. We see that the notion of α-conversion is reflected by 3.4.7. We see that the notion of β-reduction is reflected by 3.4.8.

We almost see that the substitution operator of λ-calculus is given by $[\varepsilon/x]$, by 3.4.10. What is missing is the identity,

$$[\varepsilon/x]Ap(\delta,\eta) = Ap([\varepsilon/x]\delta,[\varepsilon/x]\eta).$$

Notice that we do have this when $\varepsilon = y \in I$, by 3.4.4(iv). This identity proves very difficult to establish. We can push coordinates around so as to obtain expressions like :-

$$[\varepsilon'/x]Ap(\varepsilon,\delta) = \bigsqcup_{i=1}^{\infty} \phi_{i+2,\infty}\circ[\varepsilon'_{i+2}/x]\ Ap_i(\varepsilon_i,\delta_i)$$

and,

$$Ap([\varepsilon'/x]\varepsilon,[\varepsilon'/x]\delta) = \bigsqcup_{i=1}^{\infty} \phi_{i+1,\infty}\circ Ap_i([\varepsilon'_i/x]\varepsilon_{i-1},[\varepsilon'_i/x]\delta_{i-1}).$$

But when we try an induction, à la 3.0.5, on these expressions, it will not go through. The trouble arises from the "inexactness" mentioned in 3.3.7.

This identity is all that is needed to establish E_∞ as a model of the λ-calculus. We are forced to retreat back into the pure λ-calculus in order to get it.

We end this section by showing the existence of the I and K combinators in E_∞. We cannot yet give S because of the missing identity.

3.4.12: LEMMA:-

(i) $Ap(\lambda x.x,\varepsilon') = \varepsilon'$.

(ii) $Ap(Ap(\lambda x.\lambda y.x,\varepsilon),\varepsilon') = \varepsilon$, if $y \neq x$ and there exists $z \neq y,x$ such that z is not free in ε.

Proof:-

-(i) $Ap(\lambda x.x,\varepsilon') = [\varepsilon'/x]x$, by 3.4.8.

$= \varepsilon'$, by 3.4.10(i).

-(ii) $Ap(Ap(\lambda x.\lambda y.x,\varepsilon),\varepsilon') = Ap([\varepsilon/x](\lambda y.x),\varepsilon')$, by 3.4.8.

$= Ap(\lambda z.[\varepsilon/x][z/y]x,\varepsilon')$, by 3.4.10(iv).

$= Ap(\lambda z.[\varepsilon/x]x,\varepsilon')$, by 3.4.10(i) and 3.4.4(xii).

$= Ap(\lambda z.\varepsilon,\varepsilon')$, by 3.4.10(i).

$= \varepsilon$, by 3.4.9.

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3.5:Summary of the Construction of E_∞ and Ap :-

- I' = Simple atomic lattice of variable names (countable).

- $\left\{\begin{array}{l} E_0 = I' \\ A_0 = I' \end{array}\right\}$ and $\left\{\begin{array}{l} E_{i+1} = \lambda I.E_i + A_i \\ A_{i+1} = I' + A_i(E_{i+1}) \end{array}\right\}$.

- $$\begin{array}{l} \phi_{1,0} : \quad E_1 \longrightarrow E_0 \\ \qquad \left\{\begin{array}{l} \lambda x.\varepsilon_0 \\ \alpha_0 \end{array}\right\} \longmapsto \left\{\begin{array}{l} \bot \text{ (unless } \lambda x.\varepsilon_0 = \alpha_0 = \top) \\ \alpha_0 \end{array}\right\} \end{array}.$$

- $\phi_{0,1} : \quad E_0 \subseteq E_1$ (since $E_0 = A_0$).

- $$\begin{array}{l} \theta_{1,0} : \quad A_1 \longrightarrow A_0 \\ \qquad \left\{\begin{array}{l} x \\ \alpha_0(\varepsilon_1) \end{array}\right\} \longmapsto \left\{\begin{array}{l} x \\ \bot \text{ (unless } \alpha_0(\varepsilon_1) = x = \top) \end{array}\right\} \end{array}.$$

- $\theta_{0,1} : \quad A_0 \subseteq A_1$ (since $A_0 = I'$).

- $\phi_{i+1,i} : E_{i+1} \longrightarrow E_i$.

$$\left\{\begin{matrix} \lambda x.\varepsilon_i \\ \alpha_i \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \lambda x.\phi_{i,i-1}(\varepsilon_i) \\ \theta_{i,i-1}(\alpha_i) \end{matrix}\right\}$$

- $\theta_{i+1,i} : A_{i+1} \longrightarrow A_i$.

$$\left\{\begin{matrix} x \\ \alpha_i(\varepsilon_{i+1}) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} x \\ \theta_{i,i-1}(\alpha_i)(\phi_{i+1,i}(\varepsilon_{i+1})) \end{matrix}\right\}$$

- Similarly for $\phi_{i,i+1}$ and $\theta_{i,i+1}$.

- E_∞ = Inverse limit of $<E_{i+1},\phi_{i+1,i}>_{i=0}^{\infty}$
  A_∞ = Inverse limit of $<A_{i+1},\theta_{i+1,i}>_{i=0}^{\infty}$
  $\lambda I.E_\infty$ = Inverse limit of $<\lambda I.E_i,\phi\phi_{i+1,i}>_{i=0}^{\infty}$
  $A_\infty(E_\infty)$ = Inverse limit of $<A_i(E_{i+1}),\theta\theta_{i+1,i}>_{i=0}^{\infty}$.

- $\left\{\begin{matrix} E_\infty = \lambda I.E_\infty + A_\infty \\ A_\infty = I' + A_\infty(E_\infty) \end{matrix}\right\}$.

- $[\varepsilon_i/x]_I : I' \longrightarrow E_i$.

$$y \longmapsto \left\{\begin{matrix} y, \text{ if } x \neq y \\ \varepsilon_i, \text{ if } x = y \end{matrix}\right\}$$

- $[\varepsilon_1/x]_{E_0} = [\varepsilon_1/x]_I \in [E_0 \to E_1]$
  $[\varepsilon_2/x]_{A_0} = [\varepsilon_2/x]_I \in [A_0 \to E_2]$.

- $[\varepsilon_i'/x]_{E_{i-1}} : E_{i-1} \longrightarrow E_i$ (i ≥ 2)

$$\left\{\begin{matrix} \lambda y.\varepsilon_{i-2} \\ \alpha_{i-2} \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \lambda z.[\varepsilon_{i-1}'/x]_{E_{i-2}}[z/y]\varepsilon_{i-2}, \text{ where} \\ [\varepsilon_i'/x]_{A_{i-2}}\alpha_{i-2} \end{matrix}\right\}$$

$z \neq x$ and z is not free in $\varepsilon_{i-1}',\varepsilon_{i-2}$.

$$- \ [\varepsilon_i'/x]_{A_{i-2}} : A_{i-2} \longrightarrow E_i \qquad (i \geq 3)$$

$$\begin{Bmatrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{Bmatrix} \longmapsto \begin{Bmatrix} [\varepsilon_i'/x]_I y \\ Ap_{i-1}([\varepsilon_{i-1}'/x]_{A_{i-3}}\alpha_{i-3}, \\ \qquad [\varepsilon_{i-1}'/x]_{E_{i-2}}\varepsilon_{i-2}) \end{Bmatrix} .$$

$$- \ Ap_i : E_i \times E_i \longrightarrow E_{i+1} \qquad (i \geq 1)$$

$$\left(\begin{Bmatrix} \lambda x.\varepsilon_{i-1} \\ \alpha_{i-1} \end{Bmatrix}, \varepsilon_i'\right) \longmapsto \begin{Bmatrix} \phi_{i,i+1}([\varepsilon_i'/x]_{E_{i-1}}\varepsilon_{i-1}) \\ \alpha_{i-1}(\varepsilon_i') \end{Bmatrix} .$$

$$- \ [\varepsilon'/x]_E : E_\infty \longrightarrow E_\infty \, .$$

$$\langle\varepsilon_i\rangle_{i=0}^{\infty} \longmapsto \bigsqcup_{i=1}^{\infty} \phi_{i,\infty}([\varepsilon_i'/x]_{E_{i-1}}\varepsilon_{i-1})$$

$$- \ [\varepsilon'/x]_A : A_\infty \longrightarrow E_\infty \, .$$

$$\langle\alpha_i\rangle_{i=0}^{\infty} \longmapsto \bigsqcup_{i=2}^{\infty} \phi_{i,\infty}([\varepsilon_i'/x]_{A_{i-2}}\alpha_{i-2})$$

$$- \ Ap : E_\infty \times E_\infty \longrightarrow E_\infty \, .$$

$$(\langle\varepsilon_i\rangle_{i=0}^{\infty}, \langle\varepsilon_i'\rangle_{i=0}^{\infty}) \longmapsto \bigsqcup_{i=1}^{\infty} \phi_{i+1,\infty}(Ap_i(\varepsilon_i, \varepsilon_i'))$$

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4:E_∞-SEMANTICS OF THE λ-CALCULUS.

4.0:The Semantical Function $\mathcal{E}$:-

4.0.0:REMARK:-

We are now going to be using λ-expressions. These have a similar notation to that we have built up to describe elements of E_∞. We must be careful to realise when we are talking about elements of EXP from those of E_∞. In general, elements of EXP are enclosed within double square brackets, $\llbracket\ \rrbracket$.

4.0.1:DEF:-

Let EXP be the set of λ-expressions as defined in 0.3.1. We define the semantical function, $\mathcal{E}$, from EXP to E_∞, by the following three equations :-

$$\mathcal{E}\llbracket x\rrbracket = x \qquad (S1)$$

$$\mathcal{E}\llbracket \lambda x.\varepsilon\rrbracket = \lambda x.\mathcal{E}\llbracket\varepsilon\rrbracket \qquad (S2)$$

$$\mathcal{E}\llbracket \varepsilon(\eta)\rrbracket = \mathrm{Ap}(\mathcal{E}\llbracket\varepsilon\rrbracket,\mathcal{E}\llbracket\eta\rrbracket) \qquad (S3)$$

4.0.2:REMARK:-

On the right hand side of equations S1 and S2 above, we are using the notation of 3.4.5. We are also using the same notation for the variables in EXP and elements of E_0.

4.0.3:EXAMPLES:-

$\mathcal{E}\llbracket x\rrbracket = \langle x,x,x,x,x,x,x,x,x,x,x,x,x,........\rangle$

$\mathcal{E}\llbracket xy\rrbracket = \langle \bot,\bot,xy,xy,xy,xy,xy,xy,xy,xy,......\rangle$

$\mathcal{E}\llbracket \lambda x.y\rrbracket = \langle \bot,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,\lambda x.y,.......\rangle$

$\mathcal{E}\llbracket \lambda x.xy\rrbracket = \langle \bot,\bot,\bot,\lambda x.xy,\lambda x.xy,\lambda x.xy,\lambda x.xy,....\rangle$

$\mathcal{E}\llbracket f(yy)\rrbracket = \langle \bot,\bot,f(\bot),f(yy),f(yy),f(yy),.......\rangle$

$\mathcal{E}\llbracket \lambda y.f(yy)\rrbracket = \langle \bot,\bot,\bot,\lambda y.f(\bot),\lambda y.f(yy),..........\rangle$

$\mathcal{E}\llbracket b(\lambda c.dd)\rrbracket = \langle \bot,b(\bot),b(\bot),b(\bot),b(\lambda c.dd),........\rangle$

$\mathcal{E}[\![(\lambda y.xy)(b)]\!] \quad = \langle \bot, \bot, xb, xb, xb, xb, xb, xb, xb, xb, \ldots\ldots \rangle$

$\mathcal{E}[\![(\lambda x.xx)(\lambda x.xx)]\!] \quad = \langle \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \bot, \ldots\ldots \rangle$

$\mathcal{E}[\![Y(f)]\!] \quad = \langle \bot, \bot, f(\bot), f^2(\bot), f^3(\bot), f^4(\bot), \ldots\ldots \rangle$

<u>Proof:-</u>

- $\mathcal{E}[\![x]\!] = x = \langle x, x, x, x, x, \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![xy]\!] = Ap(x,y) = \bigsqcup_{i=1}^{\infty} \phi_{i+1,\infty} \circ Ap_i(x,y)$

  $= \bigsqcup_{i=1}^{\infty} \phi_{i+1,\infty}(xy) = \langle \bot, \bot, xy, xy, xy, xy, \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![\lambda x.y]\!] = \lambda x.\, \mathcal{E}[\![y]\!] = \lambda x.y.$
- $\mathcal{E}[\![\lambda x.xy]\!] = \lambda x. \langle \bot, \bot, xy, xy, xy, xy, \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![f(yy)]\!] = Ap(f, \langle \bot, \bot, yy, yy, yy, yy, \ldots\ldots\ldots \rangle)$

  $= \bigsqcup_{i=2}^{\infty} \phi_{i+1,\infty} \circ Ap_i(f,yy) = \bigsqcup_{i=2}^{\infty} \phi_{i+1,\infty}(f(yy))$

  $= \langle \bot, \bot, f(\bot), f(yy), f(yy), \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![\lambda y.f(yy)]\!] = \lambda y. \langle \bot, \bot, f(\bot), f(yy), f(yy), \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![b(\lambda c.dd)]\!] = Ap(b, \langle \bot, \bot, \bot, \lambda c.dd, \lambda c.dd, \ldots\ldots\ldots \rangle)$

  $= \bigsqcup_{i=3}^{\infty} \phi_{i+1,\infty} \circ Ap_i(b, \lambda c.dd) = \bigsqcup_{i=3}^{\infty} \phi_{i+1,\infty}(b(\lambda c.dd))$

  $= \langle \bot, b(\bot), b(\bot), b(\bot), b(\lambda c.dd), b(\lambda c.dd), \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![(\lambda y.xy)(b)]\!] = Ap(\lambda y.\mathcal{E}[\![xy]\!], b)$

  $= [b/y]_E \langle \bot, \bot, xy, xy, \ldots\ldots\ldots \rangle$, by 3.4.8.

  $= [b/y] \langle \bot, \bot, xy, xy, \ldots\ldots\ldots \rangle$, by 3.4.4(xii).

  $= \langle \bot, \bot, [b/y]x([b/y]y), \ldots\ldots\ldots \rangle$

  $= \langle \bot, \bot, xb, xb, xb, xb, xb, \ldots\ldots\ldots \rangle.$
- $\mathcal{E}[\![(\lambda x.xx)(\lambda x.xx)]\!] = Ap(\langle \bot, \bot, \bot, \lambda x.xx, \ldots\ldots \rangle, \langle \bot, \bot, \bot, \lambda x.xx, \ldots\ldots \rangle)$

  $= \bigsqcup_{i=3}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\lambda x.xx, \lambda x.xx).$

-Now, let $P(i) \equiv Ap_i(\lambda x.xx, \lambda x.xx) = \bot$, for $i \geq 3$.

-<u>Claim:</u> $P(3)$:-

-$Ap_3(\lambda x.xx, \lambda x.xx) = \phi_{3,4} \circ [\lambda x.xx/x]_{E_2}(xx)$

$= \phi_{3,4} \circ [\lambda x.xx/x]_{A_1}(xx)$

$$
\begin{aligned}
&= \phi_{3,4} \circ Ap_2([\bot/x]_{E_1} x, [\bot/x]_{A_0} x) \\
&= \phi_{3,4} \circ Ap_2([\bot/x]_{A_0} x, [\bot/x]_{A_0} x) \\
&= \phi_{3,4} \circ Ap_2(\bot, \bot) = \phi_{3,4}(\bot) = \bot \in E_4.
\end{aligned}
$$

-<u>Claim:</u> $P(i) \Rightarrow P(i+1)$, $\forall i \geq 3$:-

$$
\begin{aligned}
-Ap_{i+1}(\lambda x.xx, \lambda x.xx) &= \phi_{i+1,i+2} \circ [\lambda x.xx/x]_{A_{i-1}} (xx) \\
&= \phi_{i+1,i+2} \circ Ap_i([\lambda x.xx/x]_{A_{i-2}} x, \\
&\qquad\qquad [\lambda x.xx/x]_{A_{i-2}} x) \\
&= \phi_{i+1,i+2} \circ Ap_i(\lambda x.xx, \lambda x.xx) \\
&= \phi_{i+1,i+2}(\bot), \text{ by } P(i). \\
&= \bot \in E_{i+2}.
\end{aligned}
$$

-∴, by induction, $P(i)$, $\forall i \geq 3$.

-∴, $\mathcal{E}[\![(\lambda x.xx)(\lambda x.xx)]\!] = \bigsqcup_{i=3}^{\infty} \phi_{i+1,\infty}(\bot) = \bot.$

-To evaluate $\mathcal{E}[\![Y(f)]\!]$, we have first to evaluate $\mathcal{E}[\![Y]\!]$.

-Now, $Y = \lambda f.(\lambda y.f(yy))(\lambda y.f(yy))$.

-And, $\mathcal{E}[\![(\lambda y.f(yy))(\lambda y.f(yy))]\!]$

$$
\begin{aligned}
&= Ap(<\bot,\bot,\bot,\lambda y.f(\bot),\lambda y.f(yy),\ldots.>,<\bot,\bot,\bot,\lambda y.f(\bot),\lambda y.f(yy),\ldots>) \\
&= \bigsqcup_{i=4}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\lambda y.f(yy), \lambda y.f(yy)).
\end{aligned}
$$

-Suppose $i \geq 6$. Then, $Ap_i(\lambda y.f(yy),\lambda y.f(yy))$

$$
\begin{aligned}
&= \phi_{i,i+1} \circ [\lambda y.f(yy)/y]_{A_{i-2}} f(yy) \\
&= \phi_{i,i+1} \circ Ap_{i-1}([\lambda y.f(yy)/y]_{A_{i-3}} f, [\lambda y.f(yy)/y]_{A_{i-3}} (yy)) \\
&= \phi_{i,i+1} \circ Ap_{i-1}(f, Ap_{i-2}([\lambda y.f(yy)/y]_{A_{i-4}} y, [\lambda y.f(yy)/y]_{A_{i-4}} y)) \\
&= \phi_{i,i+1}(f(Ap_{i-2}(\lambda y.f(yy),\lambda y.f(yy))))).
\end{aligned}
$$

-By noting how the projections work in the above,

$$
\begin{aligned}
Ap_4(\lambda y.f(yy),\lambda y.f(yy)) &= \phi_{4,5}(f(Ap_2(\bot,\bot))) \\
&= \phi_{4,5}(f(\bot)) = f(\bot) \in E_5.
\end{aligned}
$$

-Also,

$$
\begin{aligned}
Ap_5(\lambda y.f(yy),\lambda y.f(yy)) &= \phi_{5,6}(f(Ap_3(\lambda y.f(\bot),\lambda y.f(\bot)))) \\
&= \phi_{5,6}(f(\phi_{3,4} \circ [\lambda y.f(\bot)/y]_{A_1} f(\bot))) \\
&= \phi_{5,6}(f(\phi_{3,4}(f(\bot)))), \text{ since y is}
\end{aligned}
$$

not free in $f(\perp)$.

$$= f^{2}(\perp) \in E_{6}.$$

$\therefore$, by induction, we see that, $\forall i \geq 4$,

$Ap_{i}(\lambda y.f(yy),\lambda y.f(yy)) = f^{(i-1)\div 2}(\perp)$, where "÷" means "integer divide".

$$\therefore,\ \mathcal{E}[\![(\lambda y.f(yy))(\lambda y.f(yy))]\!] = \bigsqcup_{i=4}^{\infty} \phi_{i+1,\infty}(f^{(i-1)\div 2}(\perp))$$

$$= \bigsqcup_{i=1}^{\infty} f^{i}(\perp)$$

$$= <\perp,\perp,f(\perp),f^{2}(\perp),f^{3}(\perp),........>.$$

$$\therefore,\ \mathcal{E}[\![Y]\!] = <\perp,\perp,\perp,\lambda f.f(\perp),\lambda f.f^{2}(\perp),\lambda f.f^{3}(\perp),........>.$$

$$\therefore,\ \mathcal{E}[\![Y(f)]\!] = Ap(\mathcal{E}[\![Y]\!],f) = \bigsqcup_{i=2}^{\infty} \phi_{i+1,\infty}\circ Ap_{i}(\lambda f.f^{i-2}(\perp),f)$$

$$= \bigsqcup_{i=2}^{\infty} \phi_{i,\infty}\circ[f/f]_{E_{i-1}}(f^{i-2}(\perp))$$

$$= \bigsqcup_{i=2}^{\infty} \phi_{i-1,\infty}\circ[f/f](f^{i-2}(\perp)), \text{ by 3.3.4.}$$

$$= \bigsqcup_{i=2}^{\infty} \phi_{i-1,\infty}(f^{i-2}(\perp)), \text{ by 1.2.18.}$$

$$= <\perp,\perp,f(\perp),f^{2}(\perp),f^{3}(\perp),f^{4}(\perp),........>.$$

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4.0.4:REMARK:-

Thus, it seems that expressions that are in normal form are mapped elementarily to their obvious images in E_∞. Notice that they are embedded in an E_n, where n is finite.

Also, β-convertible elements could well have the same semantics - e.g. (λy.xy)(b) and (xb).

(λx.xx)(λx.xx) is mapped to ⊥. We will show that all λ-expressions with no head normal form go to ⊥.

The Y combinator has a representation that looks like the minimal fixed point operator - but see section 7.1. The following lemma shows that all the Y_i combinators have the same semantics in E_∞ under $\mathcal{E}$, even though Y_0 is not convertible to Y_1.

Notice that E_∞ does not model η-conversion, since $\mathcal{E}[\![x]\!] \neq \mathcal{E}[\![\lambda y.xy]\!]$,

4.0.5:LEMMA:-

All the fixed point combinators, Y_i, have the same semantics in E_∞ under $\mathcal{E}$.

Proof:-

-Recall from 0.4.8 that, $\left\{\begin{array}{ll} G & = \lambda y.\lambda f.f(yf) \\ Y_0 & = Y \\ Y_{i+1} & = Y_i G,\ \forall i \geq 0 \end{array}\right\}$.

-Then, $\mathcal{E}[\![Y_1]\!] = \mathcal{E}[\![Y_0 G]\!] = \mathrm{Ap}(\mathcal{E}[\![Y]\!], \mathcal{E}[\![G]\!])$.

-By comparing with the examples in 4.0.3, we see :-

$$\mathcal{E}[\![G]\!] = \langle \bot, \bot, \bot, \bot, \lambda y.\lambda f.f(\bot), \lambda y.\lambda f.f(yf), \ldots\ldots\ldots \rangle.$$

-Also, by 4.0.3, we have :-

$$\mathcal{E}[\![Y]\!] = \langle \bot, \bot, \bot, \lambda f.f(\bot), \lambda f.f^2(\bot), \ldots\ldots\ldots \rangle.$$

-$\therefore$, $\mathcal{E}[\![Y_1]\!] = \bigsqcup_{i=5}^{\infty} \phi_{i+1,\infty} \circ \mathrm{Ap}_i(\lambda f.f^{i-2}(\bot), \lambda y.\lambda f.f(yf))$

$$= \bigsqcup_{i=5}^{\infty} \phi_{i,\infty} \circ [\lambda y.\lambda f.f(yf)/f]_{A_{i-2}} f^{i-2}(\bot).$$

-In the following, we will omit the inclusions $\phi_{i,j}$.

-Let $P(i) \equiv [\lambda y.\lambda f.f(yf)/f]_{A_{i-2}} f^{i-2}(\bot) = \lambda f.f^{i-5}(\bot) \in E_i$.

-We will establish $P(i)$, $\forall i \geq 6$.

-Note that $\lambda f.f^{i-5}(\bot) \in E_{i-3} \lhd E_i$.

-Claim: $P(6)$:-

-First we evaluate $[\lambda y.\lambda f.f(yf)/f]_{A_3} f^3(\bot)$

$= \mathrm{Ap}_4([\lambda y.\lambda f.f(\bot)/f]_{A_2} f, [\lambda y.\lambda f.f(\bot)/f]_{A_2} f^2(\bot))$

$= \mathrm{Ap}_4(\lambda y.\lambda f.f(\bot), \mathrm{Ap}_3([\bot/f]_{A_1} f, [\bot/f]_{A_1} f(\bot)))$

$= \mathrm{Ap}_4(\lambda y.\lambda f.f(\bot), \mathrm{Ap}_3(\bot, [\bot/f]_{A_1} f(\bot)))$

$= \mathrm{Ap}_4(\lambda y.\lambda f.f(\bot), \bot) = [\bot/y]_{E_3} \lambda f.f(\bot) = \lambda f.f(\bot) \in E_5$.

-Then, $[\lambda y.\lambda f.f(yf)/f]_{A_4} f^4(\bot)$

$= \mathrm{Ap}_5([\lambda y.\lambda f.f(yf)/f]_{A_3} f, [\lambda y.\lambda f.f(yf)/f]_{A_3} f^3(\bot))$

$= \mathrm{Ap}_5(\lambda y.\lambda f.f(yf), \lambda f.f(\bot))$, by above.

$= [\lambda f.f(\perp)/y]_{A_3}\ \lambda f.f(yf)$

$= \lambda g.[\lambda f.f(\perp)/y]_{A_2}\ (g(yg))$

$= \lambda g.Ap_3([\lambda f.f(\perp)/y]_{A_1} g, [\lambda f.f(\perp)/y]_{A_1} yg)$

$= \lambda g.Ap_3(g, Ap_2([\perp/y]_{A_0} y, [\perp/y]_{A_0} g))$

$= \lambda g.Ap_3(g, Ap_2(\perp, g)) = \lambda g.g(\perp) = \lambda f.f(\perp).$

-∴, we have $P(6)$.

-<u>Claim:</u> $P(i) \Rightarrow P(i+1)$, $\forall i \geq 6$:-

-Well, $[\lambda y.\lambda f.f(yf)/f]_{A_{i-1}}\ f^{i-1}(\perp)$

$= Ap_i(\lambda y.\lambda f.f(yf), [\lambda y.\lambda f.f(yf)/f]_{A_{i-2}}\ f^{i-2}(\perp))$

$= Ap_i(\lambda y.\lambda f.f(yf), \lambda f.f^{i-5}(\perp))$, by $P(i)$.

$= [\lambda f.f^{i-5}(\perp)/y]_{A_{i-2}}\ \lambda f.f(yf)$

$= \lambda g.[\lambda f.f^{i-5}(\perp)/y]_{A_{i-3}}\ g(yg)$

$= \lambda g.Ap_{i-2}(g, [\lambda f.f^{i-5}(\perp)/y]_{A_{i-4}}\ yg)$

$= \lambda g.g(Ap_{i-3}([\lambda f.f^{i-5}(\perp)/y]_{A_{i-5}}\ y, [\lambda f.f^{i-5}(\perp)/y]_{A_{i-5}}\ g))$

(Note that in the above line we have $\lambda f.f^{i-5}(\perp) \in E_{i-3}$, which just fits.)

$= \lambda g.g(Ap_{i-3}(\lambda f.f^{i-5}(\perp), g))$

$= \lambda g.g([g/f]_{A_{i-5}}\ f^{i-5}(\perp)) = \lambda g.g([g/f]f^{i-5}(\perp))$, by 3.3.4.

$= \lambda g.g(g^{i-5}(\perp)) = \lambda g.g^{i-4}(\perp) = \lambda f.f^{i-4}(\perp) \in E_{i+1}.$

-∴, we have the claim.

-∴, by induction, we have $P(i)$, $\forall i \geq 6$.

-∴, $E[\![Y_1]\!] = \bigsqcup_{i=6}^{\infty} \phi_{i,\infty} \circ [\lambda y.\lambda f.f(yf)/f]_{A_{i-2}}\ f^{i-2}(\perp)$, since the set forms a chain.

$$= \bigsqcup_{i=6}^{\infty} \phi_{i,\infty}(\lambda f.f^{i-5}(\perp)) = \bigsqcup_{i=0}^{\infty} \phi_{i+2,\infty}(\lambda f.f^{i}(\perp))$$

$$= E[\![Y]\!].$$

-∴, $E[\![Y_i]\!] = E[\![Y_{i-1}G]\!] = E[\![Y_0 GG \ldots\ldots G]\!]$

$= E[\![Y_0]\!]$, by repeated use of above.

$= E[\![Y]\!].$

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4.1:Change of Variables Operators :-

4.1.0:DEF:-

The CHANGE OF VARIABLES OPERATORS, (COVO), form the following set :-

$$\{[a_0/a_1][a_2/a_3]\ldots\ldots[a_{n-1}/a_n] \mid (n \geq 0) \wedge (a_i \text{ is a variable for all } 0 \leq i \leq n)\}.$$

4.1.1:REMARK:-

The above definition is deliberately ambiguous in that it refers to operators both in the λ-calculus and on E_∞. The context in which they are used should resolve this problem. In the following lemmas, if the context is λ-calculus, the "=" should be "$\xrightarrow{\alpha}$".

They are just the set of finite (possibly zero) compositions of single change of variable operators, [a/b]. We shall use symbols like χ, χ_i etc... to represent them.

4.1.2:LEMMA:-

COVO is closed under composition and forms a semi-group with identity [a/a].

Proof:-

-Clearly, COVO is closed under composition and composition of operators is associative.

-Also, $[a/a]\circ\chi = \chi = \chi\circ[a/a]$, either by 2.2.8(ii) if we are in E_∞, or clearly if in λ-calculus.

-Note that [a/a] = [b/b] = the zero length element. We will call this element the null element.

-Note also that COVO is not commutative and is not a group, since [a/b][b/a] = [a/a] only if b is not free in the element being operated on.

$\ddagger$

4.1.3:DEF:-

Let $\chi = [a_0/a_1][a_2/a_3]\ldots\ldots[a_{n-1}/a_n] \in$ COVO. Let x be a variable. Then, x is NOT IN χ if $x \neq a_i$, for all $0 \leq i \leq n$.

4.1.4:LEMMA:-

(i) $\chi(\lambda y.\varepsilon) = \lambda x.\chi[x/y]^{\varepsilon}$, where x is not in χ and not free in ε.

(ii) $\chi(\omega(\varepsilon)) = (\chi(\omega))(\chi(\varepsilon))$, restricted to the λ-calculus.

Proof:-

(i) -Let $\chi = [a_0/a_1]\ldots\ldots[a_{n-1}/a_n]$ and x be not in χ and not free in ε.

-Then, $\chi(\lambda y.\varepsilon) = [a_0/a_1]\ldots\ldots[a_{n-1}/a_n](\lambda y.\varepsilon)$

$= [a_0/a_1]\ldots.[a_{n-3}/a_{n-2}]\lambda x.[a_{n-1}/a_n][x/y]^{\varepsilon}$,

by 3.4.10(iv) if in E_∞, or clearly if in λ-calculus.

$= [a_0/a_1]\ldots.[a_{n-5}/a_{n-4}]\lambda x.[a_{n-3}/a_{n-2}][a_{n-1}/a_n][x/y]$

by 3.4.10(iii) if in E_∞, or clearly if in λ-calculus.

.

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$= \lambda x.[a_0/a_1]\ldots\ldots[a_{n-1}/a_n][x/y]\varepsilon$, repeating the above.

$= \lambda x.\chi[x/y]\varepsilon$.

(ii) -Trivial.

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4.1.5:COR:-

(i) $[\delta/x]\chi(\lambda y.\varepsilon) = \lambda z.[\delta/x]\chi[z/y]\varepsilon$, where $z \neq x$, not in χ and not free in δ, ε.

(ii) Restricting ourselves to the λ-calculus,

$[\delta/x]\chi(\omega(\varepsilon)) = ([\delta/x]\chi(\omega))([\delta/x]\chi(\varepsilon))$.

(iii) Restricting ourselves to E_∞,

$\chi\circ\mathbf{Ap}(\varepsilon,\delta) = \mathrm{Ap}(\chi(\varepsilon),\chi(\delta))$.

Proof:-

(i) -$[\delta/x]\chi(\lambda y.\varepsilon) = [\delta/x](\lambda z.\chi[z/y]\varepsilon)$, by 4.1.4(i).

$= \lambda z.[\delta/x]\chi[z/y](\varepsilon)$, by 3.4.10(iii) if in E_∞, or clearly if in λ-calculus.

(ii) -Trivial.

(iii) -Trivial, by repeated use of 3.4.4(iv).

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4.1.6:REMARK:-

We have introduced COVO to deal with a technical problem that occurs later. We will be doing structural inductions on the formation of expressions, $\varepsilon \in$ EXP, in which terms like $[\delta/x]\varepsilon$ are wanted in the hypothesis $\mathcal{A}[\![\varepsilon]\!]$. When we try to prove the case $(\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda y.\varepsilon]\!])$, we start off with the term $[\delta/x](\lambda y.\varepsilon)$. But this is $\lambda z.[\delta/x][z/y]\varepsilon$, for some $z \neq x$ and not free in δ or ε, and we cannot use $\mathcal{A}[\![\varepsilon]\!]$ since we have $[\delta/x][z/y]\varepsilon$ and not $[\delta/x]\varepsilon$.

Therefore, we replace $[\delta/x]\varepsilon$ with $[\delta/x]\chi(\varepsilon)$ in $\mathcal{A}[\![\varepsilon]\!]$ - i.e. an hypothesis about ε for all δ,x and χ. Then, when we start with $[\delta/x]\chi(\lambda y.\varepsilon)$, we get $\lambda z.[\delta/x]\chi[z/y]\varepsilon$, by 4.1.5(i), where $z \neq x$, not in χ and not free in δ or ε. But, now we can apply $\mathcal{A}[\![\varepsilon]\!]$ to $[\delta/x]\chi[z/y]\varepsilon$, since $\chi\circ[z/y] \in$ COVO, by 4.1.2.

The presence of χ in $\mathcal{A}[\![\varepsilon]\!]$ does not disturb too much the other two parts of the inductions, namely $\mathcal{A}[\![y]\!]$ and $(\mathcal{A}[\![\omega]\!] \wedge \mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\omega(\varepsilon)]\!])$.

We cannot extend the whole of 4.1.5(ii) to E_∞ because of the missing identity of remark 3.4.11, but have to be content with its part (iii).

4.2:The Semantical Relations $\equiv$ and $\sqsubseteq$:-

4.2.0:DEF:-

Let $\varepsilon, \varepsilon' \in$ EXP. Then,

$$\varepsilon \equiv \varepsilon' \text{ if } \mathcal{E}[\![\varepsilon]\!] = \mathcal{E}[\![\varepsilon']\!]$$

$$\text{and } \varepsilon \sqsubseteq \varepsilon' \text{ if } \mathcal{E}[\![\varepsilon]\!] \sqsubseteq \mathcal{E}[\![\varepsilon']\!].$$

4.2.1:LEMMA:-

$\equiv$ is an equivalence relation and $\sqsubseteq$ is a partial ordering (modulo $\equiv$) on EXP. Also,

$$(\varepsilon \sqsubseteq \varepsilon') \Rightarrow (\lambda x.\varepsilon \sqsubseteq \lambda x.\varepsilon')$$

$$\text{and } (\varepsilon \sqsubseteq \varepsilon') \wedge (\delta \sqsubseteq \delta') \Rightarrow (\varepsilon(\delta) \sqsubseteq \varepsilon'(\delta')),$$

together with similar results for $\equiv$. Finally, $\equiv$ and $\sqsubseteq$ are substitutive - i.e. for any context $C[\ ]$,

$$(\varepsilon \sqsubseteq \varepsilon') \Rightarrow (C[\varepsilon] \sqsubseteq C[\varepsilon'])$$

$$\text{and } (\varepsilon \equiv \varepsilon') \Rightarrow (C[\varepsilon] \equiv C[\varepsilon']).$$

Proof:-

-Trivial, because of the lattice properties of E_∞ and the monotonicity of Ap. Then, an easy induction on the structure of contexts gets the substitutivity.

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4.2.2:REMARK:-

The fact that the semantical function $\mathcal{E}$ produces substitutive relations is very important - see remark 0.7.4. Later we will define another semantical function, $\mathcal{V}$, to E_∞ whose semantics, while easily shown to give a β-model, are very difficult to prove substitutive. The trick will be to prove $\mathcal{E} = \mathcal{V}$.

Next, we rewrite the results at the end of the last chapter.

4.2.3:THEOREM:-

(i) x is not free in $\varepsilon \Rightarrow$ x is not free in $\mathcal{E}[\![\varepsilon]\!]$.

(ii) x is not free in $\varepsilon \Rightarrow [\mathcal{E}[\![\varepsilon']\!]/x]\mathcal{E}[\![\varepsilon]\!] = \mathcal{E}[\![\varepsilon]\!]$.

(iii) x is not free in $\varepsilon \Rightarrow (\lambda x.\varepsilon)(\varepsilon') \equiv \varepsilon$.

(iv) $$[\mathcal{E}[\![\varepsilon]\!]/x]\mathcal{E}[\![y]\!] = \begin{cases} \mathcal{E}[\![y]\!], \text{ if } x \neq y \\ \mathcal{E}[\![\varepsilon]\!], \text{ if } x = y \end{cases}.$$

(v) $(z \neq x)_\wedge(z$ is not free in $\varepsilon',\varepsilon) \Rightarrow$

$$([\mathcal{E}[\![\varepsilon']\!]/x]\mathcal{E}[\![\lambda y.\varepsilon]\!] = \lambda z.[\mathcal{E}[\![\varepsilon']\!]/x][z/y]\mathcal{E}[\![\varepsilon]\!]).$$

(vi) $\mathcal{E}[\![[x/y]\varepsilon]\!] = [x/y]\mathcal{E}[\![\varepsilon]\!]$.

(vii) $\varepsilon \xrightarrow{\alpha} \varepsilon' \Rightarrow \varepsilon \equiv \varepsilon'$.

(viii) $(\varepsilon \in \lambda I.EXP)_\wedge(x$ is not free in $\varepsilon) \Rightarrow (\varepsilon \equiv \lambda x.\varepsilon(x))$.

(ix) Let z be not free in $\varepsilon \in EXP$. Then,

$$\begin{aligned} \lambda z.\varepsilon(z) &\equiv (\lambda y.\lambda x.yx)(\lambda z.\varepsilon(z)) \\ &\equiv (\lambda y.\lambda x.yx)(\varepsilon). \end{aligned}$$

Proof:-

(i) -By structural induction on ε, using definition 0.3.1 and lemma 3.4.6.

(ii) -By part (i) and 3.4.4(x).

(iii) $$-\mathcal{E}[\![(\lambda x.\varepsilon)(\varepsilon')]\!] = Ap(\lambda x.\mathcal{E}[\![\varepsilon]\!],\mathcal{E}[\![\varepsilon']\!])$$

$$= \mathcal{E}[\![\varepsilon]\!], \text{ by part (i) and 3.4.9.}$$

(iv) -By 3.4.10(i).

(v) -By part (i) and 3.4.10(iv).

(vi) -This is proved by structural induction on ε. However, because of the bound variables problem described in 4.1.6, we introduce elements of COVO into the hypothesis.

-Let $\mathcal{A}[\![\varepsilon]\!] \equiv (\mathcal{E}[\![\chi(\varepsilon)]\!] = \chi \circ \mathcal{E}[\![\varepsilon]\!])$.

-Claim: $\mathcal{A}[\![x]\!]$:-

-Suppose $\chi = [a_0/b_0][a_1/b_1]......[a_n/b_n]$.

-Let $\chi_i = [a_i/b_i]......[a_n/b_n]$, for $0 \leq i \leq n$.

-Then, $E[\![\chi(x)]\!] = E[\![[a_0/b_0]\chi_1(x)]\!]$

$$= \left\{\begin{array}{l} E[\![\chi_1(x)]\!], \text{ if } b_0 \neq \chi_1(x). \\ a_0, \text{ if } b_0 = \chi_1(x). \end{array}\right\}$$

$= [a_0/b_0]E[\![\chi_1(x)]\!]$, by part (iv) above.

$= [a_0/b_0][a_1/b_1]E[\![\chi_2(x)]\!]$, similarly.

$$\vdots$$

$= \chi \circ E[\![x]\!]$, by repeating the above process.

-<u>Claim:</u> $A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!]$:-

-Choose y not in χ and not free in ε.

-Then, $E[\![\chi(\lambda x.\varepsilon)]\!] = E[\![\lambda y.\chi[y/x]\varepsilon]\!]$

$= \lambda y.E[\![\chi[y/x]\varepsilon]\!]$, by S2.

$= \lambda y.\chi[y/x]E[\![\varepsilon]\!]$, by $A[\![\varepsilon]\!]$.

$= \chi(\lambda x.E[\![\varepsilon]\!])$, by 4.1.4(i).

$= \chi \circ E[\![\lambda x.\varepsilon]\!]$, by S2.

-<u>Claim:</u> $A[\![\omega]\!] \wedge A[\![\varepsilon]\!] \Rightarrow A[\![\omega(\varepsilon)]\!]$:-

-Now, $E[\![\chi(\omega(\varepsilon))]\!] = E[\![(\chi(\omega))(\chi(\varepsilon))]\!]$, by 4.1.4(ii).

$= Ap(E[\![\chi(\omega)]\!], E[\![\chi(\varepsilon)]\!])$, by S3.

$= Ap(\chi \circ E[\![\omega]\!], \chi \circ E[\![\varepsilon]\!])$, by $A[\![\omega]\!] \wedge A[\![\varepsilon]\!]$.

$= \chi \circ Ap(E[\![\omega]\!], E[\![\varepsilon]\!])$, by 4.1.5(iii).

$= \chi \circ E[\![\omega(\varepsilon)]\!]$, by S3.

-Hence, $(\forall \varepsilon \in EXP)A[\![\varepsilon]\!]$.

(vii) -Let y be not free in ε.

-<u>Claim:</u> $\lambda x.\varepsilon \equiv \lambda y.[y/x]\varepsilon$:-

-Now, y is not free in $E[\![\varepsilon]\!]$, by part (i).

-So, $E[\![\lambda y.[y/x]\varepsilon]\!] = \lambda y.E[\![[y/x]\varepsilon]\!]$

$= \lambda y.[y/x]E[\![\varepsilon]\!]$, by part (vi).

$= \lambda x.\mathcal{E}[\![\varepsilon]\!]$, by 3.4.7.

$= \mathcal{E}[\![\lambda x.\varepsilon]\!]$.

-Suppose $\varepsilon \xrightarrow{\alpha} \varepsilon'$ by just one α-conversion. Then, ε must be of the form $C[\lambda x.\delta]$ and ε' must be of the form $C[\lambda y.[y/x]\delta]$, where y is not free in δ. Therefore, $\varepsilon \equiv \varepsilon'$, by above and 4.2.1.

-So, if $\varepsilon \xrightarrow{\alpha} \varepsilon'$ by an arbitrary number of α-conversions, we have $\varepsilon \equiv \varepsilon'$, since $\equiv$ is transitive, by 4.2.1 again.

(viii) -Let $\varepsilon = \lambda y.\varepsilon'$ and x be not free in ε.

-Choose $z \neq x$ and not free in ε'.

-Then, $\varepsilon \equiv \lambda z.[z/y]\varepsilon'$, by part (vii).

-Now, x is not free in $[z/y]\varepsilon'$.

-$\therefore$, $\mathcal{E}[\![\lambda x.\varepsilon(x)]\!] = \mathcal{E}[\![\lambda x.(\lambda z.[z/y]\varepsilon')(x)]\!]$, by 4.2.1.

$= \lambda x.\mathrm{Ap}(\lambda z.\mathcal{E}[\![[z/y]\varepsilon']\!],x)$

$= \lambda x.[x/z]\mathcal{E}[\![[z/y]\varepsilon']\!]$, by 3.4.8.

$= \lambda z.\mathcal{E}[\![[z/y]\varepsilon']\!]$, by part (i) and 3.4.7.

$= \mathcal{E}[\![\lambda z.[z/y]\varepsilon']\!]$

$= \mathcal{E}[\![\varepsilon]\!]$, by part (vii).

-$\therefore$, $\lambda x.\varepsilon(x) \equiv \varepsilon$.

(ix) -We will show first that $\lambda z.\varepsilon(z) \equiv (\lambda y.\lambda x.yx)(\varepsilon)$.

-Now, $\lambda x.yx \xrightarrow{\alpha} \lambda z.yz$.

-So, $\mathcal{E}[\![(\lambda y.\lambda x.yx)(\varepsilon)]\!] = \mathcal{E}[\![(\lambda y.\lambda z.yz)(\varepsilon)]\!]$, by part (vii).

$= \mathrm{Ap}(\lambda y.\lambda z.\mathcal{E}[\![yz]\!],\mathcal{E}[\![\varepsilon]\!])$

$= [\mathcal{E}[\![\varepsilon]\!]/y]\lambda z.\mathcal{E}[\![yz]\!]$

$= \lambda z.[\mathcal{E}[\![\varepsilon]\!]/y]\mathcal{E}[\![yz]\!]$, by 3.4.10(iii) and part (i

$= \lambda z.[\mathcal{E}[\![\varepsilon]\!]/y]<\bot,\bot,yz,yz,\ldots\ldots>$. by 4.0.3.

$= \lambda z.\bigsqcup \phi_{i,\infty}\circ[\varepsilon_i/y]yz$, where $\varepsilon_i = \phi_{\infty,i}\circ\mathcal{E}[\![\varepsilon]\!]$.

$= \lambda z.\bigsqcup \phi_{i,\infty}\circ\mathrm{Ap}_{i-1}([\varepsilon_{i-1}/y]y,[\varepsilon_{i-1}/y]z)$

$= \lambda z.\bigsqcup \phi_{i,\infty}\circ\mathrm{Ap}_{i-1}(\varepsilon_{i-1},z) = \lambda z.\mathrm{Ap}(\mathcal{E}[\![\varepsilon]\!],z)$

$= \lambda z.\mathcal{E}[\![\varepsilon(z)]\!] = \mathcal{E}[\![\lambda z.\varepsilon(z)]\!]$.

-Choose a not free in ε.

-Then, $(\lambda y.\lambda x.yx)(\lambda z.\varepsilon(z)) \equiv (\lambda y.\lambda x.yx)(\lambda a.\varepsilon(a))$, by part (vii).

$\equiv \lambda z.(\lambda a.\varepsilon(a))(z)$, by above, since z is not free in $\lambda a.\varepsilon(a)$.

$\equiv \lambda a.\varepsilon(a)$, by part (viii), since z is not free in $\lambda a.\varepsilon(a)$.

$\equiv \lambda z.\varepsilon(z)$, by part (vii).

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4.2.4:LEMMA:-

(i) $I(\varepsilon') \equiv \varepsilon'$.

(ii) $K(\varepsilon)(\varepsilon') \equiv \varepsilon$.

Proof:-

(i) -$\mathcal{E}\llbracket I(\varepsilon')\rrbracket = Ap(\mathcal{E}\llbracket\lambda x.x\rrbracket,\mathcal{E}\llbracket\varepsilon'\rrbracket)$

$= Ap(\lambda x.x,\mathcal{E}\llbracket\varepsilon'\rrbracket)$

$= \mathcal{E}\llbracket\varepsilon'\rrbracket$, by 3.4.12(i).

(ii) -$\mathcal{E}\llbracket K(\varepsilon)(\varepsilon')\rrbracket = Ap(Ap(\mathcal{E}\llbracket\lambda x.\lambda y.x\rrbracket,\mathcal{E}\llbracket\varepsilon\rrbracket),\mathcal{E}\llbracket\varepsilon'\rrbracket)$

$= Ap(Ap(\lambda x.\lambda y.x,\mathcal{E}\llbracket\varepsilon\rrbracket),\mathcal{E}\llbracket\varepsilon'\rrbracket)$

$= \mathcal{E}\llbracket\varepsilon\rrbracket$, by 3.4.12(ii), since there are only finitely many free variables in $\varepsilon \in EXP$.

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4.2.5:REMARK:-

So we see that $<E_\infty,\mathcal{E}>$ models α-conversion and looks promising as a model for β-conversion. We do not yet have,

$$\varepsilon \xrightarrow{\beta} \varepsilon' \Rightarrow \varepsilon \equiv \varepsilon',$$

since, although,

$$\mathcal{E}\llbracket(\lambda x.\varepsilon)(\varepsilon')\rrbracket = Ap(\lambda x.\mathcal{E}\llbracket\varepsilon\rrbracket,\mathcal{E}\llbracket\varepsilon'\rrbracket)$$

$$= [\mathcal{E}\llbracket\varepsilon'\rrbracket/x]\mathcal{E}\llbracket\varepsilon\rrbracket, \text{ by 3.4.8,}$$

we do not have,

$$[\mathcal{E}\llbracket\varepsilon'\rrbracket/x]\mathcal{E}\llbracket\varepsilon\rrbracket = \mathcal{E}\llbracket[\varepsilon'/x]\varepsilon\rrbracket,$$

since the structural induction on ε fails to go through in the case $\varepsilon = \eta(\delta)$, because of the missing identity mentioned in 3.4.11.

For the same reason, we cannot yet prove that,

$$S(\varepsilon)(\gamma)(\delta) \equiv (\varepsilon\delta)(\gamma\delta).$$

We claim, however, that these results are true and that we will prove them.

By 4.2.3(viii), we see that $\langle E_\infty, \mathcal{E}\rangle$ models η-conversion on abstractions. This is also what happens in Scott's model + atoms (0.7.22) and with the $P(\omega)$ model (0.7.26).

4.2.3(ix) is just playing around with the "apply" combinator, $\lambda y.\lambda x.yx$. We see that η-redexes are fixed points of "apply" and that η-redexes are semantically equivalent to their contractums when operated on by "apply". These results are necessary if $\langle E_\infty, \mathcal{E}\rangle$ is going to be a β-model.

Next, we are going to define another semantical function, $\tilde{\mathcal{E}}$. This will be closely connected with the process of taking "approximants" of λ-expressions (0.7.17).

4.3:Approximate Application and Substitution:-

4.3.0:DEF:-

$$[\varepsilon_i/x]^{\sim}_{I} := [\varepsilon_i/x]_{I}$$
$$[\varepsilon_1/x]^{\sim}_{E_0} := [\varepsilon_1/x]^{\sim}_{I}$$
$$[\varepsilon_2/x]^{\sim}_{A_0} := [\varepsilon_2/x]^{\sim}_{I}$$

For $i \geq 2$,

$$[\varepsilon_i'/x]^{\sim}_{E_{i-1}} : E_{i-1} \longrightarrow E_i$$
$$\begin{Bmatrix} \lambda y.\varepsilon_{i-2} \\ \alpha_{i-2} \end{Bmatrix} \longmapsto \begin{Bmatrix} \lambda z.[\varepsilon_{i-1}'/x]^{\sim}_{E_{i-2}}[z/y]\varepsilon_{i-2} \\ [\varepsilon_i'/x]^{\sim}_{A_{i-2}}\alpha_{i-2} \end{Bmatrix},$$

where $z \neq x$ and is not free in $\varepsilon_{i-1}', \varepsilon_{i-2}$.

For $i \geq 3$,

$$[\varepsilon_i'/x]^{\sim}_{A_{i-2}} : A_{i-2} \longrightarrow E_i$$

$$\begin{Bmatrix} y \\ \alpha_{i-3}(\varepsilon_{i-2}) \end{Bmatrix} \longmapsto \left\{\begin{array}{l} [\varepsilon_i'/x]^{\sim}_I y \\ \widetilde{Ap}_{i-1}([\varepsilon_{i-1}'/x]^{\sim}_{A_{i-3}} \alpha_{i-3}, \\ \qquad [\varepsilon_{i-1}'/x]^{\sim}_{E_{i-2}} \varepsilon_{i-2}) \end{array}\right\}.$$

For $i \geq 1$,

$$\widetilde{Ap}_i : \quad E_i \times E_i \longrightarrow E_{i+1}$$

$$\left(\begin{Bmatrix} \lambda x.\varepsilon_{i-1} \\ \alpha_{i-1} \end{Bmatrix}, \varepsilon_i'\right) \longmapsto \begin{Bmatrix} \perp \\ \alpha_{i-1}(\varepsilon_i') \end{Bmatrix}.$$

<u>4.3.1:LEMMA:-</u>

(i) The above maps are all monotone, and hence continuous, with $[\varepsilon_i/x]^{\sim}$ monotone with respect to ε_i as well.

(ii) $[\varepsilon_i/x]^{\sim}$ is doubly strict and,

$$\widetilde{Ap}_i\left(\begin{Bmatrix} \top \\ \perp \end{Bmatrix}, \varepsilon_i'\right) = \begin{Bmatrix} \top \\ \perp \end{Bmatrix}.$$

(iii) $[\varepsilon_i/x]^{\sim} \sqsubseteq [\varepsilon_i/x]$ and $\widetilde{Ap}_i \sqsubseteq Ap_i$.

(iv) $[a/b]\circ\widetilde{Ap}_i(\varepsilon_i, \varepsilon_i') = \widetilde{Ap}_i([a/b]\varepsilon_i, [a/b]\varepsilon_i')$.

(v) $(x \neq a,b) \wedge (b$ is not free in $\varepsilon_i) \Rightarrow$

$$[\varepsilon_i/x]^{\sim}[a/b] = [a/b][\varepsilon_i/x]^{\sim}.$$

(vi) (x is not free in $\begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}$) $\Rightarrow$ $\left([\varepsilon_i'/x]^{\sim}[x/y]\begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}\right.$

$$\left. = [\varepsilon_i'/y]^{\sim}\begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}\right).$$

(vii) (a is not free in $\varepsilon_i', \begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}$) $\Rightarrow$

(a is not free in $[\varepsilon_i'/x]^{\sim}\begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}$).

(viii) (a is not free in ε_i') $\Rightarrow$

(a is not free in $[\varepsilon_i'/a]^{\sim}\begin{Bmatrix} \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}$).

(ix) (a is not free in $\varepsilon_i, \varepsilon_i'$) => (a is not free in $\widetilde{Ap}_i(\varepsilon_i, \varepsilon_i')$).

(x) (x is not free in $\begin{Bmatrix}\varepsilon_{i-1}\\ \alpha_{i-2}\end{Bmatrix}$) => ($[\varepsilon_i'/x]^{\sim}\begin{Bmatrix}\varepsilon_{i-1}\\ \alpha_{i-2}\end{Bmatrix} = \begin{Bmatrix}\phi_{i-1,i}(\varepsilon_{i-1})\\ \theta_{i-2,i-1}(\alpha_{i-2})\end{Bmatrix}$).

(xi) $[x/y]^{\sim}_{E_{i-1}} = \phi_{i-1,i}\circ[x/y]$.

$[x/y]^{\sim}_{A_{i-2}} = \theta_{i-2,i-1}\circ[x/y]$.

(xii) The sets $\{\phi_{i,\infty}\circ[\varepsilon_i'/x]^{\sim}\varepsilon_{i-1} \mid i \geq 1\}$, $\{\phi_{i,\infty}\circ[\varepsilon_i'/x]^{\sim}\alpha_{i-2} \mid i \geq 2\}$ and $\{\phi_{i+1,\infty}\circ\widetilde{Ap}_i(\varepsilon_i, \varepsilon_i') \mid i \geq 1\}$ are directed chains in E_∞.

(xiii) $(y \neq x)\wedge$(y is not free in ε_i') =>

$$([\varepsilon_i'/x]^{\sim}(\lambda y.\varepsilon_{i-2}) = \lambda y.[\varepsilon_{i-1}'/x]^{\sim}\varepsilon_{i-2}).$$

Proof:-

(i),(ii) and (iii) -By a trivial induction of type 3.0.5.

(iv) -Clear.

(v) -Another trivial 3.0.5-induction, using part (iv).

(vi),(vii),(viii),(ix),(x),(xi) and (xii) -All trivial 3.0.5-inductions.

(xiii) -Choose $z \neq x$ and not free in $\varepsilon_{i-1}', \varepsilon_{i-2}$.

-Then, $[\varepsilon_i'/x]^{\sim}(\lambda y.\varepsilon_{i-2}) = \lambda z.[\varepsilon_{i-1}'/x]^{\sim}[z/y]\varepsilon_{i-2}$

$= \lambda z.[z/y][\varepsilon_{i-1}'/x]^{\sim}\varepsilon_{i-2}$, by part (v).

$= \lambda y.[\varepsilon_{i-1}'/x]^{\sim}\varepsilon_{i-2}$, since z is not free in $[\varepsilon_{i-1}'/x]^{\sim}\varepsilon_{i-2}$, by part (vii).

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4.3.2:DEF:-

$$[\varepsilon'/x]^{\sim}_{E} := \bigsqcup_{i=1}^{\infty}\phi_{i,\infty}\circ[\varepsilon_i'/x]^{\sim}_{E_{i-1}}\circ\phi_{\infty,i-1} \in [E_\infty \to E_\infty].$$

$$[\varepsilon'/x]^{\sim}_{A} := \bigsqcup_{i=2}^{\infty}\phi_{i,\infty}\circ[\varepsilon_i'/x]^{\sim}_{A_{i-2}}\circ\theta_{\infty,i-2} \in [A_\infty \to E_\infty].$$

$$\widetilde{Ap} := \bigsqcup_{i=1}^{\infty}\phi_{i+1,\infty}\circ\widetilde{Ap}_i\circ(\phi_{\infty,i}\times\phi_{\infty,i}) \in [E_\infty\times E_\infty \to E_\infty].$$

4.3.3:LEMMA:-

(i) The above maps are all continuous, with $[\varepsilon/x]^{\sim}$ monotone with respect to ε as well.

(ii) $[\varepsilon/x]^{\sim}$ is doubly strict and $\widetilde{Ap}(\left\{\begin{matrix}\top\\ \bot\end{matrix}\right\},\varepsilon') = \left\{\begin{matrix}\top\\ \bot\end{matrix}\right\}$.

(iii) $\widetilde{Ap} \sqsubseteq Ap$, $[\varepsilon/x]^{\sim}_{E} \sqsubseteq [\varepsilon/x]_{E}$ and $[\varepsilon/x]^{\sim}_{A} \sqsubseteq [\varepsilon/x]_{A}$.

(iv) $[a/b]\circ\widetilde{Ap}(\varepsilon,\varepsilon') = \widetilde{Ap}([a/b]\varepsilon,[a/b]\varepsilon')$.

(v) $(x \neq a,b)\wedge$(b is not free in ε) $\Rightarrow$

$$([\varepsilon/x]^{\sim}[a/b] = [a/b][\varepsilon/x]^{\sim}).$$

(vi) (x is not free in $\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\}$) $\Rightarrow$ $([\varepsilon'/x]^{\sim}[x/y]\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\} = [\varepsilon'/y]^{\sim}\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\})$.

(vii) (a is not free in $\varepsilon',\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\}$) $\Rightarrow$

$$(\text{a is not free in } [\varepsilon'/x]^{\sim}\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\}).$$

(viii) (a is not free in ε') $\Rightarrow$ (a is not free in $[\varepsilon'/a]^{\sim}\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\}$).

(ix) (a is not free in ε,ε') $\Rightarrow$ (a is not free in $\widetilde{Ap}(\varepsilon,\varepsilon')$).

(x) (x is not free in $\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\}$) $\Rightarrow$ $([\varepsilon'/x]^{\sim}\left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\} = \left\{\begin{matrix}\varepsilon\\ \alpha\end{matrix}\right\})$.

(xi) $[x/y]^{\sim}_{E} = [x/y]_{E}$ and $[x/y]^{\sim}_{A} = \pi_2^{-1}\circ[x/y]_{A}$.

(xii) We can compute $\widetilde{Ap}$ instantly by,

$$\widetilde{Ap}(\varepsilon,\delta) = \left\{\begin{matrix}\bot, \text{ if } \pi_2(\varepsilon) = \bot.\\ Ap(\varepsilon,\delta), \text{ if not.}\end{matrix}\right\}$$
$$= \pi_2^{-1}\circ\psi_2^{-1}\circ\xi(\pi_2(\varepsilon),\delta),$$

where,

$$\begin{aligned}\xi : A_\infty \times E_\infty &\longrightarrow A_\infty(E_\infty)\\ (\alpha,\varepsilon) &\longmapsto \langle\alpha_i(\varepsilon_{i+1})\rangle_{i=0}^{\infty}.\end{aligned}$$

(xiii) $(\varepsilon \neq \top) \Rightarrow (\widetilde{Ap}(\varepsilon,\delta) \neq \top)$.

Proof:-

(i),(ii),(iii),(iv),(v),(vi),(x) and (xi) -Trivial, from the corresponding parts of 4.3.1, noting 4.3.1(xii).

(vii),(viii) and (ix) -Also trivial, noting remark 3.4.3.

(xii) -By observation of the definition.

(xiii) -Trivial, by part (xii).

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4.3.4:LEMMA:-

(i) $[\varepsilon'/x]^{\sim}_{E}(y) = \left\{\begin{matrix} \varepsilon', & \text{if } x = y \\ y\ , & \text{if } x \neq y \end{matrix}\right\}.$

(ii) $(z \neq x) \wedge (z \text{ is not free in } \varepsilon',\varepsilon) \Rightarrow$

$$([\varepsilon'/x]^{\sim}_{E}(\lambda y.\varepsilon) = \lambda z.[\varepsilon'/x]^{\sim}_{E}[z/y]\varepsilon).$$

(iii) $(y \neq x) \wedge (y \text{ is not free in } \varepsilon') \Rightarrow$

$$([\varepsilon'/x]^{\sim}_{E}(\lambda y.\varepsilon) = \lambda y.[\varepsilon'/x]^{\sim}_{E}\varepsilon).$$

(iv) $[\varepsilon'/x]^{\sim}_{E}\widetilde{Ap}(\varepsilon,\delta) = \widetilde{Ap}([\varepsilon'/x]^{\sim}_{E}\varepsilon,[\varepsilon'/x]^{\sim}_{E}\delta).$

Proof:-

(i) and (ii) -Trivial, from their definitions.

(iii) -Trivial, by 4.3.1(xiii).

(iv) -If $\pi_2(\varepsilon) = \bot$, then $\pi_2([\varepsilon'/x]^{\sim}_{E}\varepsilon) = \bot$, also.

-∴, L.H.S. = $\bot$ = R.H.S., by 4.3.3(xii) and (ii).

-If $\pi_2(\varepsilon) \neq \bot$, then $\varepsilon = \langle\alpha_0,\langle\alpha_i\rangle_{i=0}^{\infty}\rangle$.

-∴, $\widetilde{Ap}([\varepsilon'/x]^{\sim}_{E}\varepsilon,[\varepsilon'/x]^{\sim}_{E}\delta)$

$$= \widetilde{Ap}(\bigsqcup_{i=3}^{\infty}\phi_{i-1,\infty}\circ[\varepsilon'_{i-1}/x]^{\sim}\alpha_{i-3},\bigsqcup_{j=3}^{\infty}\phi_{j-1,\infty}\circ[\varepsilon'_{j-1}/x]^{\sim}\delta_{j-2})$$

$$= \bigsqcup_{i=3}^{\infty}\ \bigsqcup_{j=3}^{\infty}\widetilde{Ap}(\phi_{i-1,\infty}\circ[\varepsilon'_{i-1}/x]^{\sim}\alpha_{i-3},\phi_{j-1,\infty}\circ[\varepsilon'_{j-1}/x]^{\sim}\delta_{j-2}),$$

by 4.3.1(xii) and 4.3.3(i).

$$= \bigsqcup_{i=3}^{\infty}\widetilde{Ap}(\phi_{i-1,\infty}\circ[\varepsilon'_{i-1}/x]^{\sim}\alpha_{i-3},\phi_{i-1,\infty}\circ[\varepsilon'_{i-1}/x]^{\sim}\delta_{i-2})$$

$$= \bigsqcup_{i=3}^{\infty}\phi_{i,\infty}\circ\widetilde{Ap}_{i-1}([\varepsilon'_{i-1}/x]^{\sim}\alpha_{i-3},[\varepsilon'_{i-1}/x]^{\sim}\delta_{i-2}), \text{ from}$$

the definition of $\widetilde{Ap}$ and $\widetilde{Ap}_{i-1}$.

$$= \bigsqcup_{i=3}^{\infty}\phi_{i,\infty}\circ[\varepsilon'_{i}/x]^{\sim}\alpha_{i-3}(\delta_{i-2}) = [\varepsilon'/x]^{\sim}_{E}\langle\bot,\bot,\langle\alpha_i(\delta_{i+1})\rangle_{i=0}^{\infty}\rangle$$

$$= [\varepsilon'/x]^{\sim}_{E}\widetilde{Ap}(\varepsilon,\delta), \text{ by 4.3.3(xii)}.$$

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4.3.5:REMARK:-

Part (iv) of the above is analogous to the identity that is missing for the "un-tilda'd" operators. So, these operators are better behaved in that respect.

We needed part (iii) for technical reasons in 4.4.3, below, and this was why 4.3.1 was so lengthy.

4.4:The Approximate Semantics $\tilde{\tilde{E}}$,$\tilde{\tilde{\sqsubseteq}}$ and $\tilde{\cong}$:-

4.4.0:DEF:-

$$\tilde{\tilde{E}}[\![x]\!] = x \qquad (\tilde{S}1)$$

$$\tilde{\tilde{E}}[\![\lambda x.\varepsilon]\!] = \lambda x.\tilde{\tilde{E}}[\![\varepsilon]\!] \qquad (\tilde{S}2)$$

$$\tilde{\tilde{E}}[\![\varepsilon(\delta)]\!] = \widetilde{Ap}(\tilde{\tilde{E}}[\![\varepsilon]\!],\tilde{\tilde{E}}[\![\delta]\!]) \qquad (\tilde{S}3)$$

4.4.1:LEMMA:-

(i) $\tilde{\tilde{E}} \sqsubseteq E$.

(ii) (x is not free in $\varepsilon \in$ EXP) => (x is not free in $\tilde{\tilde{E}}[\![\varepsilon]\!]$).

(iii) $\tilde{\tilde{E}}[\![\varepsilon]\!] \neq \top$.

Proof:-

(i),(ii) and (iii)-By structural induction, using 4.3.3(iii),(ix) and (xiii).

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4.4.2:LEMMA:-

(i) $[\tilde{\tilde{E}}[\![\varepsilon]\!]/x]^{\sim}\tilde{\tilde{E}}[\![y]\!] = \begin{cases}\tilde{\tilde{E}}[\![\varepsilon]\!], \text{ if } x = y\\ \tilde{\tilde{E}}[\![y]\!], \text{ if } x \neq y\end{cases}$.

(ii) $(z \neq x)\wedge$(z is not free in ε',ε) =>

$$([\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\lambda y.\varepsilon]\!] = \lambda z.[\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}[z/y]\tilde{\tilde{E}}[\![\varepsilon]\!]).$$

(iii) $(y \neq x)\wedge$(y is not free in ε') =>

$$([\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\lambda y.\varepsilon]\!] = \lambda y.[\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\varepsilon]\!]).$$

(iv) $[\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\varepsilon(\delta)]\!] = \widetilde{Ap}([\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\varepsilon]\!],[\tilde{\tilde{E}}[\![\varepsilon']\!]/x]^{\sim}\tilde{\tilde{E}}[\![\delta]\!])$.

Proof:-

-By using 4.4.1(ii), this is just a restatement of 4.3.4.

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4.4.3:LEMMA:-

$\tilde{E}\llbracket[\delta/x]\chi(\varepsilon)\rrbracket = [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(\varepsilon)\rrbracket$.

Proof:-

-Let $A\llbracket\varepsilon\rrbracket$ be the above statement.

-Claim: $A\llbracket y\rrbracket$:-

$$-\tilde{E}\llbracket[\delta/x]\chi(y)\rrbracket = \left\{\begin{array}{l}\tilde{E}\llbracket\delta\rrbracket, \text{ if } x = \chi(y). \\ \tilde{E}\llbracket\chi(y)\rrbracket, \text{ if } x \neq \chi(y).\end{array}\right\}$$

$$= [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(y)\rrbracket, \text{ by 4.4.2(i).}$$

-Claim: $A\llbracket\varepsilon\rrbracket \Rightarrow A\llbracket\lambda y.\varepsilon\rrbracket$:-

-Choose $z \neq x$, not in χ and not free in δ,ε.

-Then, $\tilde{E}\llbracket[\delta/x]\chi(\lambda y.\varepsilon)\rrbracket = \tilde{E}\llbracket\lambda z.[\delta/x]\chi[z/y]\varepsilon\rrbracket$

$= \lambda z.\tilde{E}\llbracket[\delta/x]\chi[z/y]\varepsilon\rrbracket$, by $\tilde{S}2$.

$= \lambda z.[\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi[z/y]\varepsilon\rrbracket$, by $A\llbracket\varepsilon\rrbracket$.

$= [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\lambda z.\chi[z/y]\varepsilon\rrbracket$, by 4.4.2(iii)

$= [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(\lambda y.\varepsilon)\rrbracket$.

-Claim: $A\llbracket\omega\rrbracket \wedge A\llbracket\varepsilon\rrbracket \Rightarrow A\llbracket\omega(\varepsilon)\rrbracket$:-

-$\tilde{E}\llbracket[\delta/x]\chi(\omega(\varepsilon))\rrbracket = \tilde{E}\llbracket([\delta/x]\chi(\omega))([\delta/x]\chi(\varepsilon))\rrbracket$

$= \tilde{Ap}(\tilde{E}\llbracket[\delta/x]\chi(\omega)\rrbracket, \tilde{E}\llbracket[\delta/x]\chi(\varepsilon)\rrbracket)$, by $\tilde{S}3$.

$= \tilde{Ap}([\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(\omega)\rrbracket, [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(\varepsilon)\rrbracket)$, by $A\llbracket\omega\rrbracket \wedge A\llbracket\varepsilon\rrbracket$.

$= [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{Ap}(\tilde{E}\llbracket\chi(\omega)\rrbracket, \tilde{E}\llbracket\chi(\varepsilon)\rrbracket)$, by 4.4.2(iv).

$= [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\chi(\omega(\varepsilon))\rrbracket$, by $\tilde{S}3$.

-∴, by structural induction, $(\forall\varepsilon \in \mathrm{EXP})A\llbracket\varepsilon\rrbracket$.

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4.4.4:THEOREM:-

$\tilde{E}\llbracket[\delta/x]\varepsilon\rrbracket = [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\varepsilon\rrbracket$.

Proof:-

-This is a special case of 4.4.3, when χ is the null COVO.

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4.4.5:COR:-

$$\tilde{E}\llbracket[a/x]\varepsilon\rrbracket = [a/x]\tilde{E}\llbracket\varepsilon\rrbracket.$$

Proof:-

-By 4.3.3(xi),4.4.4 and $\tilde{S}1$.

‡

4.4.6:DEF:-

Let $\varepsilon,\varepsilon' \in$ EXP. Then,

(i) $\varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon'$ if $\tilde{E}\llbracket\varepsilon\rrbracket \sqsubseteq \tilde{E}\llbracket\varepsilon'\rrbracket$

and (ii) $\varepsilon \cong \varepsilon'$ if $\tilde{E}\llbracket\varepsilon\rrbracket = \tilde{E}\llbracket\varepsilon'\rrbracket$.

4.4.7:LEMMA:-

As in 4.2.1, $\cong$ is an equivalence relation and $\tilde{\sqsubseteq}$ is a partial ordering modulo the equivalence classes. Also,

$$(\varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon') \Rightarrow (\lambda x.\varepsilon \mathrel{\tilde{\sqsubseteq}} \lambda x.\varepsilon')$$

$$\text{and } (\varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon')\wedge(\delta \mathrel{\tilde{\sqsubseteq}} \delta') \Rightarrow (\varepsilon(\delta) \mathrel{\tilde{\sqsubseteq}} \varepsilon'(\delta')),$$

together with similar results for $\cong$. Finally, $\tilde{\sqsubseteq}$ and $\cong$ are substitutive.

Proof:-

-Same as 4.2.1.

‡

4.4.8:LEMMA:-

$$\varepsilon \xrightarrow{\alpha} \varepsilon' \Rightarrow \varepsilon \cong \varepsilon'.$$

Proof:-

-Let y be not free in ε.

-Claim: $\lambda x.\varepsilon \cong \lambda y.[y/x]\varepsilon$:-

$$-\tilde{E}\llbracket\lambda y.[y/x]\varepsilon\rrbracket = \lambda y.\tilde{E}\llbracket[y/x]\varepsilon\rrbracket = \lambda y.[y/x]\tilde{E}\llbracket\varepsilon\rrbracket, \text{ by 4.4.5.}$$
$$= \lambda x.\tilde{E}\llbracket\varepsilon\rrbracket, \text{ by 4.4.1(ii).}$$
$$= \tilde{E}\llbracket\lambda x.\varepsilon\rrbracket.$$

-Rest of the proof is the same as that of 4.2.3(vii).

‡

4.4.9:LEMMA:-

$\varepsilon \xrightarrow{\alpha,\beta} \varepsilon' \Rightarrow \varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon'$.

Proof:-

-Let $P[\![\varepsilon]\!] \equiv (\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon') \Rightarrow (\varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon')$ - i.e. we do at most one β-reduction when going from $\varepsilon \longrightarrow \varepsilon'$.

-Clearly, $P[\![x]\!]$, since no reductions are possible.

-Claim: $P[\![\varepsilon]\!] \Rightarrow P[\![\lambda x.\varepsilon]\!]$:-

-Let $\lambda x.\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon'$.

-Then, $\lambda x.\varepsilon \xrightarrow{\alpha,1\beta} \lambda x.\varepsilon''$, where $\varepsilon' \xrightarrow{\alpha} \lambda x.\varepsilon''$ and $\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon''$.

-∴, $\tilde{E}[\![\lambda x.\varepsilon]\!] = \lambda x.\tilde{E}[\![\varepsilon]\!] \sqsubseteq \lambda x.\tilde{E}[\![\varepsilon'']\!]$, by $P[\![\varepsilon]\!]$.

$= \tilde{E}[\![\lambda x.\varepsilon'']\!] = \tilde{E}[\![\varepsilon']\!]$, by 4.4.8.

-Claim: $P[\![\varepsilon]\!] \wedge P[\![\delta]\!] \Rightarrow P[\![\varepsilon(\delta)]\!]$:-

-Let $\varepsilon(\delta) \xrightarrow{\alpha,1\beta} \eta$.

-If $\varepsilon = \lambda x.\varepsilon'$, then $\tilde{E}[\![\varepsilon(\delta)]\!] = \bot$, by 4.3.3(xii).

$\sqsubseteq \tilde{E}[\![\eta]\!]$.

-Otherwise, $\eta = \varepsilon'(\delta')$ with $\varepsilon \xrightarrow{\alpha} \varepsilon'$, $\delta \xrightarrow{\alpha,1\beta} \delta'$ or $\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon'$, $\delta \xrightarrow{\alpha} \delta'$.

-In either case, $\tilde{E}[\![\varepsilon(\delta)]\!] = \tilde{Ap}(\tilde{E}[\![\varepsilon]\!],\tilde{E}[\![\delta]\!])$

$\sqsubseteq \tilde{Ap}(\tilde{E}[\![\varepsilon']\!],\tilde{E}[\![\delta']\!])$, by 4.4.8,4.3.3(i) and $P[\![\varepsilon]\!] \wedge P[\![\delta]\!]$.

$= \tilde{E}[\![\varepsilon'(\delta')]\!] = \tilde{E}[\![\eta]\!]$.

-∴, $(\forall \varepsilon \in \mathrm{EXP}) P[\![\varepsilon]\!]$.

-Hence, the lemma, since $\tilde{\sqsubseteq}$ is transitive, by 4.4.7.

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4.4.10:COR:-

$\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$ is a directed subset of E_∞.

Proof:-

-Let $\varepsilon \xrightarrow{\alpha,\beta} \gamma$ and $\varepsilon \xrightarrow{\alpha,\beta} \delta$.

-Then, by the Church-Rosser theorem, there exists an η such that,

$$\gamma \xrightarrow{\alpha,\beta} \eta \text{ and } \delta \xrightarrow{\alpha,\beta} \eta.$$

-∴, $\gamma \tilde{\equiv} \eta$ and $\delta \tilde{\equiv} \eta$, by 4.4.9.

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4.5:Characterisation of NOH,HNF and HEAD:-

4.5.0:LEMMA:-

(i) $\phi_{\infty,0} \circ \tilde{Ap}(\varepsilon,\delta) = \perp \in E_0$.

(ii) $\phi_{\infty,0}(\lambda x.\varepsilon) = \perp \in E_0$.

(iii) $\phi_{\infty,0} \circ \tilde{E}[\![\varepsilon]\!] = \perp \iff \varepsilon \in (\lambda I.EXP \cup (EXP)(EXP))$.

Proof:-

(i) $-\phi_{\infty,0} \circ \tilde{Ap}(\varepsilon,\delta) = \phi_{\infty,0} \circ \pi_2{}^{-1} \circ \psi_2{}^{-1} \circ \xi(\pi_2(\varepsilon),\delta)$, by 4.3.3(xii).

$= \perp$.

(ii) $-\phi_{\infty,0}(\lambda x.\varepsilon) = \phi_{\infty,0}(<\perp,<\lambda x.\varepsilon_i>_{i=0}^{\infty}>)$, by 3.4.5.

$= \perp$.

(iii) $-EXP = I \cup (\lambda I.EXP \cup (EXP)(EXP))$.

$-\emptyset = I \cap (\lambda I.EXP \cup (EXP)(EXP))$.

$-\varepsilon \in I \Rightarrow \varepsilon = x \Rightarrow \phi_{\infty,0} \circ \tilde{E}[\![\varepsilon]\!] = x \neq \perp$.

$-\varepsilon \in \lambda I.EXP \Rightarrow \varepsilon = \lambda x.\varepsilon' \Rightarrow \tilde{E}[\![\varepsilon]\!] = \lambda x.\tilde{E}[\![\varepsilon']\!]$

$\Rightarrow \phi_{\infty,0} \circ \tilde{E}[\![\varepsilon]\!] = \perp$, by part (ii).

$-\varepsilon \in (EXP)(EXP) \Rightarrow \varepsilon = \omega(\delta)$

$\Rightarrow \phi_{\infty,0} \circ \tilde{E}[\![\varepsilon]\!] = \phi_{\infty,0} \circ \tilde{Ap}(\tilde{E}[\![\omega]\!],\tilde{E}[\![\delta]\!]) = \perp$, by part (i).

-Hence the result.

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4.5.1:LEMMA:-

(i) $\varepsilon \in NOH \Rightarrow \tilde{E}[\![\varepsilon]\!] = \perp$.

(ii) $\varepsilon \in HEAD \Rightarrow \tilde{E}[\![\varepsilon]\!] \neq \perp$ and $\pi_2 \circ \tilde{E}[\![\varepsilon]\!] \neq \perp$.

(iii) $\varepsilon \in HNF \Rightarrow \tilde{E}[\![\varepsilon]\!] \neq \perp$.

(iv) $\varepsilon \in HNF \backslash HEAD \Rightarrow \pi_2 \circ \tilde{E}[\![\varepsilon]\!] = \perp$.

(v) $\varepsilon \in$ HEAD $\Rightarrow E\llbracket\varepsilon\rrbracket \neq \bot$ and $\pi_2 \circ E\llbracket\varepsilon\rrbracket \neq \bot$.

(vi) $\varepsilon \in$ HNF $\Rightarrow E\llbracket\varepsilon\rrbracket \neq \bot$.

(vii) $\varepsilon \in$ HNF\HEAD $\Rightarrow \pi_2 \circ E\llbracket\varepsilon\rrbracket = \bot$.

<u>Proof:-</u>

(i) -Let $P\llbracket\varepsilon\rrbracket \equiv (\tilde{E}\llbracket\varepsilon\rrbracket = \bot)$.

-Recall the definition of NOH - 0.5.0.

-<u>Claim:</u> $P\llbracket(\lambda x.\varepsilon)(\delta)\rrbracket$:-

$-\tilde{E}\llbracket(\lambda x.\varepsilon)(\delta)\rrbracket = \widetilde{Ap}(\tilde{E}\llbracket\lambda x.\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket)$

$= \bot$, by 4.3.3(xii), since $\pi_2 \circ \tilde{E}\llbracket\lambda x.\varepsilon\rrbracket = \pi_2(\lambda x.\tilde{E}\llbracket\varepsilon\rrbracket) = \bot$.

-<u>Claim:</u> $P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\lambda x.\varepsilon\rrbracket$:-

$-\tilde{E}\llbracket\lambda x.\varepsilon\rrbracket = \lambda x.\tilde{E}\llbracket\varepsilon\rrbracket = \lambda x.\bot$, by $P\llbracket\varepsilon\rrbracket$.

$= \bot$.

-<u>Claim:</u> $P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\varepsilon(\delta)\rrbracket$:-

$-\tilde{E}\llbracket\varepsilon(\delta)\rrbracket = \widetilde{Ap}(\tilde{E}\llbracket\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket) = \widetilde{Ap}(\bot, \tilde{E}\llbracket\delta\rrbracket)$, by $P\llbracket\varepsilon\rrbracket$.

$= \bot$, by 4.3.3(ii).

-$\therefore$, by structural induction, $(\forall \varepsilon \in \text{NOH}) P\llbracket\varepsilon\rrbracket$.

(ii) -Let $P\llbracket\varepsilon\rrbracket \equiv (\tilde{E}\llbracket\varepsilon\rrbracket \neq \bot) \wedge (\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket \neq \bot)$.

-Clearly, $P\llbracket x\rrbracket$.

-<u>Claim:</u> $P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\varepsilon(\delta)\rrbracket$:-

$-\tilde{E}\llbracket\varepsilon(\delta)\rrbracket = \widetilde{Ap}(\tilde{E}\llbracket\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket) = \pi_2{}^{-1} \circ \psi_2{}^{-1} \circ \xi(\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket)$

$\neq \bot$, since $\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket \neq \bot$, by $P\llbracket\varepsilon\rrbracket$.

$-\pi_2 \circ \tilde{E}\llbracket\varepsilon(\delta)\rrbracket = \pi_2 \circ \pi_2{}^{-1} \circ \psi_2{}^{-1} \circ \xi(\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket)$

$= \psi_2{}^{-1} \circ \xi(\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket, \tilde{E}\llbracket\delta\rrbracket) \neq \bot$, as above.

-$\therefore$, by structural induction, $(\forall \varepsilon \in \text{HEAD}) P\llbracket\varepsilon\rrbracket$.

(iii) -Let $P\llbracket\varepsilon\rrbracket \equiv (\tilde{E}\llbracket\varepsilon\rrbracket \neq \bot)$.

-Now, $\varepsilon \in$ HEAD $\Rightarrow P\llbracket\varepsilon\rrbracket$, by part (ii).

-<u>Claim:</u> $P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\lambda x.\varepsilon\rrbracket$:-

$-\tilde{E}\llbracket\lambda x.\varepsilon\rrbracket = \lambda x.\tilde{E}\llbracket\varepsilon\rrbracket \neq \bot$, by $P\llbracket\varepsilon\rrbracket$.

-$\therefore$, by structural induction, $(\forall \varepsilon \in \text{HNF}) P\llbracket\varepsilon\rrbracket$.

(iv) $-\varepsilon \in \mathrm{HNF}\backslash\mathrm{HEAD} \Rightarrow (\varepsilon = \lambda x.\varepsilon') \wedge (\varepsilon' \in \mathrm{HNF})$.

$-\therefore$, $\pi_2 \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \pi_2(\lambda x.\tilde{\mathcal{E}}[\![\varepsilon']\!]) = \bot$.

(v) -By part (ii), since $\tilde{\mathcal{E}} \subseteq \mathcal{E}$, by 4.4.1(i) and π_2 is monotonic, by 2.2.16.

(vi) -By part (iii) and $\tilde{\mathcal{E}} \subseteq \mathcal{E}$.

(vii) -Same as proof of part (iv).

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4.5.2:THEOREM:-

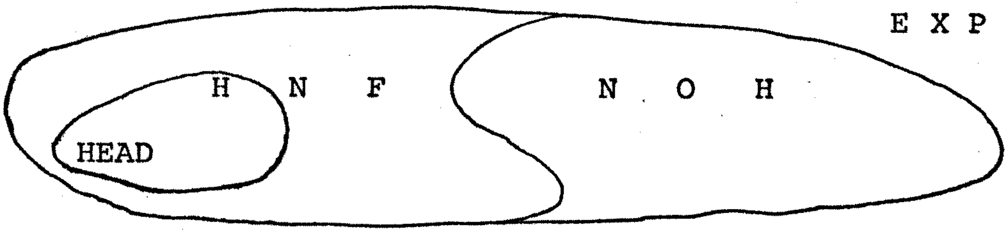
(i) $\varepsilon \in \mathrm{NOH} \Leftrightarrow \tilde{\mathcal{E}}[\![\varepsilon]\!] = \bot$.

(ii) $\varepsilon \in \mathrm{HNF} \Leftrightarrow \tilde{\mathcal{E}}[\![\varepsilon]\!] \neq \bot$.

(iii) $\varepsilon \in \mathrm{HEAD} \Leftrightarrow \pi_2 \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] \neq \bot$.

Proof:-

-By 4.5.1 and,



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4.5.3:COR:-

(i) $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = x \neq \bot \in E_i \Rightarrow \varepsilon = x$.

(ii) $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \lambda x.\varepsilon'_{i-1} \neq \bot \Rightarrow \varepsilon \xrightarrow{\alpha} \lambda y.\delta'$ and y is not free in ε'_{i-1} and $\phi_{\infty,i-1} \circ \tilde{\mathcal{E}}[\![\delta']\!] = [y/x]\varepsilon'_{i-1}$.

(iii) $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \alpha_{i-2}(\varepsilon'_{i-1}) \neq \bot \Rightarrow \varepsilon = \alpha(\varepsilon')$ and $\phi_{\infty,i-1} \circ \tilde{\mathcal{E}}[\![\alpha]\!] = \alpha_{i-2}$ and $\phi_{\infty,i-1} \circ \tilde{\mathcal{E}}[\![\varepsilon']\!] = \varepsilon'_{i-1}$.

Proof:-

(i) $-\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = x \neq \bot \Rightarrow \phi_{\infty,0} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = x \Rightarrow \varepsilon = x$, by 4.5.0(iii).

(ii) $-\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \lambda x.\varepsilon'_{i-1} \neq \bot$

$\Rightarrow \tilde{\mathcal{E}}[\![\varepsilon]\!] \neq \bot$ and $\pi_2 \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \bot$.

$\Rightarrow \varepsilon \in \mathrm{HNF}\backslash\mathrm{HEAD}$, by 4.5.2(ii) and (iii).

$\Rightarrow \varepsilon = \lambda y.\delta'$, with $\delta' \in \mathrm{HNF}$.

-But, $\phi_{\infty,i} \circ \tilde{E}\llbracket \lambda y . \delta' \rrbracket = \phi_{\infty,i}(\lambda y . \tilde{E}\llbracket \delta' \rrbracket) = \lambda y . \phi_{\infty,i-1} \circ \tilde{E}\llbracket \delta' \rrbracket$.

-∴, <u>either</u> $y = x$ <u>or</u> y is not free in ε'_{i-1} and $\phi_{\infty,i-1} \circ \tilde{E}\llbracket \delta' \rrbracket = [y/x]\varepsilon'_{i-1}$. But, if $y = x$, then $\varepsilon \xrightarrow{\alpha} \lambda y''.\delta''$, with the correct properties.

(iii) -$\phi_{\infty,i} \circ \tilde{E}\llbracket \varepsilon \rrbracket = \alpha_{i-2}(\varepsilon'_{i-1}) \neq \perp \Rightarrow \pi_2 \circ \tilde{E}\llbracket \varepsilon \rrbracket \neq \perp$

$\Rightarrow \varepsilon \in$ HEAD, by 4.5.2(iii).

-Now, $\varepsilon \notin I$, since $\varepsilon \in I \Rightarrow \varepsilon = x \Rightarrow \phi_{\infty,i} \circ \tilde{E}\llbracket \varepsilon \rrbracket = x$ - ✱.

-∴, $\varepsilon = \alpha(\varepsilon')$, where $\alpha \notin$ HEAD.

-Then, $\phi_{\infty,i} \circ \tilde{E}\llbracket \alpha(\varepsilon') \rrbracket = \phi_{\infty,i} \circ \pi_2^{-1} \circ \psi_2^{-1} \circ \xi(\pi_2 \circ \tilde{E}\llbracket \alpha \rrbracket, \tilde{E}\llbracket \varepsilon' \rrbracket)$

$= \phi_{\infty,i-1} \circ \tilde{E}\llbracket \alpha \rrbracket(\phi_{\infty,i-1} \circ \tilde{E}\llbracket \varepsilon' \rrbracket)$.

-∴, $\phi_{\infty,i-1} \circ \tilde{E}\llbracket \alpha \rrbracket = \alpha_{i-2}$ and $\phi_{\infty,i-1} \circ \tilde{E}\llbracket \varepsilon' \rrbracket = \varepsilon'_{i-1}$, since $\alpha_{i-2}(\varepsilon'_{i-1}) \neq \perp$ and $\tilde{E}\llbracket \varepsilon \rrbracket \neq \top$, by 4.4.1(iii).

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<u>4.5.4:REMARK:-</u>

The above corollary means that we know some of the syntax of an expression if we know a non-$\perp$ coordinate of its approximate semantics. This will be very useful in the next sections.

<u>4.6:The Semantical Function *V*:-</u>

<u>4.6.0:DEF:-</u>

$$V : \mathrm{EXP} \longrightarrow E_\infty$$
$$\varepsilon \longmapsto \bigsqcup\{\tilde{E}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}.$$

<u>4.6.1:THEOREM:-</u>

$\langle E_\infty, V \rangle$ is a β-model.

<u>Proof:-</u>

-Clearly, if $\varepsilon \xrightarrow{\alpha,\beta} \delta$, then $V\llbracket \varepsilon \rrbracket \sqsupseteq V\llbracket \delta \rrbracket$.

-If $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$, by the Church-Rosser theorem, there exists a δ' such that $\varepsilon' \xrightarrow{\alpha,\beta} \delta'$ and $\delta \xrightarrow{\alpha,\beta} \delta'$.

-But, then $\varepsilon' \overset{\sim}{\equiv} \delta'$, by 4.4.9, and so $V\llbracket \varepsilon \rrbracket \equiv V\llbracket \delta \rrbracket$.

-Hence, $V[\![\varepsilon]\!] = V[\![\delta]\!]$.

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4.6.2:LEMMA:-

$\langle E_\infty, V\rangle$ is solvable. In fact,

$$\varepsilon \in \mathrm{SOL} \iff V[\![\varepsilon]\!] \neq \bot.$$

Proof:-

(=>) -Let $\varepsilon \in \mathrm{SOL}$. Then, $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon' \in \mathrm{HNF}$.

-Then, $V[\![\varepsilon]\!] = V[\![\varepsilon']\!]$, by 4.6.1.

$\sqsupseteq \acute{E}[\![\varepsilon']\!] \neq \bot$, by 4.5.2(ii).

-$\therefore$, $V[\![\varepsilon]\!] \neq \bot$.

(<=) -Let $\varepsilon \in \mathrm{INSOL}$. Then, $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon' \Rightarrow \varepsilon' \in \mathrm{NOH}$.

-$\therefore$, $V[\![\varepsilon]\!] = \bigsqcup\{\bot\}$, by 4.5.2(i).

$= \bot$.

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4.6.3:REMARK:-

These simple results show E_∞ to be a very interesting object.

Recently, some work of Lévy has come to my attention - [56], where he also constructs a syntactical (semi-)lattice, $\overline{N}$. Indeed, there is probably a simple isomorphism from $\overline{N}$ to the bottom half of E_∞ ($Low(E_\infty)$ - see section 7.5) which identifies his semantics, S, with V.

However, V does not tell us anything about application in the model. Further, it is hard to prove that the semantical equivalence derived from V is substitutive - in particular,

$$(V[\![\varepsilon]\!] = V[\![\varepsilon']\!]) \wedge (V[\![\delta]\!] = V[\![\delta']\!]) \Rightarrow (V[\![\varepsilon(\delta)]\!] = V[\![\varepsilon'(\delta')]\!]).$$

We will show that $\acute{E} = V$, thus making $\langle E_\infty, E\rangle$ a model and giving us substitutivity for V. For the moment, we will show that $E \sqsubseteq V$.

4.6.4:DEF:-

Let $\begin{Bmatrix} S \\ Q \\ R \end{Bmatrix}(i) \equiv (\begin{Bmatrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix}, \delta_i \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}\llbracket\varepsilon\rrbracket, \phi_{\infty,i} \circ \tilde{E}\llbracket\delta\rrbracket) \Rightarrow (\exists \eta \in \mathrm{EXP})$ $([\delta/x]\varepsilon \xrightarrow{\alpha,\beta} \eta) \wedge ([\delta_i/x]\begin{Bmatrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{Bmatrix} \sqsubseteq \phi_{\infty,i} \circ \tilde{E}\llbracket\eta\rrbracket)$.

Let $P(i) \equiv (\varepsilon_i, \delta_i \sqsubseteq \phi_{\infty,i} \circ \tilde{E}\llbracket\varepsilon\rrbracket, \phi_{\infty,i} \circ \tilde{E}\llbracket\delta\rrbracket) \Rightarrow (\exists \eta \in \mathrm{EXP})$ $(\varepsilon(\delta) \xrightarrow{\alpha,\beta} \eta) \wedge (Ap_i(\varepsilon_i, \delta_i) \sqsubseteq \phi_{\infty,i+1} \circ \tilde{E}\llbracket\eta\rrbracket)$.

4.6.5:REMARK:-

The above definition has scope from 4.6.4 to 4.6.10.

4.6.6:LEMMA:-

$(\forall i \geq 0) S(i)$.

Proof:-

-Assume $z \neq \bot$, otherwise trivial.

-$z \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}\llbracket\varepsilon\rrbracket \Rightarrow z = \phi_{\infty,i-1} \circ \tilde{E}\llbracket\varepsilon\rrbracket$, since $\top \neq \phi_{\infty,i-1} \circ \tilde{E}\llbracket\varepsilon\rrbracket$, by 4.4.1(iii).

$\Rightarrow z = \varepsilon$, by 4.5.3(i).

-Suppose $z = x$:-

-Take $\eta = [\delta/x]x = \delta$.

-Then, $[\delta_i/x]x = \delta_i \sqsubseteq \phi_{\infty,i} \circ \tilde{E}\llbracket\delta\rrbracket$.

-Suppose $z \neq x$:-

-Take $\eta = [\delta/x]z = z$.

-Then, $[\delta_i/x]z = z = \phi_{\infty,i} \circ \tilde{E}\llbracket z\rrbracket$.

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4.6.7:LEMMA:-

$(\forall i \geq 1)(Q(i) \Rightarrow P(i))$.

Proof:-

-Again, assume $\varepsilon_i \neq \bot$, since otherwise trivial.

-Suppose $\varepsilon_i = \alpha_{i-1}$:-

-Then, $\top \neq \phi_{\infty,i} \circ \tilde{E}\llbracket\varepsilon\rrbracket = \overline{\alpha}_{i-1} \sqsupseteq \alpha_{i-1} = \varepsilon_i$.

-$\therefore$, $\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket \neq \bot$.

-Take $\eta = \varepsilon(\delta)$.

-Then, $Ap_i(\varepsilon_i,\delta_i) = \alpha_{i-1}(\delta_i) \sqsubseteq \overline{\alpha}_{i-1}(\delta_i)$

$\sqsubseteq \phi_{\infty,i}\circ\tilde{E}\llbracket\varepsilon\rrbracket(\phi_{\infty,i}\circ\tilde{E}\llbracket\delta\rrbracket)$

$= \phi_{\infty,i+1}\circ\pi_2^{-1}\circ\psi_2^{-1}\circ\xi(\pi_2\circ\tilde{E}\llbracket\varepsilon\rrbracket,\tilde{E}\llbracket\delta\rrbracket)$

$= \phi_{\infty,i+1}\circ\widetilde{Ap}(\tilde{E}\llbracket\varepsilon\rrbracket,\tilde{E}\llbracket\delta\rrbracket)$

$= \phi_{\infty,i+1}\circ\tilde{E}\llbracket\eta\rrbracket$.

-Suppose $\varepsilon_i = \lambda x.\varepsilon'_{i-1}$:-

-Then, $\top \neq \phi_{\infty,i}\circ\tilde{E}\llbracket\varepsilon\rrbracket = \lambda x.\overline{\varepsilon}'_{i-1} \sqsupseteq \lambda x.\varepsilon'_{i-1} \neq \bot$.

-∴, $\varepsilon \xrightarrow{\alpha} \lambda y.\nu$ and y is not free in $\overline{\varepsilon}'_{i-1}$ and $\phi_{\infty,i-1}\circ\tilde{E}\llbracket\nu\rrbracket = [y/x]\overline{\varepsilon}'_{i-1}$, by 4.5.3(ii).

-Then, $Ap_i(\varepsilon_i,\delta_i) = \phi_{i,i+1}\circ[\delta_i/x]\varepsilon'_{i-1}$

$\sqsubseteq \phi_{i,i+1}\circ[\delta_i/x]\overline{\varepsilon}'_{i-1}$

$= \phi_{i,i+1}\circ[\delta_i/y][y/x]\overline{\varepsilon}'_{i-1}$, since y is not free in $\overline{\varepsilon}'_{i-1}$.

$\sqsubseteq \phi_{i,i+1}\circ\phi_{\infty,i}\circ\tilde{E}\llbracket\eta\rrbracket$, by $Q(i)$, where $\varepsilon(\delta) \xrightarrow{\alpha} (\lambda y.\nu)(\delta) \xrightarrow{\beta} [\delta/y]\nu \xrightarrow{\alpha,\beta} \eta$.

-∴, the result.

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4.6.8:LEMMA:-

$(\forall i \geq 2)(Q(i-1)\wedge R(i) \Rightarrow Q(i))$.

Proof:-

-Assume $\varepsilon_{i-1} \neq \bot$, since otherwise trivial.

-Suppose $\varepsilon_{i-1} = \lambda y.\varepsilon'_{i-2}$:-

-As in the last lemma, we get $\varepsilon \xrightarrow{\alpha} \lambda z.\nu$ and z is not free in $\overline{\varepsilon}'_{i-2}$ and $\phi_{\infty,i-1}\circ\tilde{E}\llbracket\nu\rrbracket = [z/y]\overline{\varepsilon}'_{i-2} \sqsupseteq [z/y]\varepsilon'_{i-2}$.

-Assume, w.l.o.g , that $z \neq x$ and is not free in δ.

-Then, $[\delta_i/x]\varepsilon_{i-1} \sqsubseteq [\delta_i/x]\lambda y.\overline{\varepsilon}'_{i-2} = \lambda z.[\delta_{i-1}/x][z/y]\overline{\varepsilon}'_{i-2}$

$\sqsubseteq \lambda z.\phi_{\infty,i-1}\circ\tilde{E}\llbracket\eta\rrbracket$, by $Q(i-1)$, where $[\delta/x]\nu \xrightarrow{\alpha,\beta} \eta$.

$= \phi_{\infty,i}\circ\tilde{E}\llbracket\lambda z.\eta\rrbracket$, where $[\delta/x]\varepsilon \xrightarrow{\alpha,\beta} \lambda z.\eta$.

-Suppose $\varepsilon_{i-1} = \alpha_{i-2}$. Then, we have the result by $R(i)$.

$\ddagger$

4.6.9:LEMMA:-

$(\forall i \geq 3)(P(i-1) \wedge Q(i-1) \wedge R(i-1) \Rightarrow R(i))$.

Proof:-

-Assume $\alpha_{i-2} \neq \bot$, since otherwise trivial.

-Suppose $\alpha_{i-2} = \gamma_{i-3}(\varepsilon'_{i-2})$:-

-We have $\top \neq \phi_{\infty,i-1} \circ \tilde{E}[\![\varepsilon]\!] = \overline{\gamma}_{i-3}(\overline{\varepsilon}'_{i-2}) \sqsupseteq \gamma_{i-3}(\varepsilon'_{i-2}) \neq \bot$.

-$\therefore$, $\varepsilon = \gamma(\varepsilon')$ and $\phi_{\infty,i-2} \circ \tilde{E}[\![\gamma]\!] = \overline{\gamma}_{i-3} \sqsupseteq \gamma_{i-3}$ and $\phi_{\infty,i-2} \circ \tilde{E}[\![\varepsilon']\!] = \overline{\varepsilon}'_{i-2} \sqsupseteq \varepsilon'_{i-2}$, by 4.4.3(iii).

-$\therefore$, $[\delta_i/x]\alpha_{i-2} = [\delta_i/x]\gamma_{i-3}(\varepsilon'_{i-2})$

$= Ap_{i-1}([\delta_{i-1}/x]\gamma_{i-3}, [\delta_{i-1}/x]\varepsilon'_{i-2})$, where $\delta_{i-1} \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}[\![\delta]\!]$.

-$\therefore$, $[\delta_{i-1}/x]\gamma_{i-3} \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}[\![\eta]\!]$, by $R(i-1)$, where $[\delta/x]\gamma \xrightarrow{\alpha,\beta} \eta$.

-$\therefore$, $[\delta_{i-1}/x]\varepsilon'_{i-2} \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}[\![\omega]\!]$, by $Q(i-1)$, where $[\delta/x]\varepsilon' \xrightarrow{\alpha,\beta} \omega$.

-$\therefore$, $[\delta_i/x]\alpha_{i-2} \sqsubseteq \phi_{\infty,i} \circ \tilde{E}[\![\nu]\!]$, by $P(i-1)$, where $[\delta/x]\varepsilon = [\delta/x]\gamma(\varepsilon') = [\delta/x]\gamma([\delta/x]\varepsilon') \xrightarrow{\alpha,\beta} \eta(\omega) \xrightarrow{\alpha,\beta} \nu$.

-Suppose $\alpha_{i-2} = z$. Then, we have the result by $S(i)$, by 4.6.6.

$\ddagger$

4.6.10:COR:-

$(\forall i \geq 0)S(i)$, $(\forall i \geq 1)(P(i) \wedge Q(i))$ and $(\forall i \geq 2)R(i)$.

Proof:-

-Clearly, $R(2) \equiv S(2)$ and $Q(1) \equiv S(1)$.

-$\therefore$, by 3.0.5,4.6.6,4.6.7,4.6.8 and 4.6.9, we have the result.

$\ddagger$

4.6.11:THEOREM:-

$E \sqsubseteq V$.

Proof:-

-Let $A[\![\varepsilon]\!] \equiv (E[\![\varepsilon]\!] \sqsubseteq V[\![\varepsilon]\!])$.

-Clearly, $A[\![x]\!]$.

-<u>Claim:</u> $A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!]$:-

$$\begin{aligned} -E[\![\lambda x.\varepsilon]\!] &= \lambda x.E[\![\varepsilon]\!] \sqsubseteq \lambda x.V[\![\varepsilon]\!], \text{ by } A[\![\varepsilon]\!]. \\ &= \lambda x.\bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\} \\ &= \bigsqcup\{\lambda x.\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}, \text{ by 4.4.10.} \\ &= \bigsqcup\{\tilde{E}[\![\lambda x.\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\} \\ &= \bigsqcup\{\tilde{E}[\![\varepsilon'']\!] \mid \lambda x.\varepsilon \xrightarrow{\alpha,\beta} \varepsilon''\} \\ &= V[\![\lambda x.\varepsilon]\!]. \end{aligned}$$

-<u>Claim:</u> $A[\![\varepsilon]\!] \wedge A[\![\delta]\!] \Rightarrow A[\![\varepsilon(\delta)]\!]$:-

$$\begin{aligned} -\phi_{\infty,i}\circ E[\![\varepsilon]\!] &\sqsubseteq \phi_{\infty,i}\circ V[\![\varepsilon]\!], \ \forall i \geq 0, \text{ by } A[\![\varepsilon]\!]. \\ &= \phi_{\infty,i}(\bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}) \\ &= \bigsqcup\{\phi_{\infty,i}\circ\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}, \text{ by 4.4.10.} \end{aligned}$$

-∴, there exists an ε^i such that $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon^i$ and $\varepsilon_i = \phi_{\infty,i}\circ E[\![\varepsilon]\!] \sqsubseteq \phi_{\infty,i}\circ\tilde{E}[\![\varepsilon^i]\!]$, since ε_i is isolated in the finite depth lattice E_i.

-Similarly, $\forall i \geq 0$, there exists a δ^i such that $\delta \xrightarrow{\alpha,\beta} \delta^i$ and $\delta_i = \phi_{\infty,i}\circ E[\![\delta]\!] \sqsubseteq \phi_{\infty,i}\circ\tilde{E}[\![\delta^i]\!]$.

-∴, by 4.6.10, $\forall i \geq 1$, there exists an η^i such that $\varepsilon(\delta) \xrightarrow{\alpha,\beta} \varepsilon^i(\delta^i) \xrightarrow{\alpha,\beta} \eta^i$ and $Ap_i(\varepsilon_i,\delta_i) \sqsubseteq \phi_{\infty,i+1}\circ\tilde{E}[\![\eta^i]\!]$.

-Then, $E[\![\varepsilon(\delta)]\!] = Ap(E[\![\varepsilon]\!],E[\![\delta]\!])$

$$\begin{aligned} &= \bigsqcup_{i=1}^{\infty}\phi_{i+1,\infty}\circ Ap_i(\varepsilon_i,\delta_i) \\ &\sqsubseteq \bigsqcup_{i=1}^{\infty}\phi_{i+1,\infty}\circ\phi_{\infty,i+1}\circ\tilde{E}[\![\eta^i]\!], \text{ by above.} \\ &\sqsubseteq \bigsqcup_{i=1}^{\infty}\tilde{E}[\![\eta^i]\!] \sqsubseteq \bigsqcup\{\tilde{E}[\![\eta]\!] \mid \varepsilon(\delta) \xrightarrow{\alpha,\beta} \eta\} \\ &= V[\![\varepsilon(\delta)]\!]. \end{aligned}$$

-Hence, by structural induction, $(\forall\varepsilon \in EXP)A[\![\varepsilon]\!]$.

‡

4.6.12:COR:-

$\varepsilon \in \mathrm{INSOL} \Rightarrow \mathcal{E}[\![\varepsilon]\!] = \perp$.

Proof:-

-By 4.6.2 and 4.6.11.

ǂ

4.7:Characterization of $\prec$:-

4.7.0:DEF:-

The set of elements in the direct limit, $\cup\{\phi_{n,\infty}(E_n) \mid n \geq 0\}$, are FINITE elements. A λ-expression, $\varepsilon \in \mathrm{EXP}$, is FINITE if $\mathcal{E}[\![\varepsilon]\!]$ is finite.

4.7.1:LEMMA:-

$(\varepsilon \in E_\infty$ is finite$) \Leftrightarrow (\phi_{n,\infty} \circ \phi_{\infty,n}(\varepsilon) = \varepsilon$, for some $n \geq 0)$.

Proof:-

-Trivial.

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4.7.2:EXAMPLES:-

(i) Y is not finite.

(ii) Elements of INSOL are finite.

Proof:-

-By 4.0.3 and 4.6.12.

ǂ

4.7.3:LEMMA:-

$\overset{\approx}{\mathcal{E}}$ is a finite function - i.e. its image lies in the direct limit.

Proof:-

-Let $P[\![\varepsilon]\!] \equiv$ (there exists an $n \geq 0$)$(\phi_{n,\infty} \circ \phi_{\infty,n} \circ \overset{\approx}{\mathcal{E}}[\![\varepsilon]\!] = \overset{\approx}{\mathcal{E}}[\![\varepsilon]\!])$.

-Then, $P[\![x]\!]$, with $n = 0$.

-Also, clearly, $P[\![\varepsilon]\!] \Rightarrow P[\![\lambda x.\varepsilon]\!]$.

-Claim: $P\llbracket\varepsilon\rrbracket\wedge P\llbracket\delta\rrbracket \Rightarrow P\llbracket\varepsilon(\delta)\rrbracket$:-

-Let n,m be the numbers referred to in $P\llbracket\varepsilon\rrbracket, P\llbracket\delta\rrbracket$.

-Let $p = \max(n,m)$.

-Then, $\phi_{p,\infty}\circ\phi_{\infty,p}\circ\widetilde{E}\llbracket\varepsilon(\delta)\rrbracket = \phi_{p,\infty}\circ\phi_{\infty,p}\circ\widetilde{Ap}(\widetilde{E}\llbracket\varepsilon\rrbracket,\widetilde{E}\llbracket\delta\rrbracket)$

$= \widetilde{Ap}(\phi_{p,\infty}\circ\phi_{\infty,p}\circ\widetilde{E}\llbracket\varepsilon\rrbracket, \phi_{p,\infty}\circ\phi_{\infty,p}\circ\widetilde{E}\llbracket\delta\rrbracket)$,

by 4.3.3(xii).

$= \widetilde{Ap}(\widetilde{E}\llbracket\varepsilon\rrbracket,\widetilde{E}\llbracket\delta\rrbracket)$, by $P\llbracket\varepsilon\rrbracket\wedge P\llbracket\delta\rrbracket$.

$= \widetilde{E}\llbracket\varepsilon(\delta)\rrbracket$.

-$\therefore$, by structural induction, $(\forall\varepsilon \in \mathrm{EXP})P\llbracket\varepsilon\rrbracket$.

-Hence, the result, by 4.7.1.

‡

4.7.4:LEMMA:-

$\varepsilon \in \mathrm{NF} \Rightarrow \widetilde{E}\llbracket\varepsilon\rrbracket = E\llbracket\varepsilon\rrbracket = V\llbracket\varepsilon\rrbracket$.

Proof:-

-Clearly, $V\llbracket\varepsilon\rrbracket = \widetilde{E}\llbracket\varepsilon\rrbracket$, if $\varepsilon \in \mathrm{NF}$.

-But, $\widetilde{E} \sqsubseteq E \sqsubseteq V$, by 4.4.1(i) and4.6.11.

-Hence, the result.

‡

4.7.5:COR:-

Normal forms are finite.

Proof:-

-By 4.7.3 and 4.7.4.

‡

4.7.6:REMARK:-

Thus, we have justified the remark in 4.0.4 about normal forms being mapped "elementarily" by E.

We now wish to characterise $\prec$. Recall, by 0.6.11(x), that

$(\varepsilon \prec \delta) \Leftrightarrow (\forall \text{ directed } D)(\delta \sqsubseteq \bigsqcup D \Rightarrow \text{there is a } d \in D \text{ s.t. } \varepsilon \sqsubseteq d)$.

We note that, in E_i, $\prec$ is just $\sqsubseteq$, since F_i has finite depth, by 0.6.13.

4.7.7:LEMMA:-

$\mathcal{E}[\![Y]\!]$ is not isolated.

Proof:-

-Let $D = \{\phi_{i,\infty}\circ\phi_{\infty,i}\circ\mathcal{E}[\![Y]\!] \mid i \geq 0\}$.

-Then, D is directed and $\mathcal{E}[\![Y]\!] = \bigsqcup D$.

-But, by 4.0.3, there is no $d \in D$ such that $\mathcal{E}[\![Y]\!] \sqsubseteq d$.

-So, $\mathcal{E}[\![Y]\!] \not\prec \mathcal{E}[\![Y]\!]$.

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4.7.8:COR:-

$\prec$ is different from $\sqsubseteq$ in E_∞.

Proof:-

-$\mathcal{E}[\![Y]\!] \sqsubseteq \mathcal{E}[\![Y]\!]$.

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4.7.9:THEOREM:-

Let $\varepsilon,\delta \in E_\infty$ and $\varepsilon \sqsubseteq \delta$ and ε be finite. Then, $\varepsilon \prec \delta$.

Proof:-

-Let D be directed and $\delta \sqsubseteq \bigsqcup D$.

-Let n be such that $\varepsilon = \phi_{n,\infty}\circ\phi_{\infty,n}(\varepsilon)$.

-Now, $\delta_n := \phi_{\infty,n}(\delta) \sqsubseteq \phi_{\infty,n}(\bigsqcup D) = \bigsqcup\phi_{\infty,n}(D)$.

-Also, $\phi_{\infty,n}(D)$ is directed in E_n.

-But, $\varepsilon_n \sqsubseteq \delta_n$ and so $\varepsilon_n \prec \delta_n$, by remark 4.7.6.

-$\therefore$, there exists $d \in D$ such that $\varepsilon_n \sqsubseteq \phi_{\infty,n}(d)$.

-But, $\forall i \leq n$, $\varepsilon_i \sqsubseteq \phi_{\infty,i}(d)$, by monotonicity.

-And, $\forall i > n$, $\varepsilon_i = \phi_{\infty,i}(\varepsilon) = \phi_{\infty,i}\circ\phi_{n,\infty}\circ\phi_{\infty,n}(\varepsilon) = \phi_{n,i}(\varepsilon_n)$

$\sqsubseteq \phi_{n,i}\circ\phi_{\infty,n}(d)$, by monotonicity.

$\sqsubseteq \phi_{\infty,i}(d)$.

-$\therefore$, $(\forall i \geq 0)(\varepsilon_i \sqsubseteq d_i)$ - i.e. $\varepsilon \sqsubseteq d$.

-$\therefore$, $\varepsilon \prec \delta$, by remark 4.7.6.

ǂ

4.7.10:COR:-

In E_∞, finite $\equiv$ isolated.

Proof:-

(=>) $-\varepsilon \sqsubseteq \varepsilon$ and ε finite => ε isolated, by 4.7.9.

(<=) -Let $\varepsilon \prec \varepsilon$. Now, $\varepsilon = \bigsqcup_{i=0}^{\infty} \phi_{i,\infty} \circ \phi_{\infty,i}(\varepsilon)$.

$-\therefore$, $\varepsilon \sqsubseteq \phi_{n,\infty} \circ \phi_{\infty,n}(\varepsilon)$, for some $n \geq 0$, since the set is directed.

$-\therefore$, $\varepsilon = \phi_{n,\infty} \circ \phi_{\infty,n}(\varepsilon)$ - i.e. ε is finite.

ǂ

4.7.11:COR:-

For all $\varepsilon \in$ EXP and $n \geq 0$, there exists an ε' such that $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$ and $\phi_{n,\infty} \circ \phi_{\infty,n} \circ \mathcal{E}[\![\varepsilon]\!] \sqsubseteq \overset{\checkmark}{\mathcal{E}}[\![\varepsilon']\!]$.

Proof:-

$-\phi_{n,\infty} \circ \phi_{\infty,n} \circ \mathcal{E}[\![\varepsilon]\!] \sqsubseteq \mathcal{E}[\![\varepsilon]\!] \sqsubseteq \mathcal{V}[\![\varepsilon]\!]$.

-Hence, the result, since the L.H.S. is finite and so $\prec \mathcal{V}[\![\varepsilon]\!]$, by 4.7.10 and 4.7.9.

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4.7.12:COR:-

The converse of 4.7.9 is not true.

Proof:-

-By 4.6.10, $\top \prec \top$, since $\top$ is finite.

-But, $\forall \varepsilon \in E_\infty$, $\varepsilon \sqsubseteq \top$ - i.e. $\varepsilon \prec \top$, by 0.6.11(iv).

-In particular, $\mathcal{E}[\![Y]\!] \prec \top$.

-But, $\mathcal{E}[\![Y]\!]$ is not finite, by 4.7.2(i).

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4.7.13:DEF:-

Let $\varepsilon, \delta \in$ EXP. Then, $\varepsilon \prec \delta$ if $\mathcal{E}[\![\varepsilon]\!] \prec \mathcal{E}[\![\delta]\!]$. Also, ε is ISOLATED if $\mathcal{E}[\![\varepsilon]\!]$ is isolated.

4.7.14:LEMMA:-

(i) Y is not isolated.

(ii) In EXP, finite $\equiv$ isolated.

(iii) Normal forms are isolated.

(iv) Elements of INSOL are isolated.

Proof:-

-BY 4.7.7,4.7.10,4.7.5 and 4.7.2(ii).

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4.7.15:LEMMA:-

Let $\nu \sqsubseteq \widetilde{Ap}(\varepsilon,\delta) \neq \top$. Then, there exist $\varepsilon' \sqsubseteq \varepsilon$, $\delta' \sqsubseteq \delta$ such that $\nu = \widetilde{Ap}(\varepsilon',\delta')$. (N.B. We are in the lattice.)

Proof:-

-If $\nu = \bot$, take $\varepsilon' = \bot$, $\delta' = \delta$ and we are home.

-If $\nu \neq \bot$, then $\pi_2(\varepsilon) \neq \bot$ and $\varepsilon = \langle\alpha_0,\langle\alpha_i\rangle_{i=0}^{\infty}\rangle$, where $\alpha_i \in A_i \subset E_{i+1}$.

-∴, there exists an $n \geq 2$ such that $\forall i \geq n$,

$$\bot \neq \nu_i \sqsubseteq \alpha_{i-2}(\delta_{i-1}) \neq \top.$$

-So, $\forall i \geq n$, $\nu_i = \alpha'_{i-2}(\delta'_{i-1})$.

-∴, $\nu = \widetilde{Ap}(\varepsilon',\delta')$, where $\varepsilon' = \langle\alpha'_0,\langle\alpha'_i\rangle_{i=0}^{\infty}\rangle$ and $\delta' = \langle\delta'_i\rangle_{i=0}^{\infty}$ - i.e. $\varepsilon' \sqsubseteq \varepsilon$ and $\delta' \sqsubseteq \delta$.

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4.7.16:LEMMA:-

$(\nu \sqsubseteq \tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket) \Rightarrow (\nu = \tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket$, for some $\varepsilon' \in$ EXP).

Proof:-

-Let $P\llbracket\varepsilon\rrbracket$ be the lemma.

-Claim: $P\llbracket x\rrbracket$:-

-Let $\nu \sqsubseteq \tilde{\mathcal{E}}\llbracket x\rrbracket = \langle x\rangle_{i=0}^{\infty}$.

-Then, $\nu_i = \bot$ or x, $\forall i \geq 0$.

-If $\nu_i = \bot$, take $\varepsilon' = \Delta\Delta$, else take $\varepsilon' = x$.

-Claim: $P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\lambda x.\varepsilon\rrbracket$:-

-Let $\nu \sqsubseteq \tilde{\mathcal{E}}\llbracket\lambda x.\varepsilon\rrbracket = \lambda x.\tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket$.

-Then, $\nu = \lambda x.\bar{\nu}$ and $\bar{\nu} \sqsubseteq \tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket$.

-∴, $\bar{\nu} = \tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket$, by $P\llbracket\varepsilon\rrbracket$.

$-\therefore,\ \nu = \lambda x.\overline{\nu} = \lambda x.\tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket = \tilde{\mathcal{E}}\llbracket\lambda x.\varepsilon'\rrbracket.$

-<u>Claim:</u> $P\llbracket\varepsilon\rrbracket \wedge P\llbracket\delta\rrbracket \Rightarrow P\llbracket\varepsilon(\delta)\rrbracket$:-

-Let $\nu \sqsubseteq \tilde{\mathcal{E}}\llbracket\varepsilon(\delta)\rrbracket = \tilde{Ap}(\tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket,\tilde{\mathcal{E}}\llbracket\delta\rrbracket)$.

-Then, $\nu = \tilde{Ap}(\mu,\sigma)$, where $\mu \sqsubseteq \tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket$ and $\sigma \sqsubseteq \tilde{\mathcal{E}}\llbracket\delta\rrbracket$, by 4.7.15.

-So, $\mu = \tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket$ and $\sigma = \tilde{\mathcal{E}}\llbracket\delta'\rrbracket$, by $P\llbracket\varepsilon\rrbracket \wedge P\llbracket\delta\rrbracket$.

$-\therefore,\ \nu = \tilde{Ap}(\tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket,\tilde{\mathcal{E}}\llbracket\delta'\rrbracket) = \tilde{\mathcal{E}}\llbracket\varepsilon'(\delta')\rrbracket.$

$-\therefore$, by structural induction, $(\forall\varepsilon \in \mathrm{EXP})P\llbracket\varepsilon\rrbracket$.

‡

<u>4.7.17:THEOREM:-</u>

Let $\varepsilon,\delta \in \mathrm{EXP}$. Then, $(\varepsilon \prec \delta) \Leftrightarrow (\varepsilon \sqsubseteq \delta)\wedge(\varepsilon$ is finite$)$.

<u>Proof:-</u>

$(\Leftarrow)$ -By 4.7.9.

$(\Rightarrow)$ -Let $\varepsilon \prec \delta$. Then, $\mathcal{E}\llbracket\varepsilon\rrbracket \prec \mathcal{E}\llbracket\delta\rrbracket$.

-But, $\mathcal{E}\llbracket\delta\rrbracket = \bigsqcup_{i=0}^{\infty}\phi_{i,\infty}\circ\phi_{\infty,i}\circ\mathcal{E}\llbracket\delta\rrbracket$.

-So, $\mathcal{E}\llbracket\varepsilon\rrbracket \sqsubseteq \phi_{n,\infty}\circ\phi_{\infty,n}\circ\mathcal{E}\llbracket\delta\rrbracket$, for some $n \geq 0$.

-Now, $\phi_{n,\infty}\circ\phi_{\infty,n}\circ\mathcal{E}\llbracket\delta\rrbracket \sqsubseteq \tilde{\mathcal{E}}\llbracket\delta'\rrbracket$, for some δ', by 4.7.11.

-So, $\mathcal{E}\llbracket\varepsilon\rrbracket \sqsubseteq \tilde{\mathcal{E}}\llbracket\delta'\rrbracket$.

$-\therefore$, $\mathcal{E}\llbracket\varepsilon\rrbracket = \tilde{\mathcal{E}}\llbracket\varepsilon'\rrbracket$, for some ε', by 4.7.16.

-So, $\mathcal{E}\llbracket\varepsilon\rrbracket$ is finite, by 4.7.3.

-Hence, ε is finite. (We always had $\varepsilon \sqsubseteq \delta$.)

‡

<u>4.8:Characterisation of $\tilde{\sqsubseteq}$ and Normality of $<E_\infty,V>$:-</u>

<u>4.8.0:REMARK:-</u>

From now on, we are going to be working mainly within the λ-calculus. It makes sense, therefore, to give a purely syntactic characterisation of the approximate semantic relation $\tilde{\sqsubseteq}$.

The characterisation shows $\tilde{\sqsubseteq}$ to have a strong connection with Wadsworth's notion of "direct approximant" - [57].

The results we have for $\overset{\approx}{\sqsubseteq}$ (from the lattice properties) would be tedious to prove if we worked solely in the λ-calculus, using 4.8.4 as a definition.

4.8.1:LEMMA:-

$x \overset{\approx}{\sqsubseteq} \delta \Rightarrow x = \delta$.

Proof:-

-$\bot \neq \phi_{\infty,0} \circ \overset{\approx}{\mathcal{E}}[\![x]\!] = x \sqsubseteq \phi_{\infty,0} \circ \overset{\approx}{\mathcal{E}}[\![\delta]\!] \neq \top$, by 4.4.1(iii).

-$\therefore$, $x = \phi_{\infty,0} \circ \overset{\approx}{\mathcal{E}}[\![\delta]\!] \neq \bot$.

-$\therefore$, $\delta \in I$, by4.5.0(iii), and can only be x.

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4.8.2:LEMMA:-

$\varepsilon \in \lambda I.EXP$ and $\varepsilon \notin NOH$ and $\varepsilon \overset{\approx}{\sqsubseteq} \delta \Rightarrow \varepsilon \xrightarrow{\alpha} \lambda x.\varepsilon'$ and $\delta \xrightarrow{\alpha} \lambda x.\delta'$ and $\varepsilon' \overset{\approx}{\sqsubseteq} \delta'$.

Proof:-

-Let $\varepsilon = \lambda a.\varepsilon''$.

-Now, $\overset{\approx}{\mathcal{E}}[\![\varepsilon]\!] \neq \bot$, by 4.5.2(i), and so, for some $i \geq 1$,

$\bot \neq \phi_{\infty,i} \circ \overset{\approx}{\mathcal{E}}[\![\lambda a.\varepsilon'']\!] = \lambda a.\varepsilon''_{i-1} \sqsubseteq \phi_{\infty,i} \circ \overset{\approx}{\mathcal{E}}[\![\delta]\!] \neq \top$.

-$\therefore$, $\phi_{\infty,i} \circ \overset{\approx}{\mathcal{E}}[\![\delta]\!] = \lambda b.\delta''_{i-1} \neq \bot$, for some b, δ''_{i-1}.

-$\therefore$, $\delta \xrightarrow{\alpha} \lambda b.\delta''$, for some b, δ'', by 4.5.3(ii).

-Choose x not free in ε'', δ''. Then,

$\varepsilon \xrightarrow{\alpha} \lambda x.\varepsilon'$, where $\varepsilon' = [x/a]\varepsilon''$,

and $\delta \xrightarrow{\alpha} \lambda x.\delta'$, where $\delta' = [x/b]\delta''$.

-But, $\lambda x.\varepsilon' \overset{\approx}{\sqsubseteq} \lambda x.\delta'$, by 4.4.8, since $\varepsilon \overset{\approx}{\sqsubseteq} \delta$.

-i.e. $\overset{\approx}{\mathcal{E}}[\![\lambda x.\varepsilon']\!] \sqsubseteq \overset{\approx}{\mathcal{E}}[\![\lambda x.\delta']\!]$.

-i.e. $\lambda x.\overset{\approx}{\mathcal{E}}[\![\varepsilon']\!] \sqsubseteq \lambda x.\overset{\approx}{\mathcal{E}}[\![\delta']\!]$.

-i.e. $\overset{\approx}{\mathcal{E}}[\![\varepsilon']\!] \sqsubseteq \overset{\approx}{\mathcal{E}}[\![\delta']\!]$.

-i.e. $\varepsilon' \overset{\approx}{\sqsubseteq} \delta'$.

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4.8.3:LEMMA:-

$\varepsilon \in$ (EXP)(EXP) and $\varepsilon \notin$ NOH and $\varepsilon \mathrel{\tilde{\sqsubseteq}} \delta \Rightarrow \delta = \omega'(\eta')$ and $\omega \mathrel{\tilde{\sqsubseteq}} \omega'$ and $\eta \mathrel{\tilde{\sqsubseteq}} \eta'$ and where $\varepsilon = \omega(\eta)$.

<u>Proof:-</u>

-In this case, $\varepsilon \in$ HEAD. So $\pi_2 \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] \neq \bot$, by 4.5.2(iii).

-But, $\pi_2 \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] \sqsubseteq \pi_2 \circ \tilde{\mathcal{E}}[\![\delta]\!]$, by monotonicity (2.2.16), and so $\pi_2 \circ \tilde{\mathcal{E}}[\![\delta]\!] \neq \bot$ - i.e. $\delta \in$ HEAD, by 4.5.2(iii) again.

-If $\delta = x \in I$, then $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] \sqsubseteq x$, $\forall i \geq 0$, and so,

<u>either</u> $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = \bot$, $\forall i \geq 0$ - $\not{X}$.

<u>or</u> $\phi_{\infty,i} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!] = x$, $\forall i \geq 0$, $\Rightarrow \varepsilon = x$, by4.5.3(i) - $\not{X}$.

-$\therefore$, $\delta = \omega'(\eta')$, with $\omega' \in$ HEAD.

-Now, we have $\varepsilon = \omega(\eta)$, with $\omega \in$ HEAD.

-But, $\tilde{\mathcal{E}}[\![\varepsilon]\!] \sqsubseteq \tilde{\mathcal{E}}[\![\delta]\!]$.

-$\therefore$, $\tilde{Ap}(\tilde{\mathcal{E}}[\![\omega]\!], \tilde{\mathcal{E}}[\![\eta]\!]) \sqsubseteq \tilde{Ap}(\tilde{\mathcal{E}}[\![\omega']\!], \tilde{\mathcal{E}}[\![\eta']\!])$.

-Since $\pi_2 \circ \tilde{\mathcal{E}}[\![\omega]\!] \neq \bot \neq \pi_2 \circ \tilde{\mathcal{E}}[\![\omega']\!]$, by 4.5.2(iii), by the simple characterisation of Ap given in 4.3.3(xii) we must have $\tilde{\mathcal{E}}[\![\omega]\!] \sqsubseteq \tilde{\mathcal{E}}[\![\omega']\!]$ and $\tilde{\mathcal{E}}[\![\eta]\!] \sqsubseteq \tilde{\mathcal{E}}[\![\eta']\!]$.

-i.e. $\omega \mathrel{\tilde{\sqsubseteq}} \omega'$ and $\eta \mathrel{\tilde{\sqsubseteq}} \eta'$.

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4.8.4:THEOREM:-

$\varepsilon \mathrel{\tilde{\sqsubseteq}} \delta \Leftrightarrow$ <u>either</u> $\varepsilon \in$ NOH,

<u>or</u> $\varepsilon = x = \delta$,

<u>or</u> $\varepsilon \xrightarrow{\alpha} \lambda x.\varepsilon' \in$ HNF,
$\delta \xrightarrow{\alpha} \lambda x.\delta'$ and $\varepsilon' \mathrel{\tilde{\sqsubseteq}} \delta'$,

<u>or</u> $\varepsilon = \omega(\eta) \in$ HEAD,
$\delta = \omega'(\eta')$ and $\omega \mathrel{\tilde{\sqsubseteq}} \omega'$ and $\eta \mathrel{\tilde{\sqsubseteq}} \eta'$.

<u>Proof:-</u>

($\Leftarrow$) -$\varepsilon \in$ NOH $\Rightarrow \mathcal{E}[\![\varepsilon]\!] = \bot$, by 4.5.2(i).
$\sqsubseteq \tilde{\mathcal{E}}[\![\delta]\!]$.

-$\varepsilon = x = \delta \Rightarrow \tilde{\mathcal{E}}[\![\varepsilon]\!] = x = \tilde{\mathcal{E}}[\![\delta]\!]$.

-Let $\varepsilon \xrightarrow{\alpha} \lambda x.\varepsilon' \in \mathrm{HNF}$, $\delta \xrightarrow{\alpha} \lambda x.\delta'$ and $\varepsilon' \overset{\approx}{\sqsubseteq} \delta'$.

-Then, $\overset{\approx}{E}\llbracket\varepsilon\rrbracket = \overset{\approx}{E}\llbracket\lambda x.\varepsilon'\rrbracket$, by 4.4.8.

$= \lambda x.\overset{\approx}{E}\llbracket\varepsilon'\rrbracket \sqsubseteq \lambda x.\overset{\approx}{E}\llbracket\delta'\rrbracket$, since $\varepsilon' \overset{\approx}{\sqsubseteq} \delta'$.

$= \overset{\approx}{E}\llbracket\lambda x.\delta'\rrbracket = \overset{\approx}{E}\llbracket\delta\rrbracket$, by 4.4.8.

-Let $\varepsilon = \omega(\eta) \in \mathrm{HEAD}$, $\delta = \omega'(\eta')$, $\omega \overset{\approx}{\sqsubseteq} \omega'$ and $\eta \overset{\approx}{\sqsubseteq} \eta'$.

-Then, $\overset{\approx}{E}\llbracket\varepsilon\rrbracket = \overset{\approx}{E}\llbracket\omega(\eta)\rrbracket = \widetilde{Ap}(\overset{\approx}{E}\llbracket\omega\rrbracket,\overset{\approx}{E}\llbracket\eta\rrbracket)$

$\sqsubseteq \widetilde{Ap}(\overset{\approx}{E}\llbracket\omega'\rrbracket,\overset{\approx}{E}\llbracket\eta'\rrbracket)$, since $\widetilde{Ap}$ is monotone, by 4.3.3(i).

$= \overset{\approx}{E}\llbracket\omega'(\eta')\rrbracket = \overset{\approx}{E}\llbracket\delta\rrbracket$.

(=>) -Let $\varepsilon \overset{\approx}{\sqsubseteq} \delta$.

-If $\overset{\approx}{E}\llbracket\varepsilon\rrbracket = \bot$,we have $\varepsilon \in \mathrm{NOH}$, by 4.5.2(i).

-Suppose $\overset{\approx}{E}\llbracket\varepsilon\rrbracket \neq \bot$ - i.e. $\varepsilon \notin \mathrm{NOH}$.

- -$\varepsilon \in I$ => second clause of the theorem, by 4.8.1.
- -$\varepsilon \in \lambda I.\mathrm{EXP}$ => third clause, by 4.8.2.
- -$\varepsilon \in (\mathrm{EXP})(\mathrm{EXP})$ => fourth clause, by 4.8.3.

‡

<u>4.8.5:REMARK:-</u>

Using this characterisation of $\overset{\approx}{\sqsubseteq}$, we will prove that normal forms are incomparable under $\sqsubseteq$. This is fairly obvious from the examples 4.0.3.

<u>4.8.6:LEMMA:-</u>

Normal forms are maximal under $\overset{\approx}{\sqsubseteq}$ modulo $\xrightarrow{\alpha}$ - i.e. :-

$$(\varepsilon \in \mathrm{NF}) \wedge (\varepsilon \overset{\approx}{\sqsubseteq} \delta) \Rightarrow (\varepsilon \xrightarrow{\alpha} \delta).$$

<u>Proof:-</u>

-Let $A\llbracket\varepsilon\rrbracket \equiv (\varepsilon \in \mathrm{NF}) \wedge (\chi(\varepsilon) \overset{\approx}{\sqsubseteq} \delta) \Rightarrow (\chi(\varepsilon) \xrightarrow{\alpha} \delta)$.

-<u>Claim:</u> $A\llbracket y\rrbracket$:-

-Now, $\chi(y) \in I$. So, by 4.8.4, $\chi(y) = \delta$.

-<u>Claim:</u> $A\llbracket\varepsilon\rrbracket \Rightarrow A\llbracket\lambda y.\varepsilon\rrbracket$:-

-Let $\lambda y.\varepsilon \in \mathrm{NF}$ and $\chi(\lambda y.\varepsilon) \overset{\approx}{\sqsubseteq} \delta$.

-Then, $\chi(\lambda y.\varepsilon) \in \mathrm{NF} \subseteq \mathrm{HNF}$.

-$\therefore$, by 4.8.4 and 4.1.4(i), we can find a z such that z is not in χ and not free in ε and :-

$$\chi(\lambda y.\varepsilon) \xrightarrow{\alpha} \lambda z.\chi[z/y]\varepsilon,$$

$$\delta \xrightarrow{\alpha} \lambda z.\delta'$$

and $\chi[z/y]\varepsilon \mathrel{\tilde{\sqsubseteq}} \delta'$.

-But, $\varepsilon \in \mathrm{NF}$, and so, by $A[\![\varepsilon]\!]$, $\chi[z/y]\varepsilon \xrightarrow{\alpha} \delta'$.

-Thus, $\chi(\lambda y.\varepsilon) \xrightarrow{\alpha} \delta$.

-<u>Claim:</u> $A[\![\omega]\!] \wedge A[\![\varepsilon]\!] \Rightarrow A[\![\omega(\varepsilon)]\!]$:-

-Let $\omega(\varepsilon) \in \mathrm{NF}$ and $\chi(\omega(\varepsilon)) \mathrel{\tilde{\sqsubseteq}} \delta$.

-Then, $\delta = \omega'(\varepsilon')$ and $\chi(\omega) \mathrel{\tilde{\sqsubseteq}} \omega'$, $\chi(\varepsilon) \mathrel{\tilde{\sqsubseteq}} \varepsilon'$, by 4.8.4.

-But, both ω and ε are in NF.

-So, by $A[\![\omega]\!] \wedge A[\![\varepsilon]\!]$, $\chi(\omega) \xrightarrow{\alpha} \omega'$ and $\chi(\varepsilon) \xrightarrow{\alpha} \varepsilon'$.

-Thus, $\chi(\omega(\varepsilon)) \xrightarrow{\alpha} \delta$.

-$\therefore$, by structural induction, $(\forall \varepsilon \in \mathrm{EXP})A[\![\varepsilon]\!]$.

-Hence, the lemma, taking χ as null.

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<u>4.8.7:COR:-</u>

Normal forms are incomparable under $\sqsubseteq$.

<u>Proof:-</u>

-Let $\varepsilon, \delta \in \mathrm{NF}$ and $\varepsilon \sqsubseteq \delta$.

-Then, $\varepsilon \mathrel{\tilde{\sqsubseteq}} \delta$, by 4.7.4, and so $\varepsilon \xrightarrow{\alpha} \delta$, by 4.8.6.

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<u>4.8.8:LEMMA:-</u>

Normal forms are maximal under $\sqsubseteq$ modulo $\xleftarrow{\beta}$ - i.e. :-

$$(\varepsilon \in \mathrm{NF}) \wedge (\varepsilon \sqsubseteq \delta) \Rightarrow (\delta \xrightarrow{\beta} \varepsilon).$$

<u>Proof:-</u>

-Let $\varepsilon \in \mathrm{NF}$ and $\varepsilon \sqsubseteq \delta$.

-Then, $\tilde{E}[\![\varepsilon]\!] = E[\![\varepsilon]\!] \sqsubseteq E[\![\delta]\!] \sqsubseteq V[\![\delta]\!]$, by 4.7.4 and 4.6.11.

-$\therefore$, $\tilde{E}[\![\varepsilon]\!] \prec V[\![\delta]\!]$, by 4.7.3 and 4.7.9.

-Thus, $\tilde{E}[\![\varepsilon]\!] \sqsubseteq \tilde{E}[\![\delta']\!]$, for some δ' such that $\delta \xrightarrow{\alpha,\beta} \delta'$.

-But, $\varepsilon \xrightarrow{\alpha} \delta'$, by 4.8.6, and so, $\delta \xrightarrow{\alpha,\beta} \varepsilon$.

ǂ

4.8.9:THEOREM:-

$\langle E_\infty, V\rangle$ is a normal model of the λ-calculus.

Proof:-

-Let ε have a normal form but δ not have one.

-Let $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon' \in NF$.

-Suppose $V\llbracket\varepsilon\rrbracket = V\llbracket\delta\rrbracket$:-

-Then, $\tilde{E}\llbracket\varepsilon'\rrbracket = V\llbracket\varepsilon'\rrbracket = V\llbracket\varepsilon\rrbracket = V\llbracket\delta\rrbracket$.

-So, $\tilde{E}\llbracket\varepsilon'\rrbracket \sqsubseteq \tilde{E}\llbracket\delta'\rrbracket$, for some δ' s.t. $\delta \xrightarrow{\alpha,\beta} \delta'$, as in proof of 4.8.8.

-But, then $\varepsilon' \xrightarrow{\alpha} \delta'$, by 4.8.6.

-∴, $\delta \xrightarrow{\alpha,\beta} \varepsilon' \in NF$ - X .

ǂ

4.9:Requirement for $\langle E_\infty, E\rangle$ to be a β-Model:-

4.9.0:PROPERTY X:-

$(\varepsilon \xrightarrow{\alpha,\beta} \varepsilon') \Rightarrow (\tilde{E}\llbracket\varepsilon'\rrbracket \sqsubseteq E\llbracket\varepsilon\rrbracket)$.

4.9.1:LEMMA:-

$\langle E_\infty, E\rangle$ is a β-model $\Leftrightarrow$ Property X $\Leftrightarrow$ $E = V$.

Proof:-

-Suppose $\langle E_\infty, E\rangle$ is a β-model :-

-Let $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$.

-Then, $\tilde{E}\llbracket\varepsilon'\rrbracket \sqsubseteq E\llbracket\varepsilon'\rrbracket$, by 4.4.1.

$= E\llbracket\varepsilon\rrbracket$, since it is a model.

-Suppose property X :-

-Now, $V\llbracket\varepsilon\rrbracket = \bigsqcup\{\tilde{E}\llbracket\varepsilon'\rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\} \sqsubseteq E\llbracket\varepsilon\rrbracket$, by property X.

-∴, $E = V$, by 4.6.11.

-If $E = V$, then $\langle E_\infty, E\rangle$ is a β-model, by 4.6.1.

ǂ

4.9.2:LEMMA:-

$(\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon') \Rightarrow (\tilde{E}\llbracket\varepsilon'\rrbracket \sqsubseteq E\llbracket\varepsilon\rrbracket)$.

Proof:-

-We do a structural induction on ε.

-The only interesting case is $(\lambda x.\varepsilon)(\delta) \xrightarrow{\alpha,1\beta} [\delta/x]\varepsilon$.

-Then, $E[\![(\lambda x.\varepsilon)(\delta)]\!] = Ap(\lambda x.E[\![\varepsilon]\!],E[\![\delta]\!])$

$= [E[\![\delta]\!]/x]E[\![\varepsilon]\!]$, by 3.4.8.

$\sqsupseteq [\overset{''}{E}[\![\delta]\!]/x]^{\sim}\overset{''}{E}[\![\varepsilon]\!]$, by 4.3.3(iii) and 4.4.1(i).

$= \overset{''}{E}[\![[\delta/x]\varepsilon]\!]$, by 4.4.4.

ǂ

4.9.3:REMARK:-

It is a pity, but there seems to be no obvious way to extend 4.9.2 to property X.

Property X is saying that the approximate semantics of a reductum holds less information than the complete semantics of the expression that was reduced. This should be true if E, $\overset{''}{E}$ and $\sqsubseteq$ really do correspond to our intuition.

Notice that we have the reverse problem to that of Wadsworth - [58]. We know that expressions with no HNF are $\perp$ (4.6.12), but we do not have modelship. We can do $E \sqsubseteq V$, while Wadsworth had the converse.

We could, of course, prove modelship directly if we could establish the missing identity of 3.4.11 and then follow the scheme outlined in 4.2.5. We repeat this scheme here, for the sake of completeness.

4.9.4:PROPERTY Y:-

For all $\varepsilon,\delta,\eta \in E_\infty$,

$$[\varepsilon/x]Ap(\delta,\eta) = Ap([\varepsilon/x]\delta,[\varepsilon/x]\eta).$$

4.9.5:PROPERTY Z:-

For all $\varepsilon,\delta,\eta \in EXP$,

$$[E[\![\varepsilon]\!]/x]Ap(E[\![\delta]\!],E[\![\eta]\!]) = Ap([E[\![\varepsilon]\!]/x]E[\![\delta]\!],[E[\![\varepsilon]\!]/x]E[\![\eta]\!]).$$

4.9.6:LEMMA:-

Property Y => Property Z.

Proof:-

-Trivial.

ǂ

4.9.7:LEMMA:-

$<E_{\infty}, E>$ is a β-model <=> Property Z.

Proof:-

(<=) -Let $A[\![\varepsilon]\!] \equiv E[\![[\varepsilon'/x]\chi(\varepsilon)]\!] = [E[\![\varepsilon']\!]/x]E[\![\chi(\varepsilon)]\!]$.

-Then, the structural induction goes through as in the proof of lemma 4.4.3, using 4.2.3 and property Z instead of 4.4.2.

-So, in particular, we have $E[\![[\varepsilon'/x]\varepsilon]\!] = [E[\![\varepsilon']\!]/x]E[\![\varepsilon]\!]$.

-Now, let $P[\![\varepsilon]\!] \equiv (\varepsilon \xrightarrow{\alpha,1\beta} \varepsilon') \Rightarrow (\varepsilon \equiv \varepsilon')$.

-The structural induction is the same as in 4.9.2. Again, the only interesting case is $(\lambda x.\varepsilon)(\delta) \xrightarrow{\alpha,1\beta} [\delta/x]\varepsilon$.

-Then, $E[\![(\lambda x.\varepsilon)(\delta)]\!] = [E[\![\delta]\!]/x]E[\![\varepsilon']\!] = E[\![[\delta/x]\varepsilon]\!]$, by above.

-This time there is no trouble in extending $P[\![\varepsilon]\!]$ to :-

$$(\varepsilon \xrightarrow{\alpha,\beta} \varepsilon') \Rightarrow (\varepsilon \equiv \varepsilon'),$$

since ≡ is transitive.

-$\therefore$, $<E_{\infty}, E>$ is a β-model.

(=>) -Now, $(\lambda x.\delta\eta)(\varepsilon)$ β-<u>cnv</u> $((\lambda x.\delta)(\varepsilon))((\lambda x.\eta)(\varepsilon))$.

-$\therefore$, $(\lambda x.\delta\eta)(\varepsilon) \equiv ((\lambda x.\delta)(\varepsilon))((\lambda x.\eta)(\varepsilon))$.

-But, $E[\![(\lambda x.\delta\eta)(\varepsilon)]\!] = \mathrm{Ap}(\lambda x.E[\![\delta\eta]\!], E[\![\varepsilon]\!])$

$= [E[\![\varepsilon]\!]/x]E[\![\delta\eta]\!]$, by 3.4.8.

$= [E[\![\varepsilon]\!]/x]\mathrm{Ap}(E[\![\delta]\!], E[\![\eta]\!])$.

-And, $E[\![((\lambda x.\delta)(\varepsilon))((\lambda x.\eta)(\varepsilon))]\!] = \mathrm{Ap}(E[\![(\lambda x.\delta)(\varepsilon)]\!], E[\![(\lambda x.\eta)(\varepsilon)]\!]$

$= \mathrm{Ap}(\mathrm{Ap}(\lambda x.E[\![\delta]\!], E[\![\varepsilon]\!]), \mathrm{Ap}(\lambda x.E[\![\eta]\!], E[\![\varepsilon]\!]))$

$= \mathrm{Ap}([E[\![\varepsilon]\!]/x]E[\![\delta]\!], [E[\![\varepsilon]\!]/x]E[\![\eta]\!])$, by 3.4.8.

ǂ

4.10:Summary of the Main Results of this Chapter:-

$\tilde{E} \sqsubseteq E \sqsubseteq V$.

$\langle E_\infty, \tilde{E}\rangle$, $\langle E_\infty, E\rangle$ and $\langle E_\infty, V\rangle$ model "not free in".

$\langle E_\infty, \tilde{E}\rangle$ and $\langle E_\infty, E\rangle$ are substitutive α-models.

Further, their induced partial orderings, $\tilde{\sqsubseteq}$ and $\sqsubseteq$, are substitutive.

$\langle E_\infty, V\rangle$ is a normal solvable model. (Substitutivity ?)

$\langle E_\infty, E\rangle$ is a model $\Longleftrightarrow$ $E = V$ $\Longleftrightarrow$ Property X $\Longleftrightarrow$ Property Z

$\Longleftarrow$ Property Y.

$\tilde{E}\llbracket[\delta/x]\varepsilon\rrbracket = [\tilde{E}\llbracket\delta\rrbracket/x]^{\sim}\tilde{E}\llbracket\varepsilon\rrbracket$.

$(\varepsilon \xrightarrow{\beta} \varepsilon') \Rightarrow (\varepsilon \,\tilde{\sqsubseteq}\, \varepsilon')$.

$(\varepsilon \xrightarrow{1\beta} \varepsilon') \Rightarrow (\tilde{E}\llbracket\varepsilon'\rrbracket \sqsubseteq E\llbracket\varepsilon\rrbracket)$.

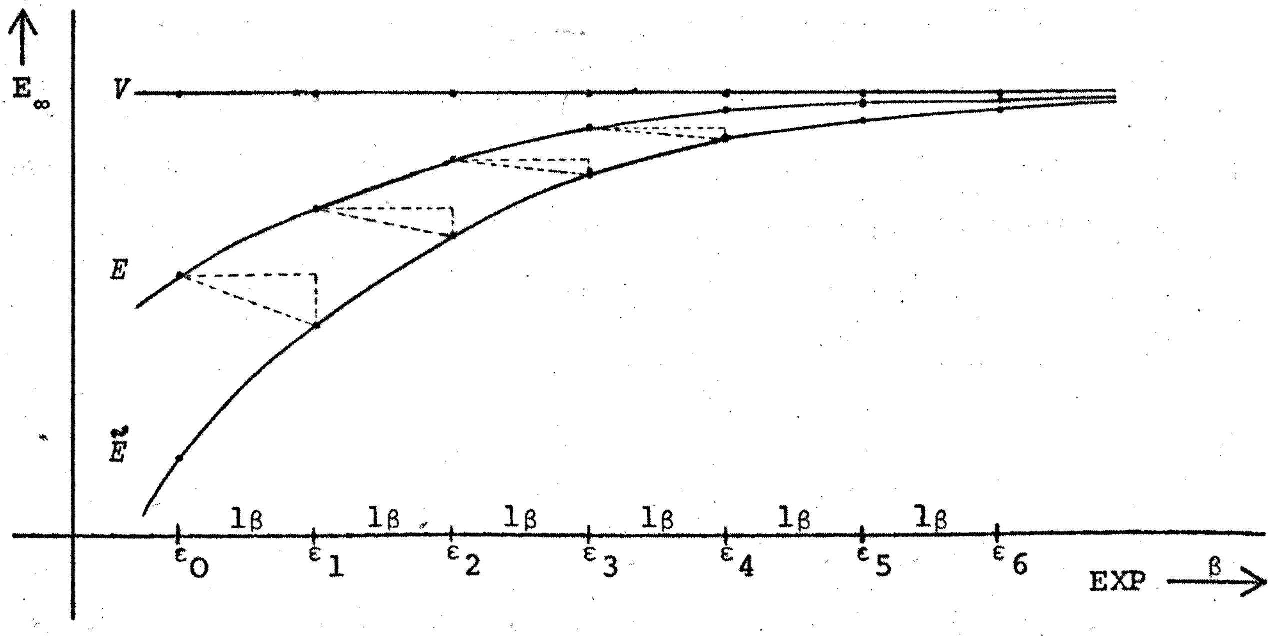
$(\varepsilon \prec \delta) \Longleftrightarrow (\varepsilon \sqsubseteq \delta)_\wedge(\varepsilon \text{ is finite})$.

β-normal forms are isolated. (N.B. finite $\equiv$ isolated)

$(\varepsilon \in \text{INSOL}) \Rightarrow ((\forall\delta \in \text{EXP})(\varepsilon \sqsubseteq \delta))_\wedge(E\llbracket\varepsilon\rrbracket = \bot)$.

$(\forall i \geq 0)(Y_i \equiv Y_0)$.

Noting that $\bigsqcup\{E\llbracket\varepsilon'\rrbracket \mid \varepsilon \xrightarrow{\beta} \varepsilon'\} = V\llbracket\varepsilon\rrbracket$, we can represent the relationship so far established between E, $\tilde{E}$ and V pictorially :-



However, we maintain that $E = V$.

5:I'TH REDUCTIONS.

5.0:I'th Application and Substitution:-

5.0.0:DEF:-

$$Ap^1(\varepsilon,\delta) = \varepsilon(\delta).$$

$$[\delta/x]^1\varepsilon = [\delta/x]\varepsilon.$$

For $i \geq 2$,

$$Ap^i(\varepsilon,\delta) = \left\{\begin{array}{l} [\delta/x]^i\varepsilon', \text{ if } \varepsilon = \lambda x.\varepsilon' \\ \varepsilon(\delta), \text{ if } \varepsilon \notin \lambda I.EXP \end{array}\right\}$$

and,

$$[\delta/x]^i\varepsilon = \left\{\begin{array}{l} [\delta/x]\varepsilon, \text{ if } \varepsilon \in I \cup NOH \\ \lambda z.[\delta/x]^{i-1}[z/y]\varepsilon', \text{ if } (*) \\ Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta), \text{ if } (**) \end{array}\right\},$$

where $(*) \equiv (\varepsilon = \lambda y.\varepsilon' \in HNF)\wedge(z \neq x$ and is not free in $\delta,\varepsilon')$

and $(**) \equiv (\varepsilon = \omega(\eta) \in HEAD)$.

5.0.1:REMARK:-

In the above definition we are trying to copy 3.0.3 as closely as possible. When computing Ap^i as opposed to Ap_i, the only differences should be those of notation - superscripts for subscripts, nothing for projections and expressions in NOH for $\perp$'s. This explains why, when $\varepsilon \in NOH$, we must do no work in computing $[\delta/x]^i\varepsilon$, so that we remain in NOH - compare with $[\delta/x]_{i-1}\perp = \perp$.

Before going any further, we must establish a few basic properties of i'th applications and substitutions - in particular that they are well-defined up to α-conversion. Compare the following induction hypothesis with that of 3.1.0. It has scope from 5.0.2 to 5.0.11 inclusive.

5.0.2:DEF:-

Let $P(\mathrm{i}) \equiv (\mathrm{Ap}^{\mathrm{i}}$ is well-defined up to α-conversion)

& $(\varepsilon \xrightarrow{\alpha} \varepsilon', \delta \xrightarrow{\alpha} \delta' \Rightarrow \mathrm{Ap}^{\mathrm{i}}(\varepsilon,\delta) \xrightarrow{\alpha} \mathrm{Ap}^{\mathrm{i}}(\varepsilon',\delta'))$

& $(\varepsilon \in \mathrm{NOH} \Rightarrow \mathrm{Ap}^{\mathrm{i}}(\varepsilon,\delta) \in \mathrm{NOH})$

& $(\mathrm{Ap}^{\mathrm{i}}([x/y]\varepsilon,[x/y]\delta) \xrightarrow{\alpha} [x/y]\mathrm{Ap}^{\mathrm{i}}(\varepsilon,\delta))$

& (a is not free in $\varepsilon,\delta \Rightarrow$ a is not free in $\mathrm{Ap}^{\mathrm{i}}(\varepsilon,\delta)$).

Let $Q(\mathrm{i}) \equiv ([\delta/x]^{\mathrm{i}}$ is well-defined up to α-conversion)

& $(\varepsilon \xrightarrow{\alpha} \varepsilon', \delta \xrightarrow{\alpha} \delta' \Rightarrow [\delta/x]^{\mathrm{i}}\varepsilon \xrightarrow{\alpha} [\delta'/x]^{\mathrm{i}}\varepsilon')$

& $(\varepsilon \in \mathrm{NOH} \Rightarrow [\delta/x]^{\mathrm{i}}\varepsilon \in \mathrm{NOH})$

& (a is not free in $\varepsilon \Rightarrow [\delta/a]^{\mathrm{i}}[a/x]\varepsilon \xrightarrow{\alpha} [\delta/x]^{\mathrm{i}}\varepsilon$)

& $(z \neq x,y) \Rightarrow ([[x/y]\delta/z]^{\mathrm{i}}[x/y]\varepsilon \xrightarrow{\alpha} [x/y][\delta/z]^{\mathrm{i}}\varepsilon)$

& (a is not free in $\varepsilon,\delta \Rightarrow$ a is not free in $[\delta/x]^{\mathrm{i}}\varepsilon$)

& (x is not free in $\varepsilon \Rightarrow [\delta/x]^{\mathrm{i}}\varepsilon \xrightarrow{\alpha} \varepsilon$).

5.0.3:LEMMA:-

$P(1) \wedge Q(1)$.

Proof:-

-All parts are obvious except for the third clause of $Q(1)$. But we already have this by 0.5.2(iii), or as follows :-

-Let $A[\![\varepsilon]\!] \equiv ([\delta/x]\chi(\varepsilon) \in \mathrm{NOH})$.

-Clearly, $A[\![(\lambda a.\mu)(\nu)]\!]$ and $A[\![\varepsilon]\!] \Rightarrow A[\![\varepsilon(\omega)]\!] \wedge A[\![\lambda a.\varepsilon]\!]$.

-∴, by structural induction, $(\forall\varepsilon \in \mathrm{NOH})A[\varepsilon]$.

ǂ

5.0.4:LEMMA:-

$Q(\mathrm{i}) \Rightarrow P(\mathrm{i}), \forall \mathrm{i} \geq 2$.

Proof:-

-Suppose $Q(\mathrm{i})$, for some $\mathrm{i} \geq 2$.

-Then, clearly Ap^{i} is well-defined up to α-conversion.

-Let $\varepsilon \xrightarrow{\alpha} \varepsilon'$ and $\delta \xrightarrow{\alpha} \delta'$. If $\varepsilon \notin \lambda\mathrm{I.EXP}$, then O.K. If not, then $\varepsilon = \lambda x.\eta$ and $\varepsilon' =$ either $\lambda x'.\eta'$ or $\lambda x.\eta''$:-

-If $\varepsilon' = \lambda x'.\eta'$, then x is not free in η' and $\eta = [x/x']\eta'$.

-Then, $Ap^{i}(\lambda x.\eta,\delta) = [\delta/x]^{i}\eta = [\delta/x]^{i}[x/x']\eta'$

$\xrightarrow{\alpha} [\delta/x']^{i}\eta'$, by $Q(i)$.

$\xrightarrow{\alpha} [\delta'/x']^{i}\eta'$, by $Q(i)$.

$= Ap^{i}(\lambda x'.\eta',\delta')$.

-On the other hand, if $\varepsilon' = \lambda x.\eta''$, then $\eta \xrightarrow{\alpha} \eta''$.

-Thus, $Ap^{i}(\lambda x.\eta,\delta) = [\delta/x]^{i}\eta \xrightarrow{\alpha} [\delta'/x]^{i}\eta''$, by $Q(i)$.

$= Ap^{i}(\lambda x.\eta'',\delta')$.

-So, either way, $Ap^{i}(\varepsilon,\delta) \xrightarrow{\alpha} Ap^{i}(\varepsilon',\delta')$.

-Let $\varepsilon \in$ NOH.

-If $\varepsilon \notin \lambda I.EXP$, then $Ap^{i}(\varepsilon,\delta) = \varepsilon(\delta) \in$ NOH also.

-If $\varepsilon = \lambda x.\varepsilon'$, then $\varepsilon' \in$ NOH and so,

$Ap^{i}(\lambda x.\varepsilon',\delta) = [\delta/x]^{i}\varepsilon' \in$ NOH, by $Q(i)$.

-The next clause is trivial if $\varepsilon \notin \lambda I.EXP$. Otherwise,

$Ap^{i}([x/y]\lambda a.\varepsilon',[x/y]\delta) \xrightarrow{\alpha} Ap^{i}(\lambda z.[x/y][z/a]\varepsilon',[x/y]\delta)$,

by above, where $z \neq x,y$ and is not free in ε'.

$= [[x/y]\delta/z]^{i}[x/y][z/a]\varepsilon'$

$\xrightarrow{\alpha} [x/y][\delta/z]^{i}[z/a]\varepsilon'$, by $Q(i)$.

$\xrightarrow{\alpha} [x/y][\delta/a]^{i}\varepsilon'$, by $Q(i)$.

$= [x/y]Ap^{i}(\lambda a.\varepsilon',\delta)$.

-Finally, let a be not free in ε,δ.

-If $\varepsilon \notin \lambda I.EXP$, then a is not free in $Ap^{i}(\varepsilon,\delta)$, trivially.

-Suppose $\varepsilon = \lambda x.\varepsilon'$. Then, $\varepsilon \xrightarrow{\alpha} \lambda x'.\varepsilon''$, where $a \neq x'$ and is not free in ε''.

-But, $Ap^{i}(\lambda x'.\varepsilon'',\delta) = [\delta/x']^{i}\varepsilon''$, in which a is not free, by $Q(i)$.

-$\therefore$, a is not free in $Ap^{i}(\varepsilon,\delta)$, since this is α-convertible to $Ap^{i}(\lambda x'.\varepsilon'',\delta)$.

ǂ

5.0.5:LEMMA:-

Suppose $Q(i-1) \wedge P(i-1)$, for some $i \geq 2$. Then, $[\delta/x]^{i}$ is well-defined up to α-conversion.

Proof:-

-Consider $[\delta/x]^{i}\varepsilon$.

-If $\varepsilon \in I \cup NOH$, then trivial.

-Suppose $\varepsilon = \lambda x.\varepsilon' \in HNF$.

-Let $z, z' \neq x$ and be not free in δ, ε' and $z \neq z'$.

-Now, z' is not free in $[z/y]\varepsilon'$, and so, by $Q(i-1)$, is not free in $[\delta/x]^{i-1}[z/y]\varepsilon'$.

-But, $[z'/z][\delta/x]^{i-1}[z/y]\varepsilon'$

$$\xrightarrow{\alpha} [[z'/z]\delta/x]^{i-1}[z'/z][z/y]\varepsilon', \text{ by } Q(i-1).$$

$$\xrightarrow{\alpha} [\delta/x]^{i-1}[z'/y]\varepsilon', \text{ since } z \text{ is not free in } \delta, \varepsilon'.$$

-∴, $\lambda z.[\delta/x]^{i-1}[z/y]\varepsilon' \xrightarrow{\alpha} \lambda z'.[\delta/x]^{i-1}[z'/y]\varepsilon'$.

-Finally, if $\varepsilon = \omega(\eta) \in HEAD$, then $[\delta/x]^{i}\varepsilon$ is well-defined up to α-conversion by $Q(i-1) \wedge P(i-1)$ - (twice).

‡

5.0.6:LEMMA:-

Suppose $Q(i-1) \wedge P(i-1)$, for some $i \geq 2$. Let $\varepsilon \xrightarrow{\alpha} \varepsilon'$ and $\delta \xrightarrow{\alpha} \delta'$. Then, $[\delta/x]^{i}\varepsilon \xrightarrow{\alpha} [\delta'/x]^{i}\varepsilon'$.

Proof:-

-If $\varepsilon \in I \cup NOH$, then trivial.

-Suppose $\varepsilon = \lambda y.\eta \in HNF$.

-If $\varepsilon' = \lambda y.\eta'$, then $\eta \xrightarrow{\alpha} \eta'$.

-Choose $z \neq x$ and not free in δ, η. Then, z is also not free in η'.

-Then, $[\delta/x]^{i-1}[z/y]\eta \xrightarrow{\alpha} [\delta'/x]^{i-1}[z/y]\eta'$, by $Q(i-1)$, since $[z/y]\eta \xrightarrow{\alpha} [z/y]\eta'$.

-∴, $[\delta/x]^{i}(\lambda y.\eta) \longrightarrow [\delta'/x]^{i}(\lambda y.\eta')$.

-On the other hand, if $\varepsilon' = \lambda y''.\eta''$, then y is not free in η'' and $[y/y'']\eta'' = \eta$.

-Choose $z \neq x$ and not free in δ,η,η''. Then,

$$
\begin{aligned}
[\delta/x]^{i}(\lambda y.\eta) &= \lambda z.[\delta/x]^{i-1}[z/y]\eta \\
&= \lambda z.[\delta/x]^{i-1}[z/y][y/y'']\eta'' \\
&\xrightarrow{\alpha} \lambda z.[\delta/x]^{i-1}[z/y'']\eta'', \text{ by } Q(i-1). \\
&= [\delta/x]^{i}(\lambda y''.\eta'').
\end{aligned}
$$

-Finally, suppose $\varepsilon = \omega(\eta) \in$ HEAD.

-Then, $\varepsilon' = \omega'(\eta')$, with $\omega \xrightarrow{\alpha} \omega'$ and $\eta \xrightarrow{\alpha} \eta'$.

-Then, $[\delta/x]^{i}\omega(\eta) \xrightarrow{\alpha} [\delta'/x]^{i}\omega'(\eta')$, by $Q(i-1) \wedge P(i-1)$.

‡

5.0.7:LEMMA:-

Suppose $Q(i-1) \wedge P(i-1)$, for some $i \geq 2$. Let a be not free in ε. Then, $[\delta/a]^{i}[a/x]\varepsilon \xrightarrow{\alpha} [\delta/x]^{i}\varepsilon$.

Proof:-

-If $\varepsilon \in I \cup$ NOH, then trivial.

-Suppose $\varepsilon = \lambda y.\varepsilon' \in$ HNF.

-Choose z and z' such that $(z \neq z') \wedge (z,z' \neq a,x) \wedge$ (z is not free in ε') $\wedge$ (z' is not free in ε',δ). Then, note that a is not free in $[z'/y]\varepsilon'$.

-∴,
$$
\begin{aligned}
[\delta/a]^{i}[a/x](\lambda y.\varepsilon') &= [\delta/a]^{i}(\lambda z.[a/x][z/y]\varepsilon') \\
&= \lambda z'.[\delta/a]^{i-1}[z'/z][a/x][z/y]\varepsilon' \\
&\xrightarrow{\alpha} \lambda z'.[\delta/a]^{i-1}[a/x][z'/y]\varepsilon' \\
&\xrightarrow{\alpha} \lambda z'.[\delta/x]^{i-1}[z'/y]\varepsilon', \text{ by } Q(i-1). \\
&= [\delta/x]^{i}(\lambda y.\varepsilon').
\end{aligned}
$$

-Finally, suppose $\varepsilon = \omega(\eta) \in$ HEAD. Then, a is not free in ω,η.

-∴,
$$
\begin{aligned}
[\delta/a]^{i}[a/x]\omega(\eta) &= [\delta/a]^{i}([a/x]\omega)([a/x]\eta) \\
&= Ap^{i-1}([\delta/a]^{i-1}[a/x]\omega,[\delta/a]^{i-1}[a/x]\eta) \\
&\xrightarrow{\alpha} Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta), \text{ by } Q(i-1) \\
&= [\delta/x]^{i}\omega(\eta). \qquad \wedge P(i-1).
\end{aligned}
$$

‡

5.0.8:LEMMA:-

Suppose $Q(i-1) \wedge P(i-1)$, for some $i \geq 2$. Then, if $z \neq x,y$,

$$[[x/y]\delta/z]^{i}[x/y]\varepsilon \xrightarrow{\alpha} [x/y][\delta/z]^{i}\varepsilon.$$

Proof:-

-If $\varepsilon \in NOH \cup I$, then so is $[x/y]\varepsilon$, by $Q(1)$, and the result is trivial.

-Suppose $\varepsilon = \lambda a.\omega \in HNF$.

-Choose b and c such that $(b \neq c) \wedge (b,c \neq x,y,z) \wedge$ (b,c are not free in ω,δ). Note that c is not free in $[x/y]\delta, [x/y][b/a]\omega$.

-Then, $[[x/y]\delta/z]^{i}[x/y](\lambda a.\omega)$

$$\xrightarrow{\alpha} [[x/y]\delta/z]^{i}(\lambda b.[x/y][b/a]\omega)$$

$$= \lambda c.[[x/y]\delta/z]^{i-1}[c/b][x/y][b/a]\omega$$

$$\xrightarrow{\alpha} \lambda c.[[x/y]\delta/z]^{i-1}[x/y][c/b][b/a]\omega$$

$$\xrightarrow{\alpha} \lambda c.[x/y][\delta/z]^{i-1}[c/a]\omega, \text{ by } Q(i-1).$$

$$= [x/y](\lambda c.[\delta/z]^{i-1}[c/a]\omega)$$

$$= [x/y][\delta/z]^{i}(\lambda a.\omega).$$

-Finally, suppose $\varepsilon = \omega(\eta) \in HEAD$.

-Then, $[[x/y]\delta/z]^{i}[x/y]\omega(\eta)$

$$= [[x/y]\delta/z]^{i}([x/y]\omega)([x/y]\eta)$$

$$= Ap^{i-1}([[x/y]\delta/z]^{i-1}[x/y]\omega, [[x/y]\delta/z]^{i-1}[x/y]\eta)$$

$$\xrightarrow{\alpha} Ap^{i-1}([x/y][\delta/z]^{i-1}\omega, [x/y][\delta/z]^{i-1}\eta),$$

by $Q(i-1) \wedge P(i-1)$.

$$\xrightarrow{\alpha} [x/y]Ap^{i-1}([\delta/z]^{i-1}\omega, [\delta/z]^{i-1}\eta), \text{ by } P(i-1).$$

$$= [x/y][\delta/z]^{i}\omega(\eta).$$

ǂ

5.0.9:LEMMA:-

Suppose $Q(i-1) \wedge P(i-1)$, for some $i \geq 2$. Let a be not free in ε,δ. Then, a is not free in $[\delta/x]^{i}\varepsilon$.

Proof:-

-If $\varepsilon \in I \cup NOH$, then trivial.

-Suppose $\varepsilon = \lambda y.\varepsilon'$. Choose $z \neq x,a$ and not free in ε'.

-Then, a is not free in $[z/y]\varepsilon'$.

-$\therefore$, a is not free in $[\delta/x]^{i-1}[z/y]\varepsilon'$, by $\mathcal{Q}(i-1)$.

-$\therefore$, a is not free in $\lambda z.[\delta/x]^{i-1}[z/y]\varepsilon' = [\delta/x]^{i}(\lambda y.\varepsilon')$.

-Finally, suppose $\varepsilon = \omega(\eta) \in HEAD$. Then, a is not free in ω,η.

-$\therefore$, a is not free in $[\delta/x]^{i-1}\omega, [\delta/x]^{i-1}\eta$, by $\mathcal{Q}(i-1)$.

-$\therefore$, a is not free in $Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)$, by $\mathcal{P}(i-1)$,

$= [\delta/x]^{i}\omega(\eta)$.

‡

5.0.10:LEMMA:-

Suppose $\mathcal{Q}(i-1)\wedge\mathcal{P}(i-1)$, for some $i \geq 2$. Then, if x is not free in ε, $[\delta/x]^{i}\varepsilon \xrightarrow{\alpha} \varepsilon$.

Proof:-

-If $\varepsilon \in I \cup NOH$, then O.K.

-If $\varepsilon = \lambda y.\varepsilon' \in HNF$, choose $z \neq x$ and not free in δ,ε'.

-Since x is not free in ε, $= \lambda y.\varepsilon'$, we have that x is not free in $[z/y]\varepsilon'$.

-Thus, $[\delta/x]^{i}(\lambda y.\varepsilon') = \lambda z.[\delta/x]^{i-1}[z/y]\varepsilon'$

$\xrightarrow{\alpha} \lambda z.[z/y]\varepsilon'$, by $\mathcal{Q}(i-1)$.

$\xrightarrow{\alpha} \lambda y.\varepsilon'$.

-If $\varepsilon = \omega(\eta) \in HEAD$, then x is not free in ω,η.

-Then, $[\delta/x]^{i}\omega(\eta) = Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)$

$\xrightarrow{\alpha} Ap^{i-1}(\omega,\eta)$, by $\mathcal{Q}(i-1)\wedge\mathcal{P}(i-1)$.

$= \omega(\eta)$, since $\omega \in HEAD$.

‡

5.0.11:THEOREM:-

$(\forall i \geq 1)(\mathcal{P}(i)\wedge\mathcal{Q}(i))$.

Proof:-

-We have $P(1)\wedge Q(1)$,by 5.0.3.

-Now, $\varepsilon \in \mathrm{NOH} \Rightarrow [\delta/x]^{i}\varepsilon = [\delta/x]\varepsilon \in \mathrm{NOH}$, by $Q(1)$.

-$\therefore$, $P(i-1)\wedge Q(i-1)\Rightarrow Q(i)$, $\forall i \geq 2$, by 5.0.5,5.0.6,5.0.7,5.0.8, 5.0.9,5.0.10 and above.

-And, $Q(i)\Rightarrow P(i)$, $\forall i \geq 2$, by 5.0.4.

-$\therefore$, $P(i-1)\wedge Q(i-1)\Rightarrow P(i)\wedge Q(i)$, $\forall i \geq 2$.

-$\therefore$, by ordinary induction, $P(i)\wedge Q(i)$, $\forall i \geq 1$.

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5.0.12:COR:-

If $z \neq x$ and is not free in δ, then, for all $i \geq 2$,

$$[\delta/x]^{i}(\lambda z.\varepsilon) \xrightarrow{\alpha} \lambda z.[\delta/x]^{i-1}\varepsilon.$$

Proof:-

-Let $z \neq x$ and be not free in δ. Choose $z' \neq x$ and not free in δ,ε.

-Then, $[\delta/x]^{i}(\lambda z.\varepsilon) = \lambda z'.[\delta/x]^{i-1}[z'/z]\varepsilon$

$\xrightarrow{\alpha} \lambda z'.[[z'/z]\delta/x]^{i-1}[z'/z]\varepsilon$, by 5.0.11, since z is not free in δ.

$\xrightarrow{\alpha} \lambda z'.[z'/z][\delta/x]^{i-1}\varepsilon$, by 5.0.11, since $x \neq z,z'$.

$\xrightarrow{\alpha} \lambda z.[\delta/x]^{i-1}\varepsilon$, since z' is not free in $[\delta/x]^{i-1}\varepsilon$, by 5.0.11.

ǂ

5.1:Preservation of $\xrightarrow{\alpha,\beta}$ and $\tilde{\sqsubseteq}$:-

5.1.0:LEMMA:-

$$(\varepsilon(\delta) \xrightarrow{\alpha,\beta} Ap^{i}(\varepsilon,\delta)) \text{ and } ([\delta/x]\varepsilon \xrightarrow{\alpha,\beta} [\delta/x]^{i}\varepsilon).$$

Proof:-

-Let $P(i)$ and $Q(i)$ be the above sentence.

-Clearly, $P(1) \wedge Q(1)$.

-Claim: $Q(i) \Rightarrow P(i)$, $\forall \geq 2$:-

-$\varepsilon \notin \lambda I.EXP \Rightarrow \varepsilon(\delta) = Ap^{i}(\varepsilon,\delta)$.

-$\varepsilon = \lambda x.\varepsilon' \Rightarrow (\lambda x.\varepsilon')(\delta) \xrightarrow{\beta} [\delta/x]\varepsilon'$

$\xrightarrow{\alpha,\beta} [\delta/x]^{i}\varepsilon'$, by $Q(i)$.

$= Ap^{i}(\lambda x.\varepsilon',\delta)$.

-Claim: $P(i-1) \wedge Q(i-1) \Rightarrow Q(i)$, $\forall \geq 2$:-

-$\varepsilon \in I \cup NOH \Rightarrow [\delta/x]\varepsilon = [\delta/x]^{i}\varepsilon$.

-$\varepsilon = \lambda y.\varepsilon' \in HNF$ and $z \neq x$ and not free in $\delta,\varepsilon' \Rightarrow$

$[\delta/x](\lambda y.\varepsilon') = \lambda z.[\delta/x][z/y]\varepsilon'$

$\xrightarrow{\alpha,\beta} \lambda z.[\delta/x]^{i-1}[z/y]\varepsilon'$, by $Q(i-1)$.

$= [\delta/x]^{i}(\lambda y.\varepsilon')$.

-$\varepsilon = \omega(\eta) \in HEAD \Rightarrow$

$[\delta/x]\omega(\eta) = ([\delta/x]\omega)([\delta/x]\eta)$

$\xrightarrow{\alpha,\beta} ([\delta/x]^{i-1}\omega)([\delta/x]^{i-1}\eta)$, by $Q(i-1)$.

$\xrightarrow{\alpha,\beta} Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)$, by $P(i-1$

$= [\delta/x]^{i}\omega(\eta)$.

-Thus, by induction, $P(i) \wedge Q(i)$, $\forall \geq 1$.

‡

5.1.1:LEMMA:-

$\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$, $\delta \xrightarrow{\alpha,\beta} \delta' \Rightarrow Ap^{i}(\varepsilon,\delta) \xrightarrow{\alpha,\beta} Ap^{i}(\varepsilon',\delta')$.

$\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$, $\delta \xrightarrow{\alpha,\beta} \delta' \Rightarrow [\delta/x]^{i}\varepsilon \xrightarrow{\alpha,\beta} [\delta'/x]^{i}\varepsilon'$.

Proof:-

-Let $P(i)$ and $Q(i)$ be the above two sentences, respectively.

-Clearly, $P(1) \wedge Q(1)$.

-Claim: $Q(i) \Rightarrow P(i)$, $\forall \geq 2$:-

-If $\varepsilon \notin \lambda I.EXP$, clearly $P(i)$.

-If not, then $\varepsilon \xrightarrow{\alpha} \lambda x.\eta$, $\varepsilon' \xrightarrow{\alpha} \lambda x.\eta'$, $\eta \xrightarrow{\alpha,\beta} \eta'$.

-Then, $Ap^{i}(\varepsilon,\delta) \xrightarrow{\alpha} Ap^{i}(\lambda x.\eta,\delta)$, by 5.0.10.

$= [\delta/x]^{i}\eta$

$\xrightarrow{\alpha,\beta} [\delta'/x]^i\eta'$, by $Q(i)$.

$= Ap^i(\lambda x.\eta',\delta')$

$\xrightarrow{\alpha} Ap^i(\varepsilon',\delta')$, by 5.0.11.

-Claim: $P(i-1) \wedge Q(i-1) \Rightarrow Q(i)$, $\forall i \geq 2$:-

-If $\varepsilon \in I \cup NOH$, then $[\delta/x]^i\varepsilon = [\delta/x]\varepsilon$

$\xrightarrow{\alpha,\beta} [\delta'/x]\varepsilon'$

$\xrightarrow{\alpha,\beta} [\delta'/x]^i\varepsilon'$, by 5.1.0.

-If $\varepsilon \in \lambda I.HNF$, $\varepsilon \xrightarrow{\alpha} \lambda y.\eta$, $\varepsilon' \xrightarrow{\alpha} \lambda y.\eta'$, $\eta \xrightarrow{\alpha,\beta} \eta'$.

-Choose $z \neq x$ and not free in δ,η,η'.

-Note that $[z/y]\eta \xrightarrow{\alpha,\beta} [z/y]\eta'$.

-Then, $[\delta/x]^i\varepsilon \xrightarrow{\alpha} [\delta/x]^i(\lambda y.\eta)$, by 5.0.11.

$= \lambda z.[\delta/x]^{i-1}[z/y]\eta$

$\xrightarrow{\alpha,\beta} \lambda z.[\delta'/x]^{i-1}[z/y]\eta'$, by $Q(i-1)$.

$= [\delta'/x]^i(\lambda y.\eta')$

$\xrightarrow{\alpha} [\delta'/x]^i\varepsilon'$, by 5.0.11.

-If $\varepsilon = \omega(\eta) \in HEAD$, then $\varepsilon' = \omega'(\eta')$ and $\omega \xrightarrow{\alpha,\beta} \omega'$, $\eta \xrightarrow{\alpha,\beta} \eta'$.

-Then, $[\delta/x]^i\varepsilon = Ap^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)$.

-But, $[\delta/x]^{i-1}\omega \xrightarrow{\alpha,\beta} [\delta'/x]^{i-1}\omega'$, by $Q(i-1)$.

-And, $[\delta/x]^{i-1}\eta \xrightarrow{\alpha,\beta} [\delta'/x]^{i-1}\eta'$, by $Q(i-1)$.

-$\therefore$, $[\delta/x]^i\varepsilon \xrightarrow{\alpha,\beta} Ap^{i-1}([\delta'/x]^{i-1}\omega',[\delta'/x]^{i-1}\eta')$, by $P(i-1)$.

$= [\delta'/x]^i\varepsilon'$.

-$\therefore$, by induction, $P(i) \wedge Q(i)$, $\forall \geq 1$.

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5.1.2:REMARK:-

In the proof of the above theorem, we see how neccessary it is that $[\delta/x]^i\varepsilon$ does nothing when $\varepsilon \in NOH$. Since, when $\varepsilon = \omega(\eta)$ and $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$, if we could not assure that $\varepsilon \in HEAD$, we would not be able to say anything about ε'.

Thus, without this clause, i'th applications and substitutions, although well-defined up to α-conversion, would behave badly with respect to β-reductions.

5.1.3:THEOREM:-

$$\mathrm{Ap}^{i}(\varepsilon,\delta) \xrightarrow{\alpha,\beta} \mathrm{Ap}^{i+1}(\varepsilon,\delta).$$

$$[\delta/x]^{i}\varepsilon \xrightarrow{\alpha,\beta} [\delta/x]^{i+1}\varepsilon.$$

Proof:-

-Let $P(i)$ and $Q(i)$ be the above two sentences, respectively.

-Claim: $P(1)$ and $Q(1)$:-

-Clearly $Q(1)$, since $[\delta/x]^{2}\varepsilon = [\delta/x]^{1}\varepsilon$.

-If $\varepsilon \notin \lambda\mathrm{I.EXP}$, $\mathrm{Ap}^{2}(\varepsilon,\delta) = \varepsilon(\delta) = \mathrm{Ap}^{1}(\varepsilon,\delta)$.

-Otherwise, $$\begin{aligned}\mathrm{Ap}^{1}(\lambda x.\varepsilon',\delta) &= (\lambda x.\varepsilon')(\delta)\\ &\xrightarrow{\beta} [\delta/x]\varepsilon'\\ &= [\delta/x]^{2}\varepsilon'\\ &= \mathrm{Ap}^{2}(\lambda x.\varepsilon',\delta).\end{aligned}$$

-Claim: $Q(i) \Rightarrow P(i), \forall i \geq 2$:-

-If $\varepsilon \notin \lambda\mathrm{I.EXP}$, $\mathrm{Ap}^{i}(\varepsilon,\delta) = \varepsilon(\delta) = \mathrm{Ap}^{i+1}(\varepsilon,\delta)$.

-Otherwise, $$\begin{aligned}\mathrm{Ap}^{i}(\lambda x.\varepsilon',\delta) &= [\delta/x]^{i}\varepsilon'\\ &\xrightarrow{\alpha,\beta} [\delta/x]^{i+1}\varepsilon', \text{ by } Q(i).\\ &= \mathrm{Ap}^{i+1}(\lambda x.\varepsilon',\delta).\end{aligned}$$

-Claim: $P(i-1) \wedge Q(i-1) \Rightarrow Q(i), \forall i \geq 2$:-

-If $\varepsilon \in \mathrm{I} \cup \mathrm{NOH}$, $[\delta/x]^{i}\varepsilon = [\delta/x]\varepsilon = [\delta/x]^{i+1}\varepsilon$.

-If $\varepsilon = \lambda y.\varepsilon' \in \mathrm{HNF}$, and $z \neq x$ and is not free in δ,ε',

$$\begin{aligned}\delta/x\ ^{i}(\lambda y.\varepsilon') &= \lambda z.[\delta/x]^{i-1}[z/y]\varepsilon\\ &\xrightarrow{\alpha,\beta} \lambda z.[\delta/x]^{i}[z/y]\varepsilon, \text{ by } Q(i-1).\\ &= [\delta/x]^{i+1}(\lambda y.\varepsilon').\end{aligned}$$

-If $\varepsilon = \omega(\eta) \in \mathrm{HEAD}$, then $[\delta/x]^{i}\omega(\eta) = \mathrm{Ap}^{i-1}([\delta/x]^{i-1}\omega,\ [\delta/x]^{i-1}\eta)$.

-But, $[\delta/x]^{i-1}\omega \xrightarrow{\alpha,\beta} [\delta/x]^{i}\omega$, by $Q(i-1)$.

-And, $[\delta/x]^{i-1}\eta \xrightarrow{\alpha,\beta} [\delta/x]^{i}\eta$, by $Q(i-1)$.

-∴, $[\delta/x]^{i}\omega(\eta) \xrightarrow{\alpha,\beta} Ap^{i-1}([\delta/x]^{i}\omega,[\delta/x]^{i}\eta)$, by 5.1.1.

$\xrightarrow{\alpha,\beta} Ap^{i}([\delta/x]^{i}\omega,[\delta/x]^{i}\eta)$, by $P(i-1)$.

$= [\delta/x]^{i+1}\omega(\eta)$.

-Thus, by induction, $P(i) \wedge Q(i)$, $\forall i \geq 1$.

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<u>5.1.4:COR:-</u>

$Ap^{i}(\varepsilon,\delta) \;\tilde{\sqsubseteq}\; Ap^{i+1}(\varepsilon,\delta)$.

$[\delta/x]^{i}\varepsilon \;\tilde{\sqsubseteq}\; [\delta/x]^{i+1}\varepsilon$.

<u>Proof:-</u>

-By 5.1.3 and 4.4.9.

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<u>5.1.5:REMARK:-</u>

Thus, we see that i'th applications and substitutions form a chain under β-reductions, and hence $\tilde{\sqsubseteq}$, with increasing i.

Finally, in this section, we check that these definitions are well behaved with respect to $\tilde{\sqsubseteq}$ - i.e. a result similar to that of 5.1.1. First, though, a trivial property of $\tilde{\sqsubseteq}$.

<u>5.1.6:LEMMA:-</u>

$\varepsilon \;\tilde{\sqsubseteq}\; \varepsilon', \; \delta \;\tilde{\sqsubseteq}\; \delta' \Rightarrow [\delta/x]\varepsilon \;\tilde{\sqsubseteq}\; [\delta'/x]\varepsilon'$.

<u>Proof:-</u>

-$\tilde{E}[\![[\delta/x]\varepsilon]\!] = [\tilde{E}[\![\delta]\!]/x]^{\sim}\tilde{E}[\![\varepsilon]\!]$, by 4.4.4.

$\sqsubseteq [\tilde{E}[\![\delta']\!]/x]^{\sim}\tilde{E}[\![\varepsilon']\!]$, by 4.3.3(i).

$= \tilde{E}[\![[\delta'/x]\varepsilon']\!]$.

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<u>5.1.7:DEF:-</u>

$P(i) \equiv (\varepsilon \;\tilde{\sqsubseteq}\; \varepsilon', \; \delta \;\tilde{\sqsubseteq}\; \delta' \Rightarrow Ap^{i}(\varepsilon,\delta) \;\tilde{\sqsubseteq}\; Ap^{i}(\varepsilon',\delta'))$.

$Q(i) \equiv (\varepsilon \;\tilde{\sqsubseteq}\; \varepsilon', \; \delta \;\tilde{\sqsubseteq}\; \delta' \Rightarrow [\delta/x]^{i}\varepsilon \;\tilde{\sqsubseteq}\; [\delta/x]^{i}\varepsilon')$.

<u>5.1.8:LEMMA:-</u>

$Q(i) \Rightarrow P(i)$, $\forall i \geq 2$.

Proof:-

-If $\varepsilon \notin \lambda$ I.EXP, $Ap^i(\varepsilon,\delta) = \varepsilon(\delta) \;\tilde{\sqsubseteq}\; \varepsilon'(\delta')$, by 4.4.7.

$\tilde{\sqsubseteq}\; Ap^i(\varepsilon',\delta')$, since there is equality if $\varepsilon' \notin \lambda$I.EXP and otherwise $\varepsilon'(\delta') \in$ NOH, which gives $\tilde{\sqsubseteq}$, by 4.8.4.

-Suppose $\varepsilon \in \lambda$ I.EXP. If $\varepsilon \in$ NOH, then so is $Ap^i(\varepsilon,\delta)$, by 5.0.11, and hence P (i), by 4.8.4.

-Otherwise, $\varepsilon \xrightarrow{\alpha} \lambda x.\eta \in$ HNF, and $\varepsilon' \xrightarrow{\alpha} \lambda x.\eta'$ and $\eta \;\tilde{\sqsubseteq}\; \eta'$, by 4.8.4.

-Then, $Ap^i(\varepsilon,\delta) \xrightarrow{\alpha} Ap^i(\lambda x.\eta,\delta)$, by 5.0.11.

$= [\delta/x]^i\eta$

$\tilde{\sqsubseteq}\; [\delta'/x]^i\eta'$, by Q (i).

$= Ap^i(\lambda x.\eta',\delta')$

$\xrightarrow{\alpha} Ap^i(\varepsilon',\delta')$, by 5.0.11.

-$\therefore$, P(i), by 4.4.8.

‡

5.1.9:LEMMA:-

$P(i-1) \wedge Q(i-1) \Rightarrow Q(i), \; \forall i \geq 2$.

Proof:-

-$\varepsilon = z \in I \Rightarrow \varepsilon' = z$, by 4.8.4.

$\Rightarrow [\delta/x]^i z = [\delta/x]z$

$\tilde{\sqsubseteq}\; [\delta'/x]z$, by 5.1.6.

$= [\delta'/x]^i z$.

-$\varepsilon \in$ NOH $\Rightarrow [\delta/x]^i\varepsilon \in$ NOH, by 5.0.11.

$\Rightarrow Q(i)$, by 4.8.4.

-$\varepsilon \in \lambda$I.HNF $\Rightarrow \varepsilon \xrightarrow{\alpha} \lambda y.\eta$, $\varepsilon' \xrightarrow{\alpha} \lambda y.\eta'$ and $\eta \;\tilde{\sqsubseteq}\; \eta'$, by 4.8.4.

-Choose $z \neq x$ and not free in $\delta,\delta',\eta,\eta'$.

-Then, $[\delta/x]^i\varepsilon \xrightarrow{\alpha} [\delta/x]^i(\lambda y.\eta)$, by 5.0.11.

$= \lambda z.[\delta/x]^{i-1}[z/y]\eta$

$\tilde{\sqsubseteq}\; \lambda z.[\delta'/x]^{i-1}[z/y]\eta$, by Q(i-1), since $[z/y]\eta \;\tilde{\sqsubseteq}\; [z/y]\eta'$, by 5.1.6.

$$= [\delta'/x]^{i}(\lambda y.\eta')$$

$$\xrightarrow{\alpha} [\delta'/x]^{i}\varepsilon', \text{ by 5.0.11.}$$

-∴, $Q(i)$, by 4.4.8.

-Finally, $\varepsilon = \omega(\eta) \in \text{HEAD} \Rightarrow \varepsilon' = \omega'(\eta')$ and $\omega \overset{\approx}{\sqsubseteq} \omega'$, $\eta \overset{\approx}{\sqsubseteq} \eta'$, by 4.8.4.

-Then, $[\delta/x]^{i}\omega(\eta) = \text{Ap}^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)$.

-But, $[\delta/x]^{i-1}\omega \overset{\approx}{\sqsubseteq} [\delta'/x]^{i-1}\omega'$, by $Q(i-1)$.

-And, $[\delta/x]^{i-1}\eta \overset{\approx}{\sqsubseteq} [\delta'/x]^{i-1}\eta'$, by $Q(i-1)$.

-∴, $[\delta/x]^{i}\omega(\eta) \overset{\approx}{\sqsubseteq} \text{Ap}^{i-1}([\delta'/x]^{i-1}\omega',[\delta'/x]^{i-1}\eta')$, by $P(i-1)$.

$$= [\delta'/x]^{i}\omega'(\eta').$$

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5.1.10:THEOREM:-

$P(i) \wedge Q(i)$, $\forall i \geq 1$.

Proof:-

-Clearly, $P(1) \wedge Q(1)$, by 4.4.7 and 5.1.6.

-∴, by induction, using 5.1.8 and 5.1.9, we get the result.

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5.2:Relationship with E_∞:-

5.2.0:REMARK:-

Since i'th application and substitution were "modelled" after the application and substitution operators of E_∞, we should expect a strong connection.

5.2.1:DEF:-

$P(i) \equiv \tilde{E}[\![\text{Ap}^{i}(\varepsilon,\delta)]\!] \sqsubseteq \text{Ap}(\tilde{E}[\![\varepsilon]\!],\tilde{E}[\![\delta]\!])$.

$Q(i) \equiv \tilde{E}[\![[\delta/x]^{i}\varepsilon]\!]' \sqsubseteq [\tilde{E}[\![\delta]\!]/x]\tilde{E}[\![\delta]\!]$.

5.2.2:LEMMA:-

$P(1) \wedge Q(1)$.

Proof:-

$$\begin{aligned}-\tilde{E}[\![Ap^1(\varepsilon,\delta)]\!] &= \tilde{E}[\![\varepsilon(\delta)]\!] = \widetilde{Ap}(\tilde{E}[\![\varepsilon]\!],\tilde{E}[\![\delta]\!]) \\ &\sqsubseteq Ap(\tilde{E}[\![\varepsilon]\!],\tilde{E}[\![\delta]\!]), \text{ by } 4.3.3(iii). \\ -\tilde{E}[\![[\delta/x]^1\varepsilon]\!] &= \tilde{E}[\![[\delta/x]\varepsilon]\!] \\ &= [\tilde{E}[\![\delta]\!]/x]^{\sim}\tilde{E}[\![\varepsilon]\!], \text{ by } 4.4.4. \\ &\sqsubseteq [\tilde{E}[\![\delta]\!]/x]\tilde{E}[\![\varepsilon]\!], \text{ by } 4.3.3(iii).\end{aligned}$$

$\nparallel$

5.2.3:LEMMA:-

$Q(i) \Longrightarrow P(i),\ \forall i \geq 2.$

Proof:-

-Suppose $\varepsilon \notin \lambda I.EXP$.

-Then, $\tilde{E}[\![Ap^i(\varepsilon,\delta)]\!] = \tilde{E}[\![\varepsilon(\delta)]\!] \sqsubseteq Ap(\tilde{E}[\![\varepsilon]\!],\tilde{E}[\![\delta]\!])$, as in the proof of 5.2.2.

-Suppose $\varepsilon = \lambda x.\varepsilon'$.

$$\begin{aligned}\text{-Then, } \tilde{E}[\![Ap^i(\lambda x.\varepsilon',\delta)]\!] &= \tilde{E}[\![[\delta/x]^i\varepsilon']\!] \\ &\sqsubseteq [\tilde{E}[\![\delta]\!]/x]\tilde{E}[\![\varepsilon']\!], \text{ by } Q(i). \\ &= Ap(\lambda x.\tilde{E}[\![\varepsilon']\!],\tilde{E}[\![\delta]\!]), \text{ by } 3.4.8. \\ &= Ap(\tilde{E}[\![\lambda x.\varepsilon']\!],\tilde{E}[\![\delta]\!]).\end{aligned}$$

$\nparallel$

5.2.4:REMARK:-

We need the following technical lemma for the other induction step. Part (i) was mentioned in 3.4.11.

5.2.5:LEMMA:-

(i) $Ap([\varepsilon'/x]\varepsilon,[\varepsilon'/x]\delta)$

$$= \bigsqcup_{i=1}^{\infty}\phi_{i+1,\infty}\circ Ap_i([\varepsilon_i'/x]\varepsilon_{i-1},[\varepsilon_i'/x]\delta_{i-1}).$$

(ii) $\pi_2(\varepsilon) \neq \perp \Longrightarrow$

$$\begin{aligned}Ap([\varepsilon'/x]\varepsilon,[\varepsilon'/x]\delta) &= [\varepsilon'/x]\widetilde{Ap}(\varepsilon,\delta) \\ &= [\varepsilon'/x]Ap(\varepsilon,\delta).\end{aligned}$$

Proof:-

(i) $-Ap([\varepsilon'/x]\varepsilon, [\varepsilon'/x]\delta)$

$$= Ap(\bigsqcup_{i=1}^{\infty}\phi_{i,\infty}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \bigsqcup_{j=1}^{\infty}\phi_{j,\infty}\circ[\varepsilon_j'/x]\delta_{j-1})$$

$$= \bigsqcup_{k=1}^{\infty}\phi_{k+1,\infty}\circ Ap_k(\phi_{\infty,k}(\bigsqcup_{i=1}^{\infty}\phi_{i,\infty}\circ[\varepsilon_i'/x]\varepsilon_{i-1}), \phi_{\infty,k}(\bigsqcup_{j=1}^{\infty}\phi_{j,\infty}\circ[\varepsilon_j'/x]\delta_{j-1}))$$

$$= \bigsqcup_{k=1}^{\infty}\phi_{k+1,\infty}\circ Ap_k(\bigsqcup_{i\geq k}\phi_{i,k}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \bigsqcup_{j\geq k}\phi_{j,k}\circ[\varepsilon_j'/x]\delta_{j-1})$$

$$= \bigsqcup_{k=1}^{\infty}\phi_{k+1,\infty}(\bigsqcup_{i\geq k}\bigsqcup_{j\geq k}Ap_k(\phi_{i,k}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \phi_{j,k}\circ[\varepsilon_j'/x]\delta_{j-1}))$$

$$= \bigsqcup_{k=1}^{\infty}\phi_{k+1,\infty}(\bigsqcup_{i\geq k}Ap_k(\phi_{i,k}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \phi_{i,k}\circ[\varepsilon_i'/x]\delta_{i-1}))$$

$$= \bigsqcup_{k=1}^{\infty}\bigsqcup_{i\geq k}\phi_{k+1,\infty}\circ Ap_k(\phi_{i,k}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \phi_{i,k}\circ[\varepsilon_i'/x]\delta_{i-1})$$

$$\sqsubseteq \bigsqcup_{k=1}^{\infty}\bigsqcup_{i\geq k}\phi_{k+1,\infty}\circ\phi_{i+1,k+1}\circ Ap_i([\varepsilon_i'/x]\varepsilon_{i-1}, [\varepsilon_i'/x]\delta_{i-1}), \text{ by 3.3.8.}$$

$$\sqsubseteq \bigsqcup_{i=1}^{\infty}\phi_{i+1,\infty}\circ Ap_i([\varepsilon_i'/x]\varepsilon_{i-1}, [\varepsilon_i'/x]\delta_{i-1})$$

$$\sqsubseteq \bigsqcup_{k=1}^{\infty}\bigsqcup_{i\geq k}\phi_{k+1,\infty}\circ Ap_k(\phi_{i,k}\circ[\varepsilon_i'/x]\varepsilon_{i-1}, \phi_{i,k}\circ[\varepsilon_i'/x]\delta_{i-1}), \text{ since}$$

this is the least upper bound of a larger set (consider i = k).

-∴, we have equality and this part is proved.

(ii) -If $\pi_2(\varepsilon) \neq \bot$, then $\varepsilon_{i-1} = \alpha_{i-2} \in A_{i-2}$, $\forall i \geq 2$.

-∴, $Ap([\varepsilon'/x]\varepsilon, [\varepsilon'/x]\delta)$

$$= \bigsqcup_{i=2}^{\infty}\phi_{i+1,\infty}\circ Ap_i([\varepsilon_i'/x]\alpha_{i-2}, [\varepsilon_i'/x]\delta_{i-1}), \text{ by part (i).}$$

$$= \bigsqcup_{i=2}^{\infty}\phi_{i+1,\infty}\circ[\varepsilon_{i+1}'/x]\alpha_{i-2}(\delta_{i-1})$$

$$= [\varepsilon'/x]\widetilde{Ap}(\varepsilon,\delta)$$

$$= [\varepsilon'/x]Ap(\varepsilon,\delta), \text{ by 4.3.3(xii).}$$

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<u>5.2.6:LEMMA:-</u>

$\mathcal{P}(i-1)\wedge\mathcal{Q}(i-1) \Rightarrow \mathcal{Q}(i)$, $\forall i \geq 2$.

<u>Proof:-</u>

-If $\varepsilon \in I \cup NOH$, then $\tilde{\mathcal{E}}[\![[\delta/x]^i\varepsilon]\!] = \tilde{\mathcal{E}}[\![[\delta/x]\varepsilon]\!] \sqsubseteq [\tilde{\mathcal{E}}[\![\delta]\!]/x]\tilde{\mathcal{E}}[\![\varepsilon]\!]$, as in the proof of 5.2.2.

-Suppose $\varepsilon = \lambda y.\varepsilon' \in \mathrm{HNF}$. Choose $z \neq x$ and not free in δ, ε'.

-Then, $\tilde{E}\llbracket[\delta/x]^{i}(\lambda y.\varepsilon')\rrbracket$

$= \tilde{E}\llbracket\lambda z.[\delta/x]^{i-1}[z/y]\varepsilon'\rrbracket$

$= \lambda z.\tilde{E}\llbracket[\delta/x]^{i-1}[z/y]\varepsilon'\rrbracket$

$\sqsubseteq \lambda z.[\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket[z/y]\varepsilon'\rrbracket$, by $Q(i-1)$.

$= [\tilde{E}\llbracket\delta\rrbracket/x]\lambda z.\tilde{E}\llbracket[z/y]\varepsilon'\rrbracket$, by 4.3.4(iii), since z is not free in $\tilde{E}\llbracket\delta\rrbracket$, by 4.4.1(ii).

$= [\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket\lambda z.[z/y]\varepsilon'\rrbracket$

$= [\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket\lambda y.\varepsilon'\rrbracket$, by 4.4.8.

-Finally, suppose $\varepsilon = \omega(\eta) \in \mathrm{HEAD}$. Then, $\omega \in \mathrm{HEAD}$ and so $\pi_2 \circ \tilde{E}\llbracket\omega\rrbracket \neq \bot$, by 4.4.2(iii).

-Then, $\tilde{E}\llbracket[\delta/x]^{i}\omega(\eta)\rrbracket = \tilde{E}\llbracket\mathrm{Ap}^{i-1}([\delta/x]^{i-1}\omega,[\delta/x]^{i-1}\eta)\rrbracket$

$\sqsubseteq \mathrm{Ap}(\tilde{E}\llbracket[\delta/x]^{i-1}\omega\rrbracket,\tilde{E}\llbracket[\delta/x]^{i-1}\eta\rrbracket)$, by $P(i-1)$.

$\sqsubseteq \mathrm{Ap}([\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket\omega\rrbracket,[\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket\eta\rrbracket)$, by $Q(i-1)$.

$= [\tilde{E}\llbracket\delta\rrbracket/x]\widetilde{\mathrm{Ap}}(\tilde{E}\llbracket\omega\rrbracket,\tilde{E}\llbracket\eta\rrbracket)$, by 5.2.5(ii).

$= [\tilde{E}\llbracket\delta\rrbracket/x]\tilde{E}\llbracket\omega(\eta)\rrbracket$.

$\nmid$

5.2.7:THEOREM:-

$P(i) \wedge Q(i), \forall i \geq 1$.

Proof:-

-By induction, using 5.2.2,5.2.3 and 5.2.6.

$\nmid$

5.2.8:REMARK:-

We have now done the work in this section that is needed for the development of the thesis, since we will obtain from this, in the next section, a sufficient condition for property X to be true.

However, to show we are on the right track, we can get a result with Ap^i on the R.H.S. of a $\sqsubseteq$ and prove the condition neccessary as well.

For this reason, and the fact that the result and its proof are almost identical to 4.6.10, we only sketch the proof here.

5.2.9:THEOREM:-

$$\text{Let } \left\{\begin{matrix} S \\ Q \\ R \end{matrix}\right\}(i) \equiv \left(\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\}, \delta_i \sqsubseteq \phi_{\infty,i-1} \circ \tilde{E}\llbracket\varepsilon\rrbracket, \phi_{\infty,i} \circ \tilde{E}\llbracket\delta\rrbracket\right)$$

$$\Rightarrow \left([\delta_i/x]\left\{\begin{matrix} z \\ \varepsilon_{i-1} \\ \alpha_{i-2} \end{matrix}\right\} \sqsubseteq \phi_{\infty,i} \circ \tilde{E}\llbracket[\delta/x]^i(\varepsilon)\rrbracket\right).$$

$$\text{Let } P(i) \equiv (\varepsilon_i, \delta_i \sqsubseteq \phi_{\infty,i} \circ \tilde{E}\llbracket\varepsilon\rrbracket, \phi_{\infty,i} \circ \tilde{E}\llbracket\delta\rrbracket)$$

$$\Rightarrow (\mathrm{Ap}_i(\varepsilon_i, \delta_i) \sqsubseteq \phi_{\infty,i+1} \circ \tilde{E}\llbracket\mathrm{Ap}^i(\varepsilon, \delta)\rrbracket).$$

Then, $(\forall i \geq 0)(S(i))$ and $(\forall i \geq 1)(P(i) \wedge Q(i))$ and $(\forall i \geq 2)(R(i)$

Proof:-

-Look at definition 4.6.4. All we have to show is that $[\delta/x]^i(\varepsilon)$, $\mathrm{Ap}^i(\varepsilon,\delta)$ are suitable choices for η such that $[\delta/x]\varepsilon \xrightarrow{\alpha,\beta} \eta$, $\varepsilon(\delta) \xrightarrow{\alpha,\beta} \eta$ respectively.

-Consider 4.6.6 :- $(S(i), \forall i \geq 0)$

$-\eta = [\delta/x]z = [\delta/x]^i z.$

-Consider 4.6.7 :- $(Q(i) \Rightarrow P(i), \forall i \geq 1)$

-If $\varepsilon_i = \alpha_{i-1}$, then $\pi_2 \circ \tilde{E}\llbracket\varepsilon\rrbracket \neq \perp$.

$-\therefore$, $\varepsilon \in$ HEAD and so $\varepsilon \notin \lambda I.\mathrm{EXP}$.

$-\therefore$, $\eta = \varepsilon(\delta) = \mathrm{Ap}^i(\varepsilon,\delta)$.

-If $\varepsilon_i = \lambda x.\varepsilon'_{i-1}$, we get $\eta = [\delta/y]^i\nu$, by $Q(i)$.

$$= \mathrm{Ap}^i(\lambda y.\nu,\delta) \xrightarrow{\alpha} \mathrm{Ap}^i(\varepsilon,\delta),$$

since $\varepsilon \xrightarrow{\alpha} \lambda y.\nu$.

-Consider 4.6.8 :- $(Q(i-1) \wedge R(i) \Rightarrow Q(i), \forall i \geq 2)$.

-If $\varepsilon_{i-1} = \lambda y.\varepsilon'_{i-2}$, we get,

$\lambda z.\eta = \lambda z.[\delta/x]^{i-1}\nu$, by $Q(i-1)$.

$\xrightarrow{\alpha} [\delta/x]^i(\lambda z.\nu)$, since $z \neq x$ and is not free in δ, by 5.0.11.

$$\xrightarrow{\alpha} [\delta/x]^{i}\varepsilon, \text{ since } \varepsilon \xrightarrow{\alpha} \lambda z.\nu.$$

-If $\varepsilon_{i-1} = \alpha_{i-2}$, then O.K. by $\mathcal{R}(i)$.

-Consider 4.6.9 :- $(\mathcal{P}(i-1) \wedge \mathcal{Q}(i-1) \wedge \mathcal{R}(i-1) \Rightarrow \mathcal{R}(i))$

-If $\alpha_{i-2} = \alpha_{i-3}(\varepsilon'_{i-2})$, we get,

$$\nu = Ap^{i-1}([\delta/x]^{i-1}\alpha,[\delta/x]^{i-1}\varepsilon'), \text{ by } \mathcal{P}(i-1) \wedge \mathcal{Q}(i-1) \wedge \mathcal{R}(i-1).$$
$$= [\delta/x]^{i}(\alpha(\varepsilon')) = [\delta/x]^{i}\varepsilon.$$

-If $\alpha_{i-2} = z$, then O.K. by $\mathcal{S}(i)$.

-∴, the theorem, by an *SPQR*-induction as in 4.6.10.

ǂ

5.3:I'th Reductions:-

5.3.0:DEF:-

$i\langle x\rangle = x \qquad (i \geq 1)$

$i\langle \lambda x.\varepsilon\rangle = \lambda x.i\langle\varepsilon\rangle$

$i\langle\varepsilon(\delta)\rangle = Ap^{i}(i\langle\varepsilon\rangle,i\langle\delta\rangle)$.

5.3.1:REMARK:-

Compare the above definition with that of the semantic function $\mathcal{E}$, 4.0.1.

We use angle brackets, < >, rather than ⟦ ⟧, since the result of applying i is another λ-expression.

We could have defined these reductions slightly differently, e.g. $i\langle\varepsilon(\delta)\rangle = Ap^{i-1}(i-1\langle\varepsilon\rangle,i-1\langle\delta\rangle)$, but this makes no difference to our thesis.

5.3.2:LEMMA:-

(i) $i\langle\varepsilon\rangle$ is well-defined up to α-conversion.

(ii) $i\langle[x/y]\varepsilon\rangle = [x/y]i\langle\varepsilon\rangle$.

(iii) a is not free in $\varepsilon \Rightarrow$ a is not free in $i\langle\varepsilon\rangle$.

(iv) $\varepsilon \xrightarrow{\alpha} \varepsilon' \Rightarrow i\langle\varepsilon\rangle \xrightarrow{\alpha} i\langle\varepsilon'\rangle$.

Proof:-

-Trivial, by doing a structural induction on ε and using 5.0.11.

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5.3.3:THEOREM:-

The i'th reductions form a chain under $\xrightarrow{\alpha,\beta}$ and $\tilde{\equiv}$.

Proof:-

-Let $P\llbracket\varepsilon\rrbracket \equiv i\langle\varepsilon\rangle \xrightarrow{\alpha,\beta} i+1\langle\varepsilon\rangle$.

-Clearly, $P\llbracket x\rrbracket$ and $(P\llbracket\varepsilon\rrbracket \Rightarrow P\llbracket\lambda x.\varepsilon\rrbracket)$.

-Claim: $P\llbracket\varepsilon\rrbracket \wedge P\llbracket\delta\rrbracket \Rightarrow P\llbracket\varepsilon(\delta)\rrbracket$:-

$$
\begin{aligned}
-i\langle\varepsilon(\delta)\rangle &= Ap^{i}(i\langle\varepsilon\rangle, i\langle\delta\rangle) \\
&\xrightarrow{\alpha,\beta} Ap^{i}(i+1\langle\varepsilon\rangle, i+1\langle\delta\rangle), \text{ by 5.1.1}, P\llbracket\varepsilon\rrbracket \text{ and } P\llbracket\delta\rrbracket. \\
&\xrightarrow{\alpha,\beta} Ap^{i+1}(i+1\langle\varepsilon\rangle, i+1\langle\delta\rangle), \text{ by 5.1.3.} \\
&= i+1\langle\varepsilon(\delta)\rangle.
\end{aligned}
$$

-$\therefore$, $(\forall\varepsilon \in \mathrm{EXP})P\llbracket\varepsilon\rrbracket$.

-Finally, $\xrightarrow{\alpha,\beta} \Rightarrow \tilde{\equiv}$, by4.4.9.

ǂ

5.3.4:EXAMPLES:-

(i) $1\langle\varepsilon\rangle = \varepsilon$.

(ii) $\varepsilon \in NF \Rightarrow i\langle\varepsilon\rangle \xrightarrow{\alpha} \varepsilon$.

(iii) $2\langle(\lambda y.xy)(b)\rangle = xb$.

(iv) $[\delta/x]\varepsilon \xrightarrow{\alpha,\beta} 2\langle(\lambda x.\varepsilon)(\delta)\rangle$.

(v) $\varepsilon \xrightarrow{\alpha,\beta} 2\langle I(\varepsilon)\rangle$.

(vi) $\varepsilon \xrightarrow{\alpha,\beta} 3\langle K(\varepsilon)(\delta)\rangle$.

(vii) $i\langle\Delta\Delta\rangle \xrightarrow{\alpha} \Delta\Delta$.

(viii) $i\langle Y\rangle = \lambda f.f^{i\div2}(\underline{spl})$, where $Y = \lambda f.\underline{spl}$ and $i\div2$ means an integer divide.

Proof:-

(i) -By a simple structural induction on ε.

(ii) -As a corollary to 5.3.3.

(iii) - $2<(\lambda y.xy)(b)> = Ap^2(2<\lambda y.xy>,2<b>)$

$= Ap^2(\lambda y.xy,b)$, by part (ii).

$= [b/y]^2(xy) = [b/y]xy = xb$.

(iv) $-2<(\lambda x.\varepsilon)(\delta)> = Ap^2(\lambda x.2<\varepsilon>,2<\delta>)$

$= [2<\delta>/x]^2 2<\varepsilon> = [2<\delta>/x]2<\varepsilon>$

$\xleftarrow{\alpha,\beta} [\delta/x]\varepsilon$, by 5.3.3.

(v) $-\varepsilon = [\varepsilon/x]x \xrightarrow{\alpha,\beta} 2<I(\varepsilon)>$, by part (iv).

(vi) $-3<K(\varepsilon)(\delta)> = Ap^3(Ap^3(K,3<\varepsilon>),3<\delta>)$

$\xleftarrow{\alpha,\beta} Ap^3(Ap^3(K,\varepsilon),\delta)$, by 5.3.3 and 5.1.1.

$= Ap^3([\varepsilon/x]^3(\lambda y.x),\delta)$

$= Ap^3(\lambda z.[\varepsilon/x]^2[z/y]x,\delta)$, where $z \neq x$ and is not free in ε.

$= Ap^3(\lambda z.[\varepsilon/x]x,\delta) = Ap^3(\lambda z.\varepsilon,\delta)$

$= [\delta/z]^2\varepsilon = [\delta/z]\varepsilon = \varepsilon$, since z is not free in ε.

(vii) -By 5.3.3, $\Delta\Delta \xrightarrow{\alpha,\beta} i<\Delta\Delta>$.

-But, $\Delta\Delta$ only reduces to itself.

-∴, $i<\Delta\Delta> \xrightarrow{\alpha} \Delta\Delta$.

(viii) $-i<Y> = i<\lambda f.\underline{spl}> = i<\lambda f.(\lambda y.f(yy))(\lambda y.f(yy))>$

$= \lambda f.Ap^i(\lambda y.f(yy),\lambda y.f(yy))$.

-But, $Ap^1(\lambda y.f(yy),\lambda y.f(yy)) = \underline{spl}$.

-And, $Ap^2(\lambda y.f(yy),\lambda y.f(yy)) = [\lambda y.f(yy)/y]^2 f(yy)$

$= [\lambda y.f(yy)/y]f(yy)$

$= f(\underline{spl})$.

-Then, for $i \geq 3$, $Ap^i(\lambda y.f(yy),\lambda y.f(yy))$

$= [\lambda y.f(yy)/y]^i f(yy)$

$= Ap^{i-1}([\lambda y.f(yy)/y]^{i-1}f,[\lambda y.f(yy)/y]^{i-1}(yy))$

$= Ap^{i-1}(f,Ap^{i-2}([\lambda y.f(yy)/y]^{i-2}y,[\lambda y.f(yy)/y]^{i-2}y))$

$= f(Ap^{i-2}(\lambda y.f(yy),\lambda y.f(yy)))$.

-$\therefore$, $\forall i \geq 1,\ Ap^{i}(\lambda y.f(yy),\lambda y.f(yy)) = f^{i \div 2}(\underline{spl})$.

-$\therefore$, $i\langle Y\rangle = \lambda f.f^{i \div 2}(\underline{spl})$.

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5.3.5:REMARK:-

Thus, we see that i'th reductions are non-trivial and, in the case of Y, go arbitrarily deep. The question is whether they always will.

Note that we have not got the result,

$$(\varepsilon \xrightarrow{\alpha,\beta} \varepsilon') \Rightarrow (i\langle\varepsilon\rangle \xrightarrow{\alpha,\beta} i\langle\varepsilon'\rangle),$$

from 5.1.1, since, if we try a structural induction on ε, we do not know the form of ε' when $\varepsilon = \omega(\delta)$.

Note also that the result corresponding to 5.1.10,

$$(\varepsilon \mathrel{\tilde{\sqsubseteq}} \varepsilon') \Rightarrow (i\langle\varepsilon\rangle \mathrel{\tilde{\sqsubseteq}} i\langle\varepsilon'\rangle) \quad - ✗ ,$$

is not true, since $I(x) \mathrel{\tilde{\sqsubseteq}} \Delta\Delta$, but $2\langle I(x)\rangle = x \mathrel{\tilde{\not\sqsubseteq}} \Delta\Delta = 2\langle\Delta\Delta\rangle$.

We also have not been able to show,

$$(\varepsilon\gamma)(\delta\gamma) \xrightarrow{\alpha,\beta} i\langle S(\varepsilon)(\delta)(\gamma)\rangle,$$

for sufficiently large i, since this would involve pushing i'th substitions through i'th applications and this seems to lead to as many troubles as trying to establish properties Y or Z.

5.3.6:THEOREM:-

$$\tilde{\mathcal{E}}[\![i\langle\varepsilon\rangle]\!] \sqsubseteq \mathcal{E}[\![\varepsilon]\!].$$

Proof:-

-Let $\mathcal{P}[\![\varepsilon]\!]$ be the above sentence.

-Clearly, $\mathcal{P}[\![x]\!]$.

-Claim: $\mathcal{P}[\![\varepsilon]\!] \Rightarrow \mathcal{P}[\![\lambda x.\varepsilon]\!]$:-

-$\tilde{\mathcal{E}}[\![i\langle\lambda x.\varepsilon\rangle]\!] = \tilde{\mathcal{E}}[\![\lambda x.i\langle\varepsilon\rangle]\!] = \lambda x.\tilde{\mathcal{E}}[\![i\langle\varepsilon\rangle]\!] \sqsubseteq \lambda x.\mathcal{E}[\![\varepsilon]\!]$, by $\mathcal{P}[\![\varepsilon]\!]$.

$= \mathcal{E}[\![\lambda x.\varepsilon]\!]$.

-Claim: $\mathcal{P}[\![\varepsilon]\!] \wedge \mathcal{P}[\![\delta]\!] \Rightarrow \mathcal{P}[\![\varepsilon(\delta)]\!]$:-

-$\tilde{\mathcal{E}}[\![i\langle\varepsilon(\delta)\rangle]\!] = \tilde{\mathcal{E}}[\![Ap^{i}(i\langle\varepsilon\rangle,i\langle\delta\rangle)]\!]$

$$\sqsubseteq \mathrm{Ap}(\tilde{E}[\![i{<}\varepsilon{>}]\!], \tilde{E}[\![i{<}\delta{>}]\!]), \text{ by } 5.2.7.$$

$$\sqsubseteq \mathrm{Ap}(E[\![\varepsilon]\!], E[\![\delta]\!]), \text{ by } P[\![\varepsilon]\!] \wedge P[\![\delta]\!].$$

$$= E[\![\varepsilon(\delta)]\!].$$

-∴, $(\forall \varepsilon \in \mathrm{EXP}) P[\![\varepsilon]\!]$.

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5.3.7:COR:-

$$\bigsqcup_{i=1}^{\infty} \tilde{E}[\![i{<}\varepsilon{>}]\!] \sqsubseteq E[\![\varepsilon]\!].$$

Proof:-

-Trivial.

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5.3.8:REMARK:-

As in the last section (see 5.2.8), we have now done the work that is needed, since we will show later that,

$$V[\![\varepsilon]\!] \sqsubseteq \bigsqcup_{i=1}^{\infty} \tilde{E}[\![i{<}\varepsilon{>}]\!],$$

and so deduce $E = V$. However, to show we are chasing a neccessary condition, we prove one more result, analogous to 4.6.11.

5.3.9:THEOREM:-

$$E[\![\varepsilon]\!] \sqsubseteq \bigsqcup_{i=1}^{\infty} \tilde{E}[\![i{<}\varepsilon{>}]\!].$$

Proof:-

-Look at the proof of 4.6.11. Let $A[\![\varepsilon]\!]$ be the theorem.

-Clearly, $A[\![x]\!]$.

-$(A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!])$ is proved as in 4.6.11, using 5.3.3 instead of 4.4.10.

-$(A[\![\varepsilon]\!] \wedge A[\![\delta]\!] \Rightarrow A[\![\varepsilon(\delta)]\!])$ is proved as in 4.6.11, only we use $j_i{<}\varepsilon{>}$, $k_i{<}\delta{>}$ instead of ε^i, δ^i. Then, by 5.2.9,

$$\mathrm{Ap}_i(\varepsilon_i, \delta_i) \sqsubseteq \phi_{\infty,i+1} \circ \tilde{E}[\![\mathrm{Ap}^i(j_i{<}\varepsilon{>}, k_i{<}\delta{>})]\!]$$

$$\sqsubseteq \phi_{\infty,i+1} \circ \tilde{E}[\![m_i{<}\varepsilon(\delta){>}]\!], \text{ where } m_i = \max(j_i, k_i, i).$$

Hence, the result.

-∴, by structural induction, $(\forall\varepsilon \in \mathrm{EXP})\mathcal{A}[\![\varepsilon]\!]$.

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5.3.10:COR:-

$$\mathcal{E}[\![\varepsilon]\!] = \bigsqcup_{i=1}^{\infty} \tilde{\mathcal{E}}[\![i<\varepsilon>]\!].$$

Proof:-

-Trivial.

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5.3.11:REMARK:-

Thus, the semantics of an expression are given by the limit of the approximate semantics of the i'th reductions. This is a stronger result than Wadsworth's [59], which we shall show carries over to the usual Scott models.

5.4:Completeness of Reduction Rules:-

5.4.0:DEF:-

Let R be a rule for defining a set of β-reductions of a λ-expression. We write $\varepsilon \xrightarrow{R} \delta$ if δ is reducible from ε by the rule R. Then, the rule is STRONGLY COMPLETE if,

$$(\varepsilon \xrightarrow{\alpha,\beta} \varepsilon') \Rightarrow (\varepsilon \xrightarrow{R} \delta) \wedge (\varepsilon' \xrightarrow{\alpha,\beta} \delta).$$

The rule is WEAKLY COMPLETE if,

$$(\varepsilon \xrightarrow{\alpha,\beta} \varepsilon') \Rightarrow (\varepsilon \xrightarrow{R} \delta) \wedge (\varepsilon' \overset{\sim}{\equiv} \delta).$$

5.4.1:LEMMA:-

(i) Strongly complete ⇒ weakly complete.

(ii) R is weakly complete ⇔ $\mathcal{V}[\![\varepsilon]\!] = \bigsqcup\{\tilde{\mathcal{E}}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{R} \varepsilon'\}$.

Proof:-

(i) -By 4.4.9.

(ii) (⇒) -Clearly, $\mathcal{V}[\![\varepsilon]\!] \sqsupseteq \bigsqcup\{\tilde{\mathcal{E}}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{R} \varepsilon'\}$.

-But, that the inequality holds the other way around, is just weak completeness.

(<=) -Let $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$. Then, $\tilde{\tilde{E}}[\![\varepsilon']\!] \sqsubseteq V[\![\varepsilon]\!]$

$$= \bigsqcup\{\tilde{\tilde{E}}[\![\delta]\!] \mid \varepsilon \xrightarrow{R} \delta\},$$

by hypothesis.

-But, $\tilde{\tilde{E}}[\![\varepsilon']\!]$ is finite, by 4.7.3, and so,

$$\tilde{\tilde{E}}[\![\varepsilon']\!] \prec \bigsqcup\{\tilde{\tilde{E}}[\![\delta]\!] \mid \varepsilon \xrightarrow{R} \delta\}.$$

-$\therefore$, $\tilde{\tilde{E}}[\![\varepsilon']\!] \sqsubseteq \tilde{\tilde{E}}[\![\delta]\!]$, for some δ such that $\varepsilon \xrightarrow{R} \delta$.

-i.e. $\varepsilon' \overset{\approx}{\sqsubseteq} \delta$, where $\varepsilon \xrightarrow{R} \delta$.

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5.4.2:(COUNTER-)EXAMPLES:-

(i) Standard reductions are strongly complete.

(ii) Normal reductions are not even weakly complete.

(iii) I'th reductions are not strongly complete.

Proof:-

(i) -If $\varepsilon \xrightarrow{\alpha,\beta} \varepsilon'$ then $\varepsilon \xrightarrow{\text{standard}} \varepsilon'$, by 0.3.40.

-Hence, strong completeness, with $\delta = \varepsilon'$.

(ii) -Let $\varepsilon = x(\Delta\Delta)(Iy)$.

-The only normal reductions are α-convertible to ε.

-But, $\varepsilon \xrightarrow{\alpha,\beta} x(\Delta\Delta)(y) \not\overset{\approx}{\sqsubseteq} \varepsilon$. (This is from Wadsworth - [60])

(iii) -Let $T = \lambda x.xxx$. Write ε^n for $\varepsilon\varepsilon\varepsilon......\varepsilon$ (n times).

-Then, $T^2 \xrightarrow{\alpha,\beta} T^n$, $\forall n \geq 2$. Also, $T^n \not\xrightarrow{\alpha,\beta} T^2$, $\forall n \geq 3$.

-But, $\Delta^2 \xrightarrow{\alpha,\beta} \Delta^2$ and $i\langle\Delta^2\rangle \xrightarrow{\alpha} \Delta^2$, by 5.3.4(vii).

-Consider $\varepsilon = (\lambda x.\Delta^2(xx))(T)$.

-Now, $\varepsilon \xrightarrow{\alpha,\beta} \Delta^2T^2 \xrightarrow{\alpha,\beta} \Delta^2T^n$, $\forall n \geq 2$.

-But, $i\langle\varepsilon\rangle = Ap^i(i\langle\lambda x.\Delta^2(xx)\rangle, i\langle T\rangle)$

$\xrightarrow{\alpha} Ap^i(\lambda x.i\langle\Delta^2(xx)\rangle, T)$, by 5.3.4(ii).

$= [T/x]^i Ap^i(i\langle\Delta^2\rangle, i\langle xx\rangle)$, $\forall i \geq 2$.

$\xrightarrow{\alpha} [T/x]^i Ap^i(\Delta^2, xx)$, by 5.3.4(ii),(vii) and 5.0.11.

$= [T/x]^i\Delta^2(xx)$, since $\Delta^2 \notin \lambda I.EXP$.

$= [T/x]\Delta^2(xx)$, since $\Delta^2(xx) \in$ NOH.

$= \Delta^2T^2$, $\forall i \geq 2$.

-∴, $\Delta^2T^n \not\xrightarrow{\alpha,\beta} i\langle\varepsilon\rangle$, $\forall i \geq 1$ and $n \geq 3$.

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5.4.3:PROPERTY M:-

I'th reductions are weakly complete.

5.4.4:THEOREM:-

Property M <=> Property X.

Proof:-

(=>) -$\varepsilon \xrightarrow{\alpha,\beta} \varepsilon' \Rightarrow \varepsilon' \overset{\sim}{\sqsubseteq} i\langle\varepsilon\rangle$, for some $i \geq 1$.

$\Rightarrow \tilde{E}[\![\varepsilon']\!] \sqsubseteq \tilde{E}[\![i\langle\varepsilon\rangle]\!]$

$\Rightarrow \tilde{E}[\![\varepsilon']\!] \sqsubseteq E[\![\varepsilon]\!]$, by 5.3.6.

(<=) -$V[\![\varepsilon]\!] = E[\![\varepsilon]\!]$, by 4.9.1.

$= \bigsqcup\{\tilde{E}[\![i\langle\varepsilon\rangle]\!] \mid i \geq 1\}$, by 5.3.10.

-∴, property M, by 5.4.1(ii).

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6:INSIDE-OUT REDUCTIONS.

6.0:Comparing Reduction Rules:-

6.0.0:REMARK:-

In this chapter, we shall be trying to prove facts about certain reduction rules. So, first of all, we generate a language in which to talk.

6.0.1:DEF:-

Let R and S be two rules for defining sets of β-reductions of λ-expressions. Then, R is STRONGLY COMPLETE RELATIVE to S if,

$$(\varepsilon \xrightarrow{S} \varepsilon') \Rightarrow (\varepsilon \xrightarrow{R} \delta) \wedge (\varepsilon' \xrightarrow{\beta} \delta).$$

Also, R is WEAKLY COMPLETE RELATIVE to S if,

$$(\varepsilon \xrightarrow{S} \varepsilon') \Rightarrow (\varepsilon \xrightarrow{R} \delta) \wedge (\varepsilon' \overset{\sim}{\equiv} \delta).$$

Further, R is STRONGLY (resp. WEAKLY) EQUIVALENT to S if R is strongly (resp. weakly) complete relative to S and S is strongly (resp. weakly) complete relative to R.

6.0.2:DEF:-

The FULL reduction rule, $\mho$, is such that,

$$\varepsilon \xrightarrow{\mho} \varepsilon' \Leftrightarrow \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'.$$

The EMPTY reduction rule, Ω, is such that,

$$\{\varepsilon' \mid \varepsilon \xrightarrow{\Omega} \varepsilon'\} = \emptyset.$$

6.0.3:LEMMA:-

(i) S is a sub-rule of $R \Rightarrow R$ is strongly complete relative to S.

(ii) Strong relative completeness (resp. equivalence) => weak relative completeness (resp. equivalence).

(iii) $\mho$ is strongly complete relative to R which is strongly complete relative to Ω.

(iv) R is strongly (resp. weakly) complete relative to $\mho$
$\Leftrightarrow$ R is strongly (resp. weakly) equivalent to $\mho$
$\Leftrightarrow$ R is strongly (resp. weakly) complete.

(v) Strong (resp. weak) equivalence is an equivalence relation on the set of reduction rules.

(vi) Strong (resp. weak) relative completeness is a partial ordering on the set of reduction rules modulo strong (resp. weak) equivalence. (Think of the relation as $\sqsupseteq$)

(vii) The set of reduction rules modulo the equivalences of part (v), respectively, forms a complete lattice under the partial orderings of (vi), respectively, with $\top = \mho$ and $\bot = \Omega$.

(viii) R is weakly complete relative to S

$$\Leftrightarrow \bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{R} \varepsilon'\} \sqsupseteq \bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{S} \varepsilon'\}.$$

(ix) R is weakly equivalent to S

$$\Leftrightarrow \bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{R} \varepsilon'\} = \bigsqcup\{\tilde{E}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{S} \varepsilon'\}.$$

<u>Proof:-</u>

(i),(ii),(iii),(iv),(v) and (vi) -Trivial.

(vii) -Let $\{[R_a] \mid a \in A\}$ be a set of strong (resp. weak) equivalence classes of reduction rules indexed by some arbitrary set A.

-Define a reduction rule, R, by :-

$$\varepsilon \xrightarrow{R} \varepsilon' \text{ iff } \varepsilon \xrightarrow{R_a} \varepsilon', \text{ for some } a \in A.$$

-<u>Claim:</u> $[R] = \bigsqcup\{[R_a] \mid a \in A\}$:-

-If $\varepsilon \xrightarrow{R_a} \varepsilon'$ then $\varepsilon \xrightarrow{R} \varepsilon'$, and so R is strongly complete relative to R_a, $\forall a \in A$ - i.e. $[R]$ is an upper bound.

-Suppose S is strongly (resp. weakly) complete relative to R_a, $\forall a \in A$.

-Let $\varepsilon \xrightarrow{R} \varepsilon'$. Then, $\varepsilon \xrightarrow{R_a} \varepsilon'$, for some $a \in A$, and so $\varepsilon \xrightarrow{S} \delta$, where $\varepsilon' \xrightarrow{\alpha,\beta}$ (resp. $\sqsubseteq$) δ.

-$\therefore$, S is strongly (resp. weakly) complete relative to R.

-Hence, we have a complete lattice since arbitrary $\bigsqcup$'s exist.

-Clearly, $\top = \mho$ and $\bot = \Omega$, by part (iii) above.

(viii) (=>) -Let R be weakly complete relative to S.

-Let $\varepsilon \xrightarrow{S} \varepsilon'$. Then, $\varepsilon \xrightarrow{R} \delta$ and $\varepsilon' \tilde{\sqsubseteq} \delta$ - i.e. $\tilde{E}[\![\varepsilon']\!] \sqsubseteq \tilde{E}[\![\delta]\!]$.

-Hence, the result.

(<=) -Assume R.H.S. and let $\varepsilon \xrightarrow{S} \varepsilon'$.

-Then, $\tilde{E}[\![\varepsilon']\!] \sqsubseteq \bigsqcup\{\tilde{E}[\![\delta]\!] \mid \varepsilon \xrightarrow{S} \delta\}$

$\sqsubseteq \bigsqcup\{\tilde{E}[\![\delta]\!] \mid \varepsilon \xrightarrow{R} \delta\}$, by assumption.

-∴, by 4.7.3 and 4.7.9,

$$\tilde{E}[\![\varepsilon']\!] \prec \bigsqcup\{\tilde{E}[\![\delta]\!] \mid \varepsilon \xrightarrow{R} \delta\}.$$

-∴, there exists a δ such that $\varepsilon \xrightarrow{R} \delta$ and $\varepsilon' \tilde{\sqsubseteq} \delta$.

-∴, R is weakly complete relative to S.

(ix) -Trivial, by part (viii).

‡

6.1:Inside-Out Reductions:-

6.1.0:REMARK:-

We are going to be working more and more inside λ-calculus. Recall the definitions of redexes, residuals and complete relative reductions and the notation developed for these in section 0.4.

The notion of inside-out reductions is contrary to standard reductions where outermost redexes are contracted first. With inside-out reductions you must contract inside redexes first.

There is no insistence, however, of contracting inner redexes - as there is in "call-by-name" - but, once a redex, $(\lambda x.\varepsilon)(\delta)$, is contracted, all residuals of any subredexes of ε or δ are protected from further contraction. If you wanted then contracted, you should have done so earlier. Thus, inside-out reductions are fairly efficient compared with other rules, as we shall see later, and we are reminded of Vuillemin's "call-by-need" mechanism - [61].

The idea of inside-out reductions was suggested to me by David Park as a means of describing how i'th reductions worked.

6.1.1:DEF:-

Let,

$$\varepsilon_0 \xrightarrow{R_1} \varepsilon_1 \xrightarrow{R_2} \varepsilon_2 \xrightarrow{R_3} \cdots\cdots \xrightarrow{R_n} \varepsilon_n,$$

be a sequence of β-reductions where R_i is the redex contracted in going from ε_{i-1} to ε_i. Then, the sequence is <u>INSIDE-OUT</u> if, whenever $1 \le i < j \le n$, the redex R_j is not the residual of any subredex of R_i relative to the subsequence $\varepsilon_{i-1} \longrightarrow \cdots \longrightarrow \varepsilon_{j-1}$. Further, the sequence remains inside-out with the insertion of α-conversions at any stage. We write,

$$\varepsilon_0 \rightsquigarrow \varepsilon_n.$$

6.1.2:EXAMPLES:-

The following reduction sequences are all inside-out :-

(i) $(\lambda x.xax)(\lambda y.y((\lambda z.z)y)) \longrightarrow (\lambda x.xax)(\lambda y.yy)$

$\longrightarrow (\lambda y.yy)(a)(\lambda y.yy) \longrightarrow aa(\lambda y.yy).$

(ii) $(\lambda f.f^3(a))((\lambda x.\lambda y.yx)b) \longrightarrow (\lambda f.f^3(a))(\lambda y.yb)$

$\longrightarrow (\lambda y.yb)((\lambda y.yb)((\lambda y.yb)(a)))$

$\longrightarrow (\lambda y.yb)((\lambda y.yb)(ab)) \longrightarrow (\lambda y.yb)(abb) \longrightarrow abbb.$

(iii) $(\lambda x.y)(\Delta\Delta) \longrightarrow y.$

(iv) $(\lambda y.\Delta(ya))(I) \longrightarrow (\lambda y.(ya)(ya))(I) \longrightarrow (Ia)(Ia)$

$\longrightarrow a(Ia) \longrightarrow aa.$

6.1.3:REMARK:-

Examples (i) and (ii) above were taken from Wadsworth - [62] — where we see that in case (i) inside-out reductions are more efficient than "normal graph reductions", while in case (ii) they are equally so.

Example (iii) is to show that inside-out reductions can terminate where the "call-by-value" mechanism would get stuck.

Example (iv) shows that inside-out reductions are not always the most efficient way, since it takes 4 reduction steps while it is possible to do it in 3. Still, "normal graph reductions" take 4 as well. Maybe, if we only insisted on the inside-out property for the "rand" of R_i, we could overcome this inefficiency. But, then we will move slightly away from our i'th reductions.

6.1.4:COUNTER-EXAMPLE:-

I'th reductions are not strongly complete relative to inside-out reductions.

Proof:-

-This is the same counter-example as in 5.4.2(iii).

-Let $\varepsilon = (\lambda x.\Delta^2(xx))(T)$, as before.

-Then, $\varepsilon \longrightarrow \Delta^2T^2 \longrightarrow \Delta^2T^n$, is inside-out, for all $n \geq 3$.

-But, $i\langle\varepsilon\rangle = \Delta^2T^2$, for all $i \geq 1$, as before.

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6.1.5:DEF:-

Let γ be a subexpression of ε. Then $\varepsilon \overset{\gamma}{\not\longrightarrow} \delta$ means that there is a β-reduction sequence from ε to δ such that none of the residuals of any subredex of γ in ε, relative to the reductions so far carried out, is ever contracted.

6.1.6:NOTATION:-

Recall that we number reduction sequences by,

$$\varepsilon \longrightarrow \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textcircled{3}$$

or,

$$\varepsilon \xrightarrow{\textcircled{3}} \delta.$$

If we concatenate sequences like,

$$\varepsilon \xrightarrow{\textcircled{3}} \delta \xrightarrow{\textcircled{4}} \gamma \xrightarrow{\textcircled{5}} \eta,$$

we get the sequence,

$$\varepsilon \xrightarrow{\textcircled{3}\,;\,\textcircled{4}\,;\,\textcircled{5}} \eta.$$

If we run sequences in parallel, we write,

$$\varepsilon(\gamma) \xrightarrow{\textcircled{3} \times \textcircled{5}} \delta(\eta).$$

We write,

$$\textcircled{3} \nearrow \{\gamma\},$$

to indicate,

$$\varepsilon \xrightarrow{\overset{\gamma}{\nearrow}} \delta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textcircled{3}$$

We generalise and combine the above definitions in the obvious way so as to allow,

$$\varepsilon \xrightarrow{\overset{\gamma}{\nearrow}\quad\overset{\gamma'}{\nearrow}} \delta,$$

$$\varepsilon \overset{\overset{\gamma}{\nearrow}}{\rightsquigarrow} \delta \text{ etc....}$$

Then, we will get sentences like,

$$\textcircled{3} \nearrow \{\gamma,\gamma'\}.$$

Finally, if,

$$(\lambda x.\varepsilon)(\delta) \xrightarrow{\textcircled{6}} [\delta/x]\varepsilon \xrightarrow{\textcircled{7}} \eta,$$

and ⑥ ; ⑦ $\nearrow \{\delta,\varepsilon\}$, we will just write,

$$[\delta/x]\varepsilon \xrightarrow{\overset{\delta}{\nearrow}\quad\overset{\varepsilon}{\nearrow}} \eta.$$

6.1.7:DEF:-

$\varepsilon \xrightarrow{\leq n} \delta$ means that there is a β-reduction sequence from ε to δ which has ≤ n β-reductions. This extends, in the obvious way to $\varepsilon \overset{\leq n}{\rightsquigarrow} \delta$ etc....

6.1.8:REMARK:-

We can read "③ $\nearrow \{\gamma\}$" as "reduction sequence ③ does not admit residuals of sub-redexes of γ" or "③ does not touch γ" or "③ bans sub-redexes of γ" or "γ is protected during ③" etc...

This notion is closely related to inside-out reductions, since if :-

$$\varepsilon \xrightarrow[\textcircled{1}]{} \delta \xrightarrow[\textcircled{2}]{'(\lambda a.\mu)(\nu)} \gamma \xrightarrow[\textcircled{3}]{} \eta,$$

is inside-out, where ② is just the contraction of the indicated sub-redex of δ, then ② ; ③ $\nearrow \{\mu,\nu\}$.

We needed the last part of the notation, 6.1.6, since ε is not neccessarily a sub-expression of $[\delta/x]\varepsilon$. Neither is δ and it might even occur repeatedly.

We now make a crucial observation about inside-out reductions which enables us to use induction on them - both structural and on the length of the sequences.

6.1.9:THEOREM:-

$$\text{(i)}\quad (\varepsilon(\delta) \longrightarrow \eta) \iff (\varepsilon \longrightarrow \varepsilon') \wedge (\delta \longrightarrow \delta') \wedge (\varepsilon'(\delta') \xrightarrow{\overset{\varepsilon'}{\nearrow}\ \overset{\delta'}{\nearrow}} \eta).$$

$$\text{(ii)}\quad (\lambda x.\varepsilon \xrightarrow{\overset{\gamma}{\nearrow}} \eta) \iff (\eta \xrightarrow{\alpha} \lambda x.\eta') \wedge (\varepsilon \xrightarrow{\overset{\gamma}{\nearrow}} \eta').$$

Proof:-

-Clear.

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6.1.10:REMARK:-

Note that in part (i) of the above theorem, either $\varepsilon'(\delta') = \eta$, or $\varepsilon' = \lambda x.\varepsilon''$ and the last reduction sequence must be of the form,

$$(\lambda x.\varepsilon'')(\delta') \longrightarrow [\delta'/x]\varepsilon'' \xrightarrow{\overset{\delta'}{\nearrow}\ \overset{\varepsilon''}{\nearrow}} \eta.$$

We end this section with an observation concerning NOH.

6.1.11:LEMMA:-

Let $\varepsilon \in$ NOH. Then,

$$([\delta/x]\varepsilon \xrightarrow{\overset{\varepsilon}{\nearrow}} \eta) \Rightarrow (\eta \in \mathrm{NOH}) \wedge (\varepsilon \in \lambda I.\mathrm{NOH} \iff \eta \in \lambda I.\mathrm{NOH}).$$

Proof:-

-Intuitively, this is clear, since the residuals of the head redex of ε are not contracted and so a head redex is preserved.

-However, as an exercise in our notation, we will prove this formally with a structural induction on the form of NOH.

-Again, we have to introduce a change of variables operator into the induction hypothesis, but this can virtually be ignored except for the abstraction case.

-Let $\mathcal{A}[\![\varepsilon]\!] \equiv ([\delta/x]\chi(\varepsilon) \xrightarrow[\nearrow]{\chi(\varepsilon)} \eta) \Rightarrow (\eta \in \mathrm{NOH}) \wedge (\varepsilon \in \lambda\mathrm{I.EXP}$

$\Leftrightarrow \eta \in \lambda\mathrm{I.EXP})$.

-First, note that $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \varepsilon \notin \mathrm{I}$.

-<u>Claim:</u> $\mathcal{A}[\![(\lambda a.\mu)(\nu)]\!]$:-

-Let $[\delta/x]\chi(\lambda a.\mu)(\nu) \xrightarrow[\nearrow]{\chi(\lambda a.\mu)(\nu)} \eta$.

-Now, $(\lambda a.\mu)(\nu) \notin \lambda\mathrm{I.EXP}$, so all we have to show is that $\eta \in \mathrm{NOH}\backslash\lambda\mathrm{I.NOH}$.

-But, $[\delta/x]\chi(\lambda a.\mu)(\nu) = (\lambda z.[\delta/x]\chi[z/a]\mu)([\delta/x]\chi(\nu))$, where $z \neq x$, not in χ and not free in δ,μ.

-This external redex is not contracted and so its residual, still external, will remain in η - i.e. $\eta \in \mathrm{NOH}\backslash\lambda\mathrm{I.NOH}$.

-<u>Claim:</u> $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\varepsilon(\omega)]\!]$:-

-Let $[\delta/x]\chi(\varepsilon(\omega)) \xrightarrow[\nearrow]{\chi(\varepsilon(\omega))} \eta$.

-Now, $\varepsilon \notin \mathrm{I}$, since $\mathcal{A}[\![\varepsilon]\!]$.

-If $\varepsilon \in \lambda\mathrm{I.EXP}$, we have the result by the first claim.

-∴, suppose $\varepsilon \in (\mathrm{EXP})(\mathrm{EXP})$.

-Now, $\varepsilon(\omega) \notin \lambda\mathrm{I.EXP}$ and so, again, we must show $\eta \in \mathrm{NOH}\backslash\lambda\mathrm{I.NOH}$.

-But, $[\delta/x]\chi(\varepsilon(\omega)) = ([\delta/x]\chi(\varepsilon))([\delta/x]\chi(\omega))$, where $[\delta/x]\chi(\varepsilon) \notin \lambda\mathrm{I.EXP}$. Therefore, the reductions to η must lie internal, within these two parts, at least until the left one becomes an abstraction - i.e. until,

$$[\delta/x]\chi(\varepsilon) \xrightarrow[\nearrow]{\chi(\varepsilon)} \varepsilon' \in \lambda\mathrm{I.EXP}.$$

-But, this is impossible, by $\mathcal{A}[\![\varepsilon]\!]$, since $\varepsilon \notin \lambda\mathrm{I.EXP}$.

-Further, all such $\varepsilon' \in \mathrm{NOH}$ and so $\eta \in (\mathrm{NOH})(\mathrm{EXP})$.

-∴, $\eta \in \mathrm{NOH}\backslash\lambda\mathrm{I.NOH}$.

-<u>Claim:</u> $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda y.\varepsilon]\!]$:-

-Let $[\delta/x]\chi(\lambda y.\varepsilon) \xrightarrow[\nearrow]{\chi(\lambda y.\varepsilon)} \eta$.

-But, $[\delta/x]\chi(\lambda y.\varepsilon) = \lambda z.[\delta/x]\chi[z/y]\varepsilon$, where $z \neq x$, not in χ, and not free in δ,ε.

-∴, $[\delta/x]\chi[z/y]\varepsilon \xrightarrow[\nearrow]{\chi[z/y]\varepsilon} \bar{\eta}$ and $\eta \xrightarrow{\alpha} \lambda z.\bar{\eta} \in \lambda\mathrm{I.EXP}$.

-$\therefore$, by $A[\![\varepsilon]\!]$, $\overline{\eta} \in \mathrm{NOH}$. Thus, $\eta \in \mathrm{NOH}$.

-$\therefore$, by structural induction, $(\forall \varepsilon \in \mathrm{NOH})A[\![\varepsilon]\!]$.

-So, letting χ be the null COVO, we get the lemma.

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6.1.12:COR:-

$$(\varepsilon \in \mathrm{NOH})\wedge(\varepsilon(\delta) \xrightarrow{\overset{\varepsilon}{\nearrow}} \eta) \Rightarrow (\eta \in \mathrm{NOH}).$$

Proof:-

-If $\varepsilon \notin \lambda\mathrm{I.EXP}$, then $\eta = \varepsilon(\delta') \in (\mathrm{NOH})(\mathrm{EXP}) \subset \mathrm{NOH}$.

-If $\varepsilon = \lambda x.\varepsilon'$, with $\varepsilon' \in \mathrm{NOH}$, then the reduction is of the form,

$$(\lambda x.\varepsilon')(\delta) \xrightarrow{\alpha,\beta} (\lambda x.\varepsilon')(\delta') \longrightarrow [\delta'/x]\varepsilon' \xrightarrow{\overset{\varepsilon'}{\nearrow}} \eta.$$

-$\therefore$, $\eta \in \mathrm{NOH}$, by 6.1.11.

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6.2:Weak Equivalence of I'th and Inside-Out Reductions:-

6.2.0:LEMMA:-

$$\varepsilon(\delta) \xrightarrow{\overset{\varepsilon}{\nearrow}\quad\overset{\delta}{\nearrow}} \mathrm{Ap}^{i}(\varepsilon,\delta).$$

$$[\delta/x](\varepsilon) \xrightarrow{\overset{\varepsilon}{\nearrow}\quad\overset{\delta}{\nearrow}} [\delta/x]^{i}(\varepsilon).$$

Proof:-

-Let $P(i)$ and $Q(i)$ be the above two sentences respectively.

-Then, clearly, $P(1)\wedge Q(1)$.

-Claim: $Q(i) \Rightarrow P(i)$, $\forall i \geq 2$:-

-If $\varepsilon \notin \lambda\mathrm{I.EXP}$, then trivial, since $\varepsilon(\delta) = \mathrm{Ap}^{i}(\varepsilon,\delta)$.

-If $\varepsilon = \lambda x.\varepsilon'$, then $\mathrm{Ap}^{i}(\lambda x.\varepsilon',\delta) = [\delta/x]^{i}\varepsilon'$.

-But, $(\lambda x.\varepsilon')(\delta) \longrightarrow [\delta/x]\varepsilon' \xrightarrow{\overset{\delta}{\nearrow}\quad\overset{\varepsilon'}{\nearrow}} [\delta/x]^{i}\varepsilon'$, by $Q(i)$, and it is inside-out, by 6.1.9(i).

-$\therefore$, $(\lambda x.\varepsilon')(\delta) \xrightarrow{\overset{\lambda x.\varepsilon'}{\nearrow}\quad\overset{\delta}{\nearrow}} \mathrm{Ap}^{i}(\lambda x.\varepsilon',\delta)$.

-Claim: $P(i-1)\wedge Q(i-1) \Rightarrow Q(i)$, $\forall i \geq 2$:-

-If $\varepsilon \in \mathrm{I} \cup \mathrm{NOH}$, then trivial, since $[\delta/x]\varepsilon = [\delta/x]^{i}\varepsilon$.

-If $\varepsilon = \lambda y.\varepsilon' \in \mathrm{HNF}$, then, choosing $z \neq x$ and not free in δ,ε',

$$[\delta/x](\lambda y.\varepsilon') = \lambda z.[\delta/x][z/y]\varepsilon'$$

$$\xrightarrow{\overset{[z/y]\varepsilon'}{\nearrow}\qquad\overset{\delta}{\nearrow}}\lambda z.[\delta/x]^{i-1}[z/y]\varepsilon', \text{ by } Q(i-1).$$

$$= [\delta/x]^{i}(\lambda y.\varepsilon').$$

-Therefore, we have,

$$[\delta/x](\lambda z.[z/y]\varepsilon')\xrightarrow{\overset{\lambda z.[z/y]\varepsilon'}{\nearrow}\qquad\overset{\delta}{\nearrow}}[\delta/x]^{i}(\lambda z.[z/y]\varepsilon'$$

-i.e.,

$$[\delta/x](\lambda y.\varepsilon')\xrightarrow{\overset{\lambda y.\varepsilon'}{\nearrow}\qquad\overset{\delta}{\nearrow}}[\delta/x]^{i}(\lambda y.\varepsilon').$$

-Finally, if $\varepsilon = \omega(\eta) \in$ HEAD, then,

$$[\delta/x]\omega \xrightarrow{\overset{\delta}{\nearrow}\,\overset{\omega}{\nearrow}\;\textcircled{1}}[\delta/x]^{i-1}\omega, \text{ by } Q(i-1),$$

$$\text{and } [\delta/x]\eta \xrightarrow{\overset{\delta}{\nearrow}\,\overset{\eta}{\nearrow}\;\textcircled{2}}[\delta/x]^{i-1}\eta, \text{ by } Q(i-1).$$

-But, by $P(i-1)$,

$$([\delta/x]^{i-1}\omega)([\delta/x]^{i-1}\eta)\xrightarrow{\overset{[\delta/x]^{i-1}\omega}{\nearrow}\quad\textcircled{3}\quad\overset{[\delta/x]^{i-1}\eta}{\nearrow}}[\delta/x]^{i}\omega(\eta).$$

-Then, by 6.1.9(i), ① × ② ; ③ is inside-out.

-But, also, ① × ② ; ③ $\nearrow \{\delta,\omega,\eta\}$, since residuals of sub-redexes of δ,ω,η relative to ① × ② must be sub-redexes of $[\delta/x]^{i-1}\omega$ and $[\delta/x]^{i-1}\eta$, which are protected during ③ .

-$\therefore$, ① × ② ; ③ $\nearrow \{\delta,\omega(\eta)\}$, since $\omega(\eta)$ is not a redex, being an element of HEAD.

-$\therefore$, by induction, $(\forall i \geq 1)(P(i) \wedge Q(i))$.

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6.2.1:COR:-

I'th reductions are inside-out.

Proof:-

-Let $A[\![\varepsilon]\!] \equiv (\varepsilon \rightsquigarrow i\langle\varepsilon\rangle)$.

-Clearly, $A[\![x]\!]$ and $(A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!])$.

-Claim: $A[\![\omega]\!] \wedge A[\![\eta]\!] \Rightarrow A[\![\omega(\eta)]\!]$:-

-Trivially, $\omega(\eta) \rightsquigarrow 1\langle\omega(\eta)\rangle$.

-For all $i \geq 2$,

$$\begin{tikzcd}
\omega(\eta) \arrow[r, squiggly] & (i\langle\omega\rangle)(i\langle\eta\rangle)\text{, by } \mathcal{A}[\![\omega]\!]_{\wedge}\mathcal{A}[\![\eta]\!]. \arrow[d, squiggly, "i\langle\omega\rangle \nearrow" {pos=0.25}, "i\langle\eta\rangle \nearrow" {pos=0.6}] \\
& Ap_i(i\langle\omega\rangle, i\langle\eta\rangle) \arrow[d, phantom, "\shortparallel"] \\
& i\langle\omega(\eta)\rangle.
\end{tikzcd}$$

-∴, by 6.1.9(i), $\omega(\eta) \rightsquigarrow i\langle\omega(\eta)\rangle$.

-So, by structural induction, $(\forall\varepsilon \in EXP)\mathcal{A}[\![\varepsilon]\!]$.

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6.2.2:COR:-

Inside-out reductions are strongly complete relative to i'th reductions.

Proof:-

-By 6.0.3(i) and 6.2.1.

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6.2.3:REMARK:-

The above result is not surprising since this was the motivation for introducing inside-out reductions.

However, we now get the bonus of i'th reductions "dominating" inside-out reductions, thus establishing weak equivalence.

6.2.4:LEMMA:-

$(\ \varepsilon(\delta) \xrightarrow{\overset{\varepsilon}{\nearrow}\ \ \overset{\delta}{\nearrow}} \eta) \Rightarrow (\eta \cong Ap^i(\varepsilon,\delta)$, for some $i \geq 1)$.

$([\delta/x](\varepsilon) \xrightarrow{\overset{\varepsilon}{\nearrow}\ \ \overset{\delta}{\nearrow}} \eta) \Rightarrow (\eta \cong [\delta/x]^i\varepsilon$, for some $i \geq 1)$.

Proof:-

-Let $P(n)$ and $Q(n)$ be the above two sentences respectively, but with "$\leq n$" written above the L.H.S. reduction sequences.

-Clearly, $P(0)$ and $Q(0)$, with $i = 1$.

-<u>Claim:</u> $\mathcal{Q}(n-1) \Rightarrow \mathcal{P}(n),\ \forall n \geq 1$:-

-If $\varepsilon \notin \lambda I.\mathrm{EXP}$, then trivial since $\varepsilon(\delta) = \eta = \mathrm{Ap}^1(\varepsilon,\delta)$.

-If $\varepsilon = \lambda x.\varepsilon'$, then the reduction must be,

$$(\lambda x.\varepsilon')(\delta) \longrightarrow [\delta/x]\varepsilon' \rightsquigarrow \overset{\delta}{\nearrow} \overset{\varepsilon'}{\nearrow} \xrightarrow{<\, n} \eta.$$

-∴, $\eta \overset{\sim}{\equiv} [\delta/x]^i\varepsilon'$, for some $i \geq 1$, by $\mathcal{Q}(n-1)$.

$= \mathrm{Ap}^i(\lambda x.\varepsilon',\delta)$.

-<u>Claim:</u> $\mathcal{P}(n) \Rightarrow \mathcal{Q}(n),\ \forall n \geq 1$:-

-We do a structural induction on ε. Again the bound variable causes a little problem in the abstraction case.

-Let $A[\![\varepsilon]\!] \equiv ([\delta/x]\chi(\varepsilon) \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi(\varepsilon)}{\nearrow} \xrightarrow{\leq\, n} \eta)$

$\Rightarrow (\eta \overset{\sim}{\equiv} [\delta/x]^i\chi(\varepsilon)$, for some $i \geq 1)$.

-Then, $\mathcal{Q}(n) \equiv (\forall\varepsilon \in \mathrm{EXP})A[\![\varepsilon]\!]$.

-Clearly, $A[\![y]\!]$, since $\eta = [\delta/x]\chi(\varepsilon)$ in this case.

-<u>Claim:</u> $A[\![\varepsilon]\!] \Rightarrow A[\![\lambda y.\varepsilon]\!]$:-

-Let $[\delta/x]\chi(\lambda y.\varepsilon) \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi(\lambda y.\varepsilon)}{\nearrow} \xrightarrow{\leq\, n} \eta$.

-By 6.1.9(ii),

$[\delta/x]\chi[z/y]\varepsilon \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi[z/y]\varepsilon}{\nearrow} \xrightarrow{\leq\, n} \bar{\eta}$, where $z \neq x$, not in χ and is not free in δ,ε. Also, $\eta \xrightarrow{\alpha} \lambda z.\bar{\eta}$.

-By $A[\![\varepsilon]\!]$, $\bar{\eta} \overset{\sim}{\equiv} [\delta/x]^{i-1}\chi[z/y]\varepsilon$, for some $i \geq 2$.

-∴, $\eta \overset{\sim}{\equiv} \lambda z.[\delta/x]^{i-1}\chi[z/y]\varepsilon$, by 5.0.4.

$= [\delta/x]^i(\lambda y.\varepsilon)$.

-<u>Claim:</u> $A[\![\omega]\!] \wedge A[\![\varepsilon]\!] \Rightarrow A[\![\omega(\varepsilon)]\!]$:-

-Let $[\delta/x]\chi(\omega(\varepsilon)) \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi(\omega(\varepsilon))}{\nearrow} \xrightarrow{\leq\, n} \eta$.

-If $\omega(\varepsilon) \in \mathrm{NOH}$, then so is $\chi(\omega(\varepsilon))$ and also, by 6.1.11, η. Thus, the result is trivial. (N.B. This is where the induction breaks down if we tried to prove a "strong" version - i.e. using $\xrightarrow{\alpha,\beta}$ instead of $\overset{\sim}{\equiv}$.)

-∴, we suppose $\omega(\varepsilon) \in \mathrm{HEAD}$. Then, by 6.1.9(i),

$([\delta/x]\chi(\omega) \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi(\omega)}{\nearrow} \xrightarrow{\leq\, n} \mu), ([\delta/x]\chi(\varepsilon) \rightsquigarrow \overset{\delta}{\nearrow} \overset{\chi(\varepsilon)}{\nearrow} \xrightarrow{\leq\, n} \nu)$ and $(\mu(\nu) \rightsquigarrow \overset{\mu}{\nearrow} \overset{\nu}{\nearrow} \xrightarrow{\leq\, n} \eta)$.

Thus, $\eta \overset{\sim}{\equiv} Ap^{i}(\mu, \nu)$, by $P(n)$.

$\overset{\sim}{\equiv} Ap^{i}([\delta/x]^{j}\chi(\omega), [\delta/x]^{k}\chi(\varepsilon))$, by $\mathcal{A}[\![\omega]\!] \wedge \mathcal{A}[\![\varepsilon]\!]$ and 5.1.10.

$\overset{\sim}{\equiv} Ap^{i}([\delta/x]^{m}\chi(\omega), [\delta/x]^{m}\chi(\varepsilon))$, where $m = \max(i,j,k)$, by 5.2.10.

$\overset{\sim}{\equiv} Ap^{m}([\delta/x]^{m}\chi(\omega), [\delta/x]^{m}\chi(\varepsilon))$, by 5.1.4.

$= [\delta/x]^{m+1}\chi(\omega(\varepsilon))$, since $\chi(\omega)(\chi(\varepsilon)) \in$ HEAD.

-∴, by structural induction, $(\forall\varepsilon \in EXP)\mathcal{A}[\![\varepsilon]\!]$.

-∴, by ordinary induction, $(\forall n \geq 0)(P(n) \wedge Q(n))$.

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6.2.5:THEOREM:-

I'th reductions are weakly complete relative to inside-out reductions.

Proof:-

-Let $\mathcal{A}[\![\varepsilon]\!] \equiv (\varepsilon \rightsquigarrow \delta) \Rightarrow (\delta \overset{\sim}{\equiv} i\langle\varepsilon\rangle)$, for some $i \geq 1$.

-Clearly, $\mathcal{A}[\![x]\!]$.

-Claim: $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda x.\varepsilon]\!]$:-

-Let $\lambda x.\varepsilon \rightsquigarrow \delta$.

-Then, $\varepsilon \rightsquigarrow \delta'$ and $\delta \xrightarrow{\alpha} \lambda z.\delta'$, by 6.1.9(ii).

-∴, $\delta \cong \lambda x.\delta' \overset{\sim}{\equiv} \lambda x.i\langle\varepsilon\rangle$, by $\mathcal{A}[\![\varepsilon]\!]$, $= i\langle\lambda x.\varepsilon\rangle$.

-Claim: $\mathcal{A}[\![\varepsilon]\!] \wedge \mathcal{A}[\![\delta]\!] \Rightarrow \mathcal{A}[\![\varepsilon(\delta)]\!]$:-

-Let $\varepsilon(\delta) \rightsquigarrow \eta$.

-Then, $\varepsilon \rightsquigarrow \varepsilon'$, $\delta \rightsquigarrow \delta'$ and $\varepsilon'(\delta') \rightsquigarrow \overset{\varepsilon'}{\nearrow} \overset{\delta'}{\nearrow} \rightsquigarrow \eta$, by 6.1.9(i).

-∴, $\eta \overset{\sim}{\equiv} Ap^{i}(\varepsilon',\delta')$, by 6.2.4.

$\overset{\sim}{\equiv} Ap^{i}(j\langle\varepsilon\rangle,k\langle\delta\rangle)$, by $\mathcal{A}[\![\varepsilon]\!] \wedge \mathcal{A}[\![\delta]\!]$ and 5.1.10.

$\overset{\sim}{\equiv} Ap^{m}(m\langle\varepsilon\rangle,m\langle\delta\rangle)$, where $m = \max(i,j,k)$, by 5.3.3,5.1.10 and 5.1.4.

$= m\langle\varepsilon(\delta)\rangle$.

-So, by structural induction, $(\forall\varepsilon \in EXP)\mathcal{A}[\![\varepsilon]\!]$.

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6.2.6:COR:-

Inside-out reductions are weakly equivalent to i'th reductions.

Proof:-

-By 6.2.2,6.0.3(ii) and 6.2.5.

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6.2.7:REMARK:-

Thus, inside-out reductions have turned out to be a particularly relevant type of reduction to study when seeking results about i'th reductions.

6.2.8:PROPERTY A:-

Inside-out reductions are strongly complete.

6.2.9:PROPERTY B:-

Inside-out reductions are weakly complete.

6.2.10 LEMMA:-

Property B <=> Property M.

Proof:-

-By 6.0.3(iv) and 6.2.6.

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6.2.11:REMARK:-

In the rest of this chapter, we are chasing property B, since it seems that our notion of weak completeness is precisely what we need, while strong completeness is redundant.

If inside-out (or i'th) reductions are weakly complete, then, given any reduction, we can match the result symbol by symbol with the result of an inside-out reduction except, possibly, for parts that are in NOH. If these parts eventually emerge into HNF, then we can match it, but if they stay stuck in NOH no matter how you try to reduce it, these parts are unsolvable and not important and so what does it matter if we cannot match them?

We end this section with a crucial Church-Rosser property for $\rightsquigarrow$'s. It is very easy to prove here, but if we could strengthen it by replacing the $\overset{\sim}{\equiv}$ with an $\xrightarrow{\alpha,\beta}$, the development in the rest of this chapter would establish property A.

We use the term "weak" to describe results involving $\overset{\sim}{\equiv}$. The corresponding result involving $\xrightarrow{\alpha,\beta}$, we will call "strong".

6.2.12:LEMMA:- (Weak Church-Rosser Theorem for $\rightsquigarrow$'s.)

Let $\varepsilon \rightsquigarrow \delta$ and $\varepsilon \rightsquigarrow \gamma$. Then, there exists an η such that $\varepsilon \rightsquigarrow \eta$ and $\delta,\gamma \overset{\sim}{\equiv} \eta$.

Proof:-

-By 6.2.5, $\delta \overset{\sim}{\equiv} i\langle\varepsilon\rangle$ and $\gamma \overset{\sim}{\equiv} j\langle\varepsilon\rangle$, for some $i,j \geq 1$.

-So, $\delta,\gamma \overset{\sim}{\equiv} k\langle\varepsilon\rangle$, where $k = \max(i,j)$, by 5.3.3.

-But, $\varepsilon \rightsquigarrow k\langle\varepsilon\rangle$, by 6.2.1.

-Take $\eta := k\langle\varepsilon\rangle$.

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6.3:Weak Parallel Moves:-

6.3.0:DEF:-

$$P(n) \equiv (\varepsilon(\delta) \xrightarrow{\overset{\varepsilon}{\nearrow}\ \overset{\delta}{\nearrow}\ \ \leq n} \eta) \wedge (\delta \overset{\sim}{\equiv} \delta') \wedge (\varepsilon \overset{\sim}{\equiv} \varepsilon')$$
$$\Rightarrow (\varepsilon'(\delta') \xrightarrow{\overset{\varepsilon'}{\nearrow}\ \overset{\delta'}{\nearrow}\ \ \leq n} \eta') \wedge (\eta \overset{\sim}{\equiv} \eta').$$
$$Q(n) \equiv ([\delta/x]\varepsilon \xrightarrow{\overset{\varepsilon}{\nearrow}\ \overset{\delta}{\nearrow}\ \ \leq n} \eta) \wedge (\delta \overset{\sim}{\equiv} \delta') \wedge (\varepsilon \overset{\sim}{\equiv} \varepsilon')$$
$$\Rightarrow ([\delta'/x]\varepsilon' \xrightarrow{\overset{\varepsilon'}{\nearrow}\ \overset{\delta'}{\nearrow}\ \ \leq n} \eta') \wedge (\eta \overset{\sim}{\equiv} \eta').$$

6.3.1:LEMMA:-

(i) $P(0) \wedge Q(0)$.

(ii) $Q(n-1) \Rightarrow P(n)$, $\forall n \geq 1$.

Proof:-

(i) -Trivial.

(ii) - If $\varepsilon \not= \lambda I.EXP$, then trivial since $\eta = \varepsilon(\delta)$.

-If $\varepsilon = \lambda x.\omega$, the reduction must be,

$$(\lambda x.\omega)(\delta) \longrightarrow [\delta/x]\omega \overset{\overset{\delta}{\nearrow}\;\overset{\omega}{\nearrow}\quad\le n-1}{\rightsquigarrow} \eta.$$

-Now, $\lambda x.\omega \overset{\sim}{\equiv} \varepsilon'$. If $\lambda x.\omega \in$ NOH, then $\omega \in$ NOH and so $\eta \in$ NOH, by 6.1.11. Hence, the result, taking $\eta' := \varepsilon'(\delta')$.

-Otherwise, by 4.8.4, $\lambda x.\omega \xrightarrow{\alpha} \lambda y.\gamma$, $\varepsilon' \xrightarrow{\alpha} \lambda y.\gamma'$ and $\gamma \overset{\sim}{\equiv} \gamma'$.

-Since we can insert α-conversions into inside-out reductions, we have,

$$(\lambda x.\omega)(\delta) \xrightarrow{\alpha} (\lambda y.\gamma)(\delta) \xrightarrow{\beta} [\delta/y]\gamma \overset{\overset{\delta}{\nearrow}\;\overset{\gamma}{\nearrow}\quad\le n-1}{\rightsquigarrow} \eta.$$

-By $Q(n-1)$, $[\delta'/y]\gamma' \overset{\overset{\delta'}{\nearrow}\;\overset{\gamma'}{\nearrow}\quad\le n-1}{\rightsquigarrow} \eta'$ and $\eta \overset{\sim}{\equiv} \eta'$.

-∴, $(\lambda y.\gamma')(\delta') \overset{\overset{\delta'}{\nearrow}\;\overset{\gamma'}{\nearrow}\quad\le n}{\rightsquigarrow} \eta'$.

-∴, $\varepsilon'(\delta') \overset{\overset{\varepsilon'}{\nearrow}\;\overset{\delta'}{\nearrow}\quad\le n}{\rightsquigarrow} \eta'$.

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6.3.2:REMARK:-

This section is just to establish a technical result which is necessary later.

We complete the last stage of the induction with another structural induction. Again, we have the difficulty with bound variables.

6.3.3:DEF:-

$$\text{Let } \mathcal{A}[\![\varepsilon]\!] \equiv ((\delta \overset{\sim}{\equiv} \delta') \wedge (\chi(\varepsilon) \overset{\sim}{\equiv} \varepsilon') \wedge ([\delta/x]\chi(\varepsilon) \overset{\overset{\delta}{\nearrow}\;\overset{\chi(\varepsilon)}{\nearrow}\;\le n}{\rightsquigarrow} \eta)$$
$$\Rightarrow (\eta \overset{\sim}{\equiv} \eta') \wedge ([\delta'/x]\varepsilon' \overset{\overset{\delta'}{\nearrow}\;\overset{\varepsilon'}{\nearrow}\quad\le n}{\rightsquigarrow} \eta')).$$

6.3.4:LEMMA:-

(i) $\mathcal{A}[\![y]\!]$.

(ii) $\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda y.\varepsilon]\!]$.

Proof:-

(i) -Trivial, since there can be no reductions.

(ii) -Let $\delta \overset{\sim}{\equiv} \delta'$, $\chi(\lambda y.\varepsilon) \overset{\sim}{\equiv} \varepsilon'$ and $[\delta/x]\chi(\lambda y.\varepsilon) \overset{\overset{\delta}{\nearrow}\;\overset{\chi(\lambda y.\varepsilon)}{\nearrow}\;\le n}{\rightsquigarrow} \eta$.

-If $\lambda y.\varepsilon \in$ NOH, then so is $\chi(\lambda y.\varepsilon)$ and, thus, η, by 6.1.11.

-Then, the result is trivial, taking $\eta' := [\delta'/x]\varepsilon'$.

-If $\lambda y.\varepsilon \in$ HNF, then there exists $z \neq x$, not in χ, not free in δ, δ' and ε such that $[\delta/x]\chi(\lambda y.\varepsilon) = \lambda z.[\delta/x]\chi[z/y]\varepsilon$ and $\chi[z/y]\varepsilon \overset{\sim}{\equiv} \varepsilon''$,

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-Hence, $Q(n)$, with χ as the null change of variables operator.

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6.3.7:THEOREM:-

$(\forall n \geq 0)(P(n) \wedge Q(n))$.

Proof:-

-By ordinary induction, using 6.3.1 and 6.3.6.

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6.4:Commutativity of Reduction Path Diagrams:-

6.4.0:RECALL:-

Let R_ε be a set of sub-redexes of ε. Then,

$$\varepsilon \xrightarrow{R_\varepsilon} \delta,$$

means that ε reduces to δ by a complete relative reduction of R_ε.

We know, by Curry - [63] - and 0.4.13, that the order in which we contract the redexes in R_ε does not matter. We could, for instance, make the reduction standard. Alternatively, we could make it inside-out, merely by contracting the innermost ones first, thus obtaining what Curry calls a "minimal relative reduction" - [64]. Again, we note the efficiency of inside-out reductions.

Suppose we also have,

$$\varepsilon = \varepsilon_0 \xrightarrow{1\beta} \varepsilon_1 \xrightarrow{1\beta} \varepsilon_2 \xrightarrow{1\beta} \cdots\cdots \xrightarrow{1\beta} \varepsilon_n \qquad \textcircled{1},$$

then $R_\varepsilon/$① is the set of residuals of the redexes in R_ε relative to the sequence ①.

Let $\textcircled{1i}$ be the initial part of ① as far as ε_i. Then, there is a parallel sequence,

$$\delta = \delta_0 \xrightarrow{\beta} \delta_1 \xrightarrow{\beta} \delta_2 \xrightarrow{\beta} \cdots\cdots \xrightarrow{\beta} \delta_n \qquad \textcircled{2},$$

such that,

$$\varepsilon_i \xrightarrow{R_\varepsilon/\textcircled{1i}} \delta_i,$$

by 0.4.14.

Note, also, that if ③ is a sequence that extends ①, then $(R_\varepsilon/\text{①})/\text{③} = R_\varepsilon/\text{①};\text{③}$.

6.4.1:REMARK:-

We shall be interested in finding circumstances in which the fact that ①, in 6.4.0 above, is inside-out implies the same about ②.

This is not true in general. For example, let :-

$$\varepsilon = (\lambda x.I(xab))(K) \longrightarrow (\lambda x.xab)(K) \longrightarrow Kab \longrightarrow (\lambda y.a)(b) \longrightarrow a.$$

Then, this is inside-out and, in fact, is the only possible way we can have $\varepsilon \rightsquigarrow a$.

Now, let $R_\varepsilon = \{\varepsilon\}$ and we have $\varepsilon \xrightarrow{R_\varepsilon} I(Kab) =: \delta$. The parallel reduction sequence is then :-

$$\delta = \quad I(Kab) \longrightarrow Kab \longrightarrow Kab \longrightarrow (\lambda y.a)(b) \longrightarrow a,$$

which is not inside-out.

The circumstances we are looking for will be a strong version of 6.3.7. Before we get on to this, however, we have to introduce a concept of "commutativity" of residuals amongst the complex reduction path diagrams that can be drawn. Curry called this notion "equivalent reduction paths" - [65] - and his "property E" provides our first example. The need for such a concept in the present proof was first shown by J.R.Hindley, and we will remark upon this later (6.7.6).

First, though, we give an "obvious" technical result about residuals.

6.4.2:LEMMA:-

A redex cannot be the residual of two distinct redexes relative to the same reduction path.

Proof:-

-Let $A[\![\varepsilon]\!] \equiv (S,R \text{ are sub-redexes of } \varepsilon) \wedge (S \neq R) \wedge (\varepsilon \xrightarrow{\alpha,1\beta\ \text{①}} \delta)$

$$\Rightarrow (\{S\}/\text{①} \cap \{R\}/\text{①} = \emptyset).$$

-Clearly, $\mathcal{A}[\![x]\!]$ and $(\mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\lambda x.\varepsilon]\!])$.

-Claim: $\mathcal{A}[\![\omega]\!] \wedge \mathcal{A}[\![\varepsilon]\!] \Rightarrow \mathcal{A}[\![\omega(\varepsilon)]\!]$:-

-If $\omega(\varepsilon) \xrightarrow{\alpha,1\beta} \omega'(\varepsilon)$ or $\omega(\varepsilon')$, then clear: by induction if both R and S are in either ω or ε, or trivially if R and S are split between ω and ε.

-If $\omega(\varepsilon) = (\lambda x.\bar{\omega})(\varepsilon) \longrightarrow [\varepsilon/x]\bar{\omega}$, then the result is also clear.

-Hence, by structural induction, $(\forall \varepsilon \in \mathrm{EXP})\mathcal{A}[\![\varepsilon]\!]$.

-Let $P(n) \equiv (S,R \text{ are sub-redexes of } \varepsilon) \wedge (S \neq R) \wedge (\varepsilon \xrightarrow{\leq n \quad \textcircled{2}} \delta)$

$$\Rightarrow (\{S\}/\textcircled{2} \cap \{R\}/\textcircled{2} = \emptyset).$$

-Clearly, $P(0)$.

-Claim: $P(n-1) \Rightarrow P(n)$, $\forall n \geq 1$:-

-Now, $\varepsilon \xrightarrow{\leq n-1 \quad \textcircled{3}} \delta' \xrightarrow{\alpha,1\beta \quad \textcircled{4}} \delta$, where $\textcircled{2} = \textcircled{3};\textcircled{4}$.

-By $P(n-1)$, $\{S\}/\textcircled{3} \cap \{R\}/\textcircled{3} = \emptyset$.

-Take any $S' \in \{S\}/\textcircled{3}$ and $R' \in \{R\}/\textcircled{3}$. Then, $S' \neq R'$, and they are sub-redexes of δ'.

-So, by $\mathcal{A}[\![\delta']\!]$ above, $\{S'\}/\textcircled{4} \cap \{R'\}/\textcircled{4} = \emptyset$.

$$-\therefore,\ \{S\}/\textcircled{2} \cap \{R\}/\textcircled{2} = (\{S\}/\textcircled{3})/\textcircled{4} \cap (\{R\}/\textcircled{3})/\textcircled{4}$$
$$= \cup\{\{S'\}/\textcircled{4} \cap \{R'\}/\textcircled{4} \mid S' \in \{S\}/\textcircled{3},\ R' \in \{R\}/\textcircled{3}\}$$
$$= \emptyset.$$

-Thus, by induction, we have $(\forall n \geq 0)P(n)$.

-Hence, the lemma.

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6.4.3:DEF:-

Suppose we have a diagram of reduction paths. Let ε be on it and let R_ε be some set of sub-redexes of ε. Then, the diagram COMMUTES FOR R_ε IN ε if, for any δ on the diagram and paths ① and ② such that,

$$\begin{tikzcd}\varepsilon \arrow[r, bend left, "\textcircled{1}"] \arrow[r, bend right, "\textcircled{2}"'] & \delta,\end{tikzcd}$$

we have $(\forall \rho \in R_\varepsilon)(\{\rho\}/\textcircled{1} = \{\rho\}/\textcircled{2})$.

6.4.4:NOTATION:-

We write $\textcircled{\varepsilon}$ to denote the set of all sub-redexes of ε. If a diagram has a unique source expression, ε - i.e. for all δ on the diagram, there is a path from ε to δ - we say that the diagram <u>COMMUTES</u> if it commutes for $\textcircled{\varepsilon}$ in ε.

6.4.5 LEMMA:-

Let D be a diagram which commutes for R_ε in ε and let

$$\begin{tikzcd}\varepsilon \arrow[r, bend left, "\textcircled{1}"] \arrow[r, bend right, "\textcircled{2}"'] & \delta.\end{tikzcd}$$

Then, $(R_\varepsilon/\textcircled{1} = R_\varepsilon/\textcircled{2})$. Also, if $R'_\varepsilon \subset R_\varepsilon$, then D commutes for R'_ε in ε.

<u>Proof:-</u>

-Trivial.

6.4.6:EXAMPLES:-

(i) Suppose $\begin{tikzcd}\varepsilon \arrow[r, bend left, "\textcircled{1}"] \arrow[r, bend right, "\textcircled{2}"'] & \delta\end{tikzcd}$, where ① and ② are complete relative reductions of some R_ε in ε. Then, the diagram commutes.

(ii) Diagrams with loops tend not to commute - e.g. :-

$$\begin{tikzcd}\Delta\Delta \arrow[loop left, "\textcircled{1}"] \arrow[loop right, "\textcircled{2}"]\end{tikzcd},$$

where ① is one β-reduction and ② is just α-conversions.

(iii) The following diagram does not commute for {ΔΔ} in Δ(ΔΔ) :-

$$\begin{tikzcd}\Delta(\Delta\Delta) \arrow[r, "\textcircled{1}"] & (\Delta\Delta) \arrow[loop above, "\textcircled{2}"] & (\Delta\Delta) \arrow[loop above, "\textcircled{3}"]\end{tikzcd},$$

where ② and ③ are the contractions of the left and right redex, ΔΔ, respectively.

(iv) The following diagram commutes for $\{Ix\}$ in $(\lambda a.b)(Ix)$:-

$$\begin{tikzcd}
(\lambda a.b)(Ix) \arrow[rr, "\textcircled{1}"] \arrow[dr, "\textcircled{3}"'] & & (\lambda a.b)(x) \arrow[dl, "\textcircled{2}"] \\
& b &
\end{tikzcd}$$

Proof:-

(i) -Let $\rho \in \textcircled{$\varepsilon$}$.

-If $\rho \in R_\varepsilon$, then $\{\rho\}/\textcircled{1} = \{\rho\}/\textcircled{2} = \emptyset$, since $\textcircled{1}$ and $\textcircled{2}$ are complete relative reductions.

-If $\rho \notin R_\varepsilon$, then $\{\rho\}/\textcircled{1} = \{\rho\}/\textcircled{2}$, by 0.4.13.

(ii) $-\{\Delta\Delta\}/\textcircled{1} = \emptyset \neq \{\Delta\Delta\} = \{\Delta\Delta\}/\textcircled{2}$.

(iii) $-\{\Delta\Delta\}/\textcircled{1};\textcircled{2};\textcircled{3} = \emptyset \neq \{\Delta\Delta\} = \{\Delta\Delta\}/\textcircled{1};\textcircled{2}$.

(iv) $-\{Ix\}/\textcircled{1};\textcircled{2} = \emptyset = \{Ix\}/\textcircled{3}$.

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6.4.7:REMARK:-

6.4.6(i) is just a restatement of Curry's (strong) property E (0.4.13).

6.4.6(ii) is due to Hindley. Note that $\textcircled{2} \not\nearrow \{\Delta\Delta\}$ but $\textcircled{1}$ contracts $\Delta\Delta$.

In 6.4.6(iii), both $\textcircled{1};\textcircled{2}$ and $\textcircled{1};\textcircled{2};\textcircled{3}$ contract a residual of $\Delta\Delta$, yet the diagram still does not commute.

In 6.4.6(iv), we see that commutativity does not imply that a residual of a redex is contracted along one path iff it is contracted along all paths.

6.4.8:LEMMA:-

If $\left(\begin{tikzcd} \varepsilon \arrow[r, bend left, "\textcircled{1}"] \arrow[r, bend right, "\textcircled{2}"'] & \delta \end{tikzcd}\right)$ commutes for R_ε in ε, then

$$\left(\begin{tikzcd} \varepsilon' \arrow[r, "\textcircled{3}"] & \varepsilon \arrow[r, bend left, "\textcircled{1}"] \arrow[r, bend right, "\textcircled{2}"'] & \delta \arrow[r, "\textcircled{4}"] & \delta' \end{tikzcd}\right)$$

commutes for $R_{\varepsilon'}$ in ε', where $(R_{\varepsilon'}/\ 3\ ) \subseteq R_\varepsilon$.

Proof:-

-Let $\gamma' \in R_{\varepsilon'}$.

-Then, $\{\gamma'\}/\textcircled{3};\textcircled{1};\textcircled{4} = ((\{\gamma'\}/\textcircled{3})/\textcircled{1})/\textcircled{4}$

$= ((\{\gamma'\}/\textcircled{3})/\textcircled{2})/\textcircled{4}$, by 6.4.5, since $\{\gamma'\}/\textcircled{3} \subseteq R_{\varepsilon'}/\textcircled{3} \subseteq R_{\varepsilon}$.

$= \{\gamma'\}/\textcircled{3};\textcircled{2};\textcircled{4}$.

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6.4.9:COR:-

Consider :-

$$\begin{tikzcd}
\varepsilon \arrow[r, "\textcircled{1}"] \arrow[d, "\textcircled{3}"'] \arrow[dr, phantom, "A"] & \delta \arrow[r, "\textcircled{2}"] \arrow[d, "\textcircled{4}"'] \arrow[dr, phantom, "B"] & \gamma \arrow[d, "\textcircled{5}"] \\
\varepsilon' \arrow[r, "\textcircled{6}"'] & \delta' \arrow[r, "\textcircled{7}"'] & \gamma'
\end{tikzcd}$$.

Then, if square A commutes for R_ε in ε and square B commutes for R_δ in δ and $R_\varepsilon/\textcircled{1} \subset R_\delta$, the whole diagram commutes for R_ε in ε.

Proof:-

-Let $\rho \in R_\varepsilon$.

-Then, $\{\rho\}/\textcircled{1};\textcircled{2};\textcircled{5} = \{\rho\}/\textcircled{1};\textcircled{4};\textcircled{7}$, by 6.4.8.

$= \{\rho\}/\textcircled{3};\textcircled{6};\textcircled{7}$, by 6.4.8.

-Hence, the result.

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6.4.10:COR:-

Consider :-

$$\begin{tikzcd}
\varepsilon_0 \arrow[r, "\textcircled{1}"] \arrow[d] \arrow[dr, phantom, "A_1"] & \varepsilon_1 \arrow[r, "\textcircled{2}"] \arrow[d, dash] \arrow[dr, phantom, "A_2"] & \varepsilon_2 \arrow[r, "\textcircled{3}"] \arrow[d] \arrow[dr, phantom, "A_3 \cdots\cdots\cdots\cdots"] & \cdots\cdots \arrow[r, "\textcircled{n-1}"] \arrow[dr, phantom, "A_{n-1}"] & \varepsilon_{n-1} \arrow[r, "\textcircled{n}"] \arrow[d] \arrow[dr, phantom, "A_n"] & \varepsilon_n \arrow[d] \\
\delta_0 \arrow[r] & \delta_1 \arrow[r] & \delta_2 \arrow[r] & \cdots\cdots \arrow[r] & \delta_{n-1} \arrow[r] & \delta_n
\end{tikzcd}$$

Suppose square A_i commutes for $R_{\varepsilon_{i-1}}$ in ε_{i-1} and $(R_{\varepsilon_{i-1}}/\textcircled{i}) \subseteq R_{\varepsilon_i}$, where $1 \leq i \leq n$. Then, the whole diagram commutes for R_{ε_0} in ε_0.

Proof:-

-By 6.4.9, square $A_1;A_2$ commutes for R_{ε_0} in ε_0.

-By 6.4.9, square $(A_1;A_2);A_3$ commutes for R_{ε_0} in ε_0.

-Hence, by repeated application of 6.4.9, we get the result.

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6.4.11:COR:-

Consider how "parallel" sequences are constructed as in corollary 0.4.14 (or 6.4.0) :-

$$\begin{tikzcd}
\varepsilon_0 \arrow[r, "\textcircled{1}"] \arrow[d, "R_{\varepsilon_0}"'] \arrow[dr, phantom, "A_1"] & \varepsilon_1 \arrow[r, "\textcircled{2}"] \arrow[d, "R_{\varepsilon_0}/\textcircled{1}"] & \varepsilon_2 \arrow[r, "\textcircled{3}"] \arrow[d, "R_{\varepsilon_0}/\textcircled{1};\textcircled{2}"] \arrow[dr, phantom, "\cdots\cdot"] & \cdots\cdots\cdot\cdot \arrow[r, "\textcircled{n}"] \arrow[dr, phantom, "A_n"] & \varepsilon_n \arrow[d, "R_{\varepsilon_0}/\textcircled{1};\ldots;\textcircled{n}"] \\
\delta_0 \arrow[r] & \delta_1 \arrow[r] & \delta_2 \arrow[r] & \cdots\cdots\cdot\cdot \arrow[r] & \delta_n .
\end{tikzcd}$$

Then, the whole diagram commutes.

Proof:-

-Each reduction $\textcircled{i}$ is just one β-reduction, say of the redex σ_{i-1} in ε_{i-1}.

-The two ways around each square A_i are merely two ways of carrying out a complete relative reduction of $R_{\varepsilon_0}/\textcircled{1};\ldots;\textcircled{i-1}$ $\cup \{\sigma_{i-1}\}$.

-So, by 6.4.6(i), each square A_i commutes.

-∴, by 6.4.10, the whole diagram commutes, since, clearly, $(\textcircled{\varepsilon_{i-1}}/\textcircled{i}) \subseteq \textcircled{\varepsilon_i}$.

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6.4.12:REMARK:-

We now return to the question of when the construction of such parallel sequences preserves inside-outness, as well as commutes residuals.

6.4.13:LEMMA:-

Let $(\delta_0 \to \delta_1 \to \ldots\ldots \to \delta_m)$ be inside-out. Then, so is $([\delta_0/x]\varepsilon \to [\delta_1/x]\varepsilon \to \ldots\ldots \to [\delta_m/x]\varepsilon)$.

Proof:-

-Let $\mathcal{A}[\![\varepsilon]\!] \equiv ([\delta_0/x]\chi(\varepsilon) \to \ldots\ldots \to [\delta_m/x]\chi(\varepsilon))$ is inside-out.

-Then, the structural induction goes through very easily.

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6.4.14:LEMMA:-

Let $(\varepsilon_0 \longrightarrow \varepsilon_1 \longrightarrow \ldots\ldots \longrightarrow \varepsilon_n)$ be inside-out. Then, so is $([\delta/x]\varepsilon_0 \longrightarrow [\delta/x]\varepsilon_1 \longrightarrow \ldots\ldots \longrightarrow [\delta/x]\varepsilon_n)$.

Proof:-

-This is clear, since merely replacing free x's with δ's in the first sequence has no effect on the reductions.

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6.4.15:THEOREM:-

Let $\varepsilon_0 \xrightarrow{\textcircled{1}} \varepsilon_n$ and $\delta_0 \xrightarrow{\textcircled{2}} \delta_m$. Then, we may construct the following diagram :-

$$\begin{tikzcd}
(\lambda y.\varepsilon_0)(\delta_0) \arrow[r, "\textcircled{5}"] \arrow[d, shift right=2] \arrow[d, shift left=2, "\textcircled{3}"'] & (\lambda y.\varepsilon_n)(\delta_m) \arrow[d, "\textcircled{4}"] \\
{[\delta_0/y]\varepsilon_0} \arrow[r, "\textcircled{6}"] & {[\delta_m/y]\varepsilon_n}.
\end{tikzcd}$$

such that it commutes.

Further, if ① and ② are inside-out, then so are ⑤ and, in particular, ⑥.

Finally, if R_ε and R_δ are sets of sub-redexes of ε_0 and δ_0 respectively, such that $\textcircled{1} \nearrow R_\varepsilon$ and $\textcircled{2} \nearrow R_\delta$, then $\textcircled{5} \nearrow (R_\varepsilon \cup R_\delta)$ and, in particular, $\textcircled{6} \nearrow (R_\varepsilon \cup R_\delta)$.

Proof:-

-We construct $\textcircled{5} := \textcircled{null} \times \textcircled{2} ; \textcircled{1} \times \textcircled{null}$, where $\textcircled{null}$ is the null-β-reduction sequence - i.e. :-

$$(\lambda y.\varepsilon_0)(\delta_0) \longrightarrow (\lambda y.\varepsilon_0)(\delta_m) \;\ldots\ldots\; \textcircled{null} \times \textcircled{2}$$

$$\text{and } (\lambda y.\varepsilon_0)(\delta_m) \longrightarrow (\lambda y.\varepsilon_n)(\delta_m) \;\ldots\ldots\; \textcircled{1} \times \textcircled{null} .$$

-Then, construct ③, ④ and ⑥ in the usual parallel way of corollary 0.4.14, where $R = \{(\lambda y.\varepsilon_0)(\delta_0)\}$.

-By 6.4.11, the diagram commutes.

-Now, $\textcircled{6} = \textcircled{7} ; \textcircled{8}$, where :-

$$[\delta_0/y]\varepsilon_0 \longrightarrow [\delta_m/y]\varepsilon_0 \;\ldots\ldots\ldots\; \textcircled{7}$$

$$\text{and } [\delta_m/y]\varepsilon_0 \longrightarrow [\delta_m/y]\varepsilon_n \;\ldots\ldots\ldots\; \textcircled{8} .$$

-So, if ① and ② are inside-out, then so are ⑦ and ⑧, by 6.4.13 and 6.4.14.

-But the residuals of all sub-redexes of the redexes contracted during ⑦ mst be sub-redexes of the various occurences of δ_m in $[\delta_m/y]\varepsilon_0$.

-But, no sub-redexes of the occurences of δ_m in $[\delta_m/y]\varepsilon_0$ are ever contracted during ⑧.

-Hence, ⑥ is inside-out as well.

-Now, suppose ① $\nearrow R_\varepsilon$ and ② $\nearrow R_\delta$. Then, clearly, ⑤ $\nearrow (R_\varepsilon \cup R_\delta)$.

-Consider any reduction step in ⑤, e.g. ⑩ where :-

$$\begin{tikzcd}
(\lambda y.\varepsilon_0)(\delta_0) \arrow[r, "\textcircled{9}"] \arrow[d, "\textcircled{3}"'] \arrow[dr, phantom, "A"] & \mu \arrow[r, "\pi", "\textcircled{10}"'] \arrow[d, "\textcircled{12}"'] & \nu \arrow[r, "\textcircled{11}"] \arrow[d, "\textcircled{13}"] & (\lambda y.\varepsilon_n)(\delta_m) \arrow[d, "\textcircled{4}"] \\
{[\delta_0/y]\varepsilon_0} \arrow[r, "\textcircled{14}"'] & \mu' \arrow[r, "\textcircled{15}"'] & \nu' \arrow[r, "\textcircled{16}"'] & {[\delta_m/y]\varepsilon_n}.
\end{tikzcd}$$

-We have ⑤ = ⑨ ; ⑩ ; ⑪ and ⑥ = ⑭ ; ⑮ ; ⑯ and ⑩ is just the contraction of the sub-redex, π, in μ.

-<u>Define:</u> $\textcircled{$R_\varepsilon$} := \cup\{\textcircled{$\rho$} \mid \rho \in R_\varepsilon\}$.

-Now, $\pi \notin (\textcircled{R_ε} \cup \textcircled{R_δ})/$⑨, since ⑤ $\nearrow (R_\varepsilon \cup R_\delta)$.

-By construction, ⑮ is an inside-out complete relative reduction of $\{\pi\}/$ ⑫ .

-∴, we have $\{\pi\}/$ ⑫ $\cap (\textcircled{$R_\varepsilon$} \cup \textcircled{$R_\delta$})/$⑨ ; ⑫ $= \emptyset$, since otherwise there would be a redex in $\{\pi\}/$ ⑫ that was the residual of two distinct sub-redexes of μ relative to ⑫ , namely π and one from $(\textcircled{$R_\varepsilon$} \cup \textcircled{$R_\delta$})/$⑨ - ✗ to 6.4.2.

-Now, square A commutes, by 6.4.11, and so,

$$\{\pi\}/\textcircled{12} \cap (\textcircled{R_ε} \cup \textcircled{R_δ})/\textcircled{3};\textcircled{14} = \emptyset.$$

-Thus, no residual of a sub-redex of $(R_\varepsilon \cup R_\delta)$ relative to ③;⑭ is contracted during ⑮ .

-We can show this at each step of the construction of the diagram and so ③;⑥ $(R_\varepsilon \cup R_\delta)$ - i.e. ⑥ $\nearrow (R_\varepsilon \cup R_\delta)$.

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6.4.16:REMARK:-

So we have found one example of where inside-outness is preserved. In the next section, we shall use this in chasing property B.

6.5:Properties C,D and E:-

6.5.0:PROPERTY C:-

$$(\varepsilon \xrightarrow{\alpha,1\beta} \delta \rightsquigarrow \eta) \Rightarrow (\varepsilon \rightsquigarrow \eta') \wedge (\eta \overset{\sim}{\equiv} \eta').$$

6.5.1:LEMMA:-

Property C <=> Property B.

Proof:-

(<=) -Trivial, since $\varepsilon \xrightarrow{\alpha,\beta} \eta$.

(=>) -Let $\varepsilon \xrightarrow{\alpha,\beta} \delta$.

-Then, $\varepsilon = \varepsilon_0 \rightarrow \varepsilon_1 \rightarrow \ldots\ldots \rightarrow \varepsilon_{n-1} \rightarrow \varepsilon_n = \delta$, where $\varepsilon_i \xrightarrow{\alpha,1\beta} \varepsilon_{i+1}$.

-Now, $\varepsilon_{n-1} \rightsquigarrow \varepsilon_n$.

-So, $\varepsilon_{n-2} \rightsquigarrow \varepsilon_n^{\ 1}$, where $\varepsilon_n \overset{\sim}{\equiv} \varepsilon_n^{\ 1}$, by property C.

-And, $\varepsilon_{n-3} \rightsquigarrow \varepsilon_n^{\ 2}$, where $\varepsilon_n^{\ 1} \overset{\sim}{\equiv} \varepsilon_n^{\ 2}$, by property C.

-Hence, the result, by repeated use of property C.

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6.5.2:PROPERTY D:-

$$([\delta/x]\varepsilon \rightsquigarrow \eta) \Rightarrow (\delta \rightsquigarrow \delta') \wedge (\varepsilon \rightsquigarrow \varepsilon')$$
$$\wedge([\delta'/x]\varepsilon' \rightsquigarrow \overset{\delta'}{\nearrow} \rightsquigarrow \overset{\varepsilon'}{\nearrow} \rightsquigarrow \eta') \wedge (\eta \overset{\sim}{\equiv} \eta').$$

6.5.3:LEMMA:-

Property D <=> Property C.

Proof:-

(<=) -Let $[\delta/x]\varepsilon \rightsquigarrow \eta \in$ HNF - otherwise trivial ($\eta' = [\delta/x]\varepsilon$).

-Then, $(\lambda x.\varepsilon)(\delta) \xrightarrow{\alpha,1\beta} [\delta/x]\varepsilon \rightsquigarrow \eta$.

-$\therefore$, $(\lambda x.\varepsilon)(\delta) \rightsquigarrow \eta'$, where $\eta \overset{\sim}{\equiv} \eta'$, by property C.

-∴, $\lambda x.\varepsilon \rightsquigarrow \lambda x.\varepsilon'$, $\delta \rightsquigarrow \delta'$ and $(\lambda x.\varepsilon')(\delta') \overset{\varepsilon' \quad \delta'}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta'$, by 6.1.9(i). But, $(\eta \in \mathrm{HNF}) \Rightarrow (\eta' \in \mathrm{HNF}) \Rightarrow (\eta' \neq (\lambda x.\varepsilon')(\delta'))$.

-∴, $[\delta'/x]\varepsilon' \overset{\varepsilon' \quad \delta'}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta'$.

$(\Rightarrow)$ -Let $A[\![\varepsilon]\!]$ be such that $(\forall\varepsilon \in \mathrm{EXP})A[\![\varepsilon]\!] \equiv$ Property C.

-Clearly, $A[\![x]\!]$ and $(A[\![\varepsilon]\!] \Rightarrow A[\![\lambda x.\varepsilon]\!])$.

-<u>Claim:</u> $A[\![\mu]\!] \wedge A[\![\nu]\!] \Rightarrow A[\![\mu(\nu)]\!]$:-

-Let $\mu(\nu) \xrightarrow{\alpha,1\beta} \delta \rightsquigarrow \eta$.

-<u>Case 1:</u> $\mu = \lambda x.\mu'$ and $\delta = [\nu/x]\mu'$:-

-Then, $[\nu/x]\mu' \rightsquigarrow \eta$.

-By property D, $\nu \rightsquigarrow \nu''$, $\mu' \rightsquigarrow \mu''$, $\eta \stackrel{\sim}{\equiv} \eta''$ and $[\nu''/x]\mu'' \overset{\nu'' \quad \mu''}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta''$.

-∴, $\mu \rightsquigarrow \lambda x.\mu''$ and $(\lambda x.\mu'')(\nu'') \overset{\mu'' \quad \nu''}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta''$.

-∴, by 6.1.9(i), $\mu(\nu) \rightsquigarrow \eta''$.

-<u>Case 2:</u> $\delta = \mu'(\nu)$ and $\mu \xrightarrow{\alpha,1\beta} \mu'$:-

-Now, $\mu'(\nu) \rightsquigarrow \eta$.

-By 6.1.9(i), $\mu' \rightsquigarrow \mu''$, $\nu \rightsquigarrow \nu''$ and $\mu''(\nu'') \overset{\mu'' \quad \nu''}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta$.

-But, $\mu \xrightarrow{\alpha,1\beta} \mu' \rightsquigarrow \mu''$.

-∴, $\mu \rightsquigarrow \mu'''$ and $\mu'' \stackrel{\sim}{\equiv} \mu'''$, by $A[\![\mu]\!]$.

-But, $\mu'''(\nu'') \overset{\mu''' \quad \nu''}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta'$, where $\eta \stackrel{\sim}{\equiv} \eta'$, by the weak parallel moves theorem, 6.3.7.

-So, $\mu(\nu) \rightsquigarrow \eta'$, by 6.1.9(i).

-<u>Case 3:</u> $\delta = \mu(\nu')$ and $\nu \xrightarrow{\alpha,1\beta} \nu'$:-

-Same as case 2, using $A[\![\nu]\!]$ instead of $A[\![\mu]\!]$.

-Hence, claim is established.

-∴, by structural induction, property C.

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<u>6.5.4:PROPERTY E:-</u>

$$([\delta/x]\omega(\varepsilon) \overset{\omega \quad \varepsilon \quad \delta}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta) \Rightarrow (\omega(\varepsilon) \overset{\omega \quad \varepsilon}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \rho) \wedge (\eta \stackrel{\sim}{\equiv} \eta') \wedge ([\delta/x]\rho \overset{\delta \quad \rho}{\rightsquigarrow\nearrow\rightsquigarrow\nearrow\rightsquigarrow} \eta').$$

6.5.5:LEMMA:-

Property E => Property D.

Proof:-

-Let $A\llbracket\varepsilon\rrbracket \equiv ([\delta/x]\chi(\varepsilon) \rightsquigarrow \eta) \Rightarrow (\delta \rightsquigarrow \delta') \wedge (\chi(\varepsilon) \rightsquigarrow \varepsilon')$
$\wedge([\delta'/x]\varepsilon' \overset{\delta' \quad \varepsilon'}{\rightsquigarrow} \eta') \wedge (\eta \overset{\sim}{\equiv} \eta')$.

-Clearly, $A\llbracket y\rrbracket$.

-Claim: $A\llbracket\varepsilon\rrbracket \Rightarrow A\llbracket\lambda y.\varepsilon\rrbracket$:-

-Let $[\delta/x]\chi(\lambda y.\varepsilon) \rightsquigarrow \eta$.

-Choose $z \neq x$, not in χ and not free in δ,ε.

-Then, $[\delta/x]\chi(\lambda y.\varepsilon) = \lambda z.[\delta/x]\chi[z/y]\varepsilon$, by 4.1.5(i).

-And, $[\delta/x]\chi[z/y]\varepsilon \rightsquigarrow \overline{\eta}$, where $\eta \xrightarrow{\alpha} \lambda z.\overline{\eta}$, by 6.1.9(ii)

-$\therefore$, by $A\llbracket\varepsilon\rrbracket$, $\delta \rightsquigarrow \delta'$, $\chi[z/y]\varepsilon \rightsquigarrow \varepsilon'$, $\overline{\eta} \overset{\sim}{\equiv} \overline{\eta}'$ and
$[\delta'/x]\varepsilon' \overset{\delta' \quad \varepsilon'}{\rightsquigarrow} \overline{\eta}'$.

-So, $\lambda z.[\delta'/x]\varepsilon' \overset{\delta' \quad \varepsilon'}{\rightsquigarrow} \lambda z.\overline{\eta}'$, by 6.1.9(ii).

-Now, since z cannot be free in δ', $\lambda z.[\delta'/x]\varepsilon' = [\delta'/x](\lambda z.\varepsilon'$

-Also, $\lambda z.\chi[z/y]\varepsilon \rightsquigarrow \lambda z.\varepsilon'$, by 6.1.9(ii), and $\lambda z.\chi[z/y]\varepsilon$
$= \chi(\lambda y.\varepsilon)$.

-Finally, $\eta \cong \lambda z.\overline{\eta} \overset{\sim}{\equiv} \lambda z.\overline{\eta}'$, and so we have $A\llbracket\lambda y.\varepsilon\rrbracket$.

-Claim: $A\llbracket\omega\rrbracket \wedge A\llbracket\varepsilon\rrbracket \Rightarrow A\llbracket\omega(\varepsilon)\rrbracket$:-

-Let $[\delta/x]\chi(\omega(\varepsilon)) \rightsquigarrow \eta$.

-Then, by 6.1.9(i), $[\delta/x]\chi(\omega) \rightsquigarrow \mu$, $[\delta/x]\chi(\varepsilon) \rightsquigarrow \nu$
and $\mu(\nu) \overset{\mu \quad \nu}{\rightsquigarrow} \eta$.

-$\therefore$, $\delta \rightsquigarrow \delta'$, $\chi(\omega) \rightsquigarrow \omega'$, $[\delta'/x]\omega' \overset{\delta' \quad \omega'}{\rightsquigarrow} \mu'$
and $\mu \overset{\sim}{\equiv} \mu'$, by $A\llbracket\omega\rrbracket$.

-$\therefore$, $\delta \rightsquigarrow \delta''$, $\chi(\varepsilon) \rightsquigarrow \varepsilon'$, $[\delta''/x]\varepsilon' \overset{\delta'' \quad \varepsilon'}{\rightsquigarrow} \nu'$
and $\nu \overset{\sim}{\equiv} \nu'$, by $A\llbracket\varepsilon\rrbracket$.

-Now, by the weak Church-Rosser theorem, 6.2.12, we have
$\delta \rightsquigarrow \delta'''$, where $\delta',\delta'' \overset{\sim}{\equiv} \delta'''$.

-Then, by the weak parallel moves theorem, 6.3.7, $\mu' \overset{\sim}{\equiv} \mu''$,
$\nu' \overset{\sim}{\equiv} \nu''$, $[\delta'''/x]\omega' \overset{\delta''' \quad \omega'}{\rightsquigarrow} \mu''$, $[\delta'''/x]\varepsilon' \overset{\delta''' \quad \varepsilon'}{\rightsquigarrow} \nu''$.

-Further, $\mu''(\nu'') \xrightarrow{\mu''\ \ \nu''} \eta'$, where $\eta \overset{\sim}{\equiv} \eta'$, again by 6.3.7.

-Next, by 6.4.15, $[\delta/x]\chi(\omega) \longrightarrow [\delta'''/x]\omega'$ and $[\delta/x]\chi(\varepsilon) \longrightarrow [\delta'''/x]\varepsilon'$.

-We have :-

$$\begin{tikzcd}
{[\delta/x]\chi(\omega(\varepsilon)) = (} & {[\delta/x]\chi(\omega))} \arrow[d, squiggly] & & {([\delta/x]\chi(\varepsilon))} \arrow[d, squiggly] \\
{[\delta'''/x]\omega'(\varepsilon') = (} & {[\delta'''/x]\omega')} \arrow[d, squiggly, "\delta'''\ \omega'"] & & {([\delta'''/x]\varepsilon')} \arrow[d, squiggly, "\delta'''\ \varepsilon'"] \\
 & \mu'' & (\arrow[d, squiggly, "\mu''\ \nu''"] & \nu''\ ) \\
 & & \eta'. &
\end{tikzcd}$$

-In particular, $[\delta'''/x]\omega'(\varepsilon') \xrightarrow{\delta'''\ \ \omega'\ \ \varepsilon'} \eta'$, by 6.1.9(i).

-∴, by property E, $\omega'(\varepsilon') \xrightarrow{\omega'\ \ \varepsilon'} \rho$, $\eta' \overset{\sim}{\equiv} \eta''$ and $[\delta'''/x]\rho \xrightarrow{\delta'''\ \ \rho} \eta''$.

-Thence, $\chi(\omega(\varepsilon)) \longrightarrow \rho$, by 6.1.9(i), and $\eta \overset{\sim}{\equiv} \eta''$, giving us $\mathcal{A}[\![\omega(\varepsilon)]\!]$.

-∴, by structural induction, $(\forall\varepsilon \in \mathrm{EXP})\mathcal{A}[\![\varepsilon]\!]$, and so property D.

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6.5.6:REMARK:-

Note that in the proof of 6.5.5 we used the weak Church-Rosser theorem. If we had a strong version, we could develop strong versions of properties C,D and E, relating them back to property A, the strong form of property B. This would be useful because we will eventually prove the strong version of property E.

Summing up the situation at present, it is :-

$\langle E_\infty, \mathcal{E}\rangle$ is a β-model of the λ-calculus

$$\begin{tikzcd}
\Leftrightarrow X \arrow[r, Leftrightarrow] & Z \arrow[r, Leftrightarrow] & M \arrow[r, Leftrightarrow] & B \arrow[r, Leftrightarrow] & C \arrow[r, Leftrightarrow] & D \\
 & Y \arrow[u, Rightarrow] & & A \arrow[u, Rightarrow] & & E \arrow[u, Rightarrow]
\end{tikzcd}$$

6.6:Strong Parallel Moves:-

6.6.0:NOTATION:-

Let R_ε and R_δ be sets of sub-redexes of ε and δ respectively. Let $(\lambda x.\varepsilon)(\delta) \xrightarrow{\textcircled{1}} [\delta/x]\varepsilon$. Then, we will allow ourselves the license of writing just R_ε and R_δ for $R_\varepsilon/\textcircled{1}$ and $R_\delta/\textcircled{1}$, when the context makes it clear to which expression the sub-redexes belong.

6.6.1:REMARK:-

Let $\varepsilon \overset{R_\varepsilon}{\rightsquigarrow} \varepsilon'$ and $\delta \overset{R_\delta}{\rightsquigarrow} \delta'$. Then, by 6.4.15, we can build up the diagram :-

$$\begin{tikzcd}
(\lambda y.\varepsilon)(\delta) \arrow[r, "\textcircled{1}"] \arrow[d, squiggly, "R_\varepsilon"'] \arrow[d, squiggly, "\textcircled{3}\ R_\delta"] & {[\delta/x]\varepsilon} \arrow[d, squiggly, "\textcircled{4}"] \\
(\lambda y.\varepsilon')(\delta') \arrow[r, "\textcircled{2}"] & {[\delta'/x]\varepsilon'.}
\end{tikzcd}$$

Note that ③;② is a complete relative (inside-out) reduction of $R_\varepsilon \cup R_\delta \cup \{(\lambda y.\varepsilon)(\delta)\}$.

By the way the diagram was constructed, ①;④ is also a complete relative reduction of $R_\varepsilon \cup R_\delta \cup \{(\lambda y.\varepsilon)(\delta)\}$, and so we can say that ④ is a complete relative (inside-out) reduction of $R_\varepsilon \cup R_\delta$, sub-redexes of $[\delta/x]\varepsilon$.

Thus, it makes sense to write,

$$[\delta/x]\varepsilon \overset{R_\delta \cup R_\varepsilon}{\rightsquigarrow} [\delta'/x]\varepsilon'.$$

6.6.2:DEF:-

$\mathcal{P}(n) \equiv (\varepsilon(\delta) \overset{\varepsilon\ \nearrow\ \delta\ \nearrow\ \textcircled{1}\ \leq n}{\rightsquigarrow} \eta) \wedge (\varepsilon \overset{R_\varepsilon}{\rightsquigarrow} \varepsilon') \wedge (\delta \overset{R_\delta}{\rightsquigarrow} \delta') \Rightarrow$ (we can construct the diagram :-

$$\begin{tikzcd}
\varepsilon(\delta) \arrow[r, squiggly, "\overset{\varepsilon}{\nearrow}\overset{\delta}{\nearrow}\ \textcircled{1}\ \leq n"] \arrow[d, squiggly, "R_\varepsilon \cup R_\delta"'] & \eta \arrow[d, squiggly, "(R_\varepsilon \cup R_\delta)/\textcircled{1}"] \\
\varepsilon'(\delta') \arrow[r, squiggly, "\overset{\varepsilon'}{\nearrow}\overset{\delta'}{\nearrow}"] & \eta',
\end{tikzcd}$$

such that it commutes).

$\mathcal{Q}(n) \equiv ([\delta/x]\varepsilon \xrightarrow{\overset{\delta}{\nearrow}\overset{\varepsilon}{\nearrow}\ \textcircled{2}\ \leq n} \eta) \wedge (\varepsilon \xrightarrow{R_\varepsilon} \varepsilon') \wedge (\delta \xrightarrow{R_\delta} \delta') \Rightarrow$ (we can construct the diagram :-

$$\begin{tikzcd}
{[\delta/x]\varepsilon} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\overset{\varepsilon}{\nearrow}\ \textcircled{2}\ \leq n"] \arrow[d, squiggly, "R_\delta \cup R_\varepsilon"'] & \eta \arrow[d, squiggly, "(R_\delta \cup R_\varepsilon)/\textcircled{2}"] \\
{[\delta'/x]\varepsilon'} \arrow[r, squiggly, "\overset{\delta'}{\nearrow}\overset{\varepsilon'}{\nearrow}"] & \eta',
\end{tikzcd}$$

such that it commutes).

__6.6.3:LEMMA:-__

(i) $\mathcal{P}(0) \wedge \mathcal{Q}(0)$.

(ii) $\mathcal{Q}(n-1) \Rightarrow \mathcal{P}(n)$, $\forall n \geq 1$.

__Proof:-__

(i) -Trivial, taking $\eta' = \varepsilon'(\delta')$, $[\delta'/x]\varepsilon'$ respectively.

(ii) -Let $(\varepsilon(\delta) \xrightarrow{\overset{\varepsilon}{\nearrow}\overset{\delta}{\nearrow}\ \textcircled{1}\ \leq n} \eta) \wedge (\varepsilon \xrightarrow{R_\varepsilon} \varepsilon')$
$\wedge (\delta \xrightarrow{R_\delta} \delta')$.

-If $\varepsilon(\delta)$ is not a redex, then ① is trivial - i.e. $\varepsilon(\delta) \overset{\alpha}{—} \eta$, and so the construction is trivial as in part (i).

-Suppose $\varepsilon = \lambda x.\bar{\varepsilon}$. Consider the diagram :-

$$\begin{tikzcd}
(\lambda x.\bar{\varepsilon})(\delta) \arrow[r, "\textcircled{4}"] \arrow[d, squiggly, "R_{\bar{\varepsilon}} \cup R_\delta"'] \arrow[dr, phantom, "A"] & {[\delta/x]\bar{\varepsilon}} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\overset{\bar{\varepsilon}}{\nearrow}\ \textcircled{3}\ \leq n-1"] \arrow[d, squiggly, "R_\delta \cup R_{\bar{\varepsilon}}"] \arrow[dr, phantom, "B"] & \eta \arrow[d, squiggly, "(R_\delta' \cup R_{\bar{\varepsilon}})/\textcircled{3}"] \\
(\lambda x.\bar{\varepsilon}')(\delta') \arrow[r, "\textcircled{6}"'] & {[\delta'/x]\bar{\varepsilon}'} \arrow[r, squiggly, "\overset{\delta'}{\nearrow}\overset{\bar{\varepsilon}'}{\nearrow}", "\textcircled{5}"'] & \eta'.
\end{tikzcd}$$

-We must have $(R_\varepsilon = R_{\bar{\varepsilon}}) \wedge (\bar{\varepsilon} \xrightarrow{R_{\bar{\varepsilon}}} \bar{\varepsilon}') \wedge (\lambda x.\bar{\varepsilon}' \xrightarrow{\alpha} \varepsilon')$.

-Also, ① = ④;③.

-Construct square A by 6.4.15 and remark 6.6.1.

-Construct square B by $\mathcal{Q}(n-1)$.

-But, both squares A and B commute, and so, by 6.4.9, the whole diagram commutes.

-Further, by 6.1.9(i), ⑥;⑤ is inside-out.

-Finally, $(R_\delta \cup R_{\bar{\varepsilon}})/\textcircled{3} = (R_\delta \cup R_{\bar{\varepsilon}})/\textcircled{4};\textcircled{3}$, by 6.6.0.

$$= (R_\varepsilon \cup R_\delta)/\textcircled{1}.$$

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6.6.4:DEF:-

$$A[\![\varepsilon]\!] \equiv ([\delta/x]\chi(\varepsilon) \xrightarrow{\nearrow^{\delta}\ \nearrow^{\chi(\varepsilon)}\ \textcircled{7}\ \leq n} \eta)$$

$\wedge(\chi(\varepsilon) \xrightarrow{R_{\chi(\varepsilon)}} \varepsilon') \wedge (\delta \xrightarrow{R_\delta} \delta') \Rightarrow$ (we can construct :-

$$\begin{tikzcd}
{[\delta/x]\chi(\varepsilon)} \arrow[r, squiggly, "\nearrow^{\delta}\ \nearrow^{\chi(\varepsilon)}\ \textcircled{7}\ \leq n"] \arrow[d, squiggly, "R_\delta \cup R_{\chi(\varepsilon)}"'] & \eta \arrow[d, squiggly, "(R_\delta \cup R_{\chi(\varepsilon)})/\textcircled{7}"] \\
{[\delta'/x]\varepsilon'} \arrow[r, squiggly, dashed, "\nearrow^{\delta'}\ \nearrow^{\varepsilon'}"] & \eta',
\end{tikzcd}$$

such that it commutes).

6.6.5:REMARK:-

As usual, n is fixed in $A[\![\varepsilon]\!]$ (along with ε), while δ, x, χ, $R_\delta, R_{\chi(\varepsilon)}$ and η are free. We have,

$$(\forall\varepsilon \in \text{EXP})A[\![\varepsilon]\!] \Rightarrow Q(n).$$

6.6.6:LEMMA:-

(i) $A[\![y]\!]$.

(ii) $A[\![\varepsilon]\!] \Rightarrow A[\![\lambda y.\varepsilon]\!]$.

Proof:-

(i) -Trivial, since ⑦ can only be α-conversions, since $[\delta/x]\chi(y)$ is either a variable or δ, whose sub-redexes are protected during ⑦.

(ii) -Let $([\delta/x]\chi(\lambda y.\varepsilon) \xrightarrow{\nearrow^{\delta}\ \nearrow^{\chi(\lambda y.\varepsilon)}\ \textcircled{8}\ \leq n} \eta)$

$\wedge(\delta \xrightarrow{R_\delta} \delta') \wedge (\chi(\lambda y.\varepsilon) \xrightarrow{R_{\chi(\lambda y.\varepsilon)}} \varepsilon')$.

-Choose $z \neq x$, not in χ and not free in δ, ε.

-Then, $[\delta/x]\chi(\lambda y.\varepsilon) = \lambda z.[\delta/x]\chi[z/y]\varepsilon$.

-Also, $\chi(\lambda y.\varepsilon) = \lambda z.\chi[z/y]\varepsilon$.

-Also, $\chi[z/y]\varepsilon \xrightarrow{R_{\chi[z/y]\varepsilon}} \bar{\varepsilon}'$, where $\varepsilon' \xrightarrow{\alpha} \lambda z.\bar{\varepsilon}'$ and $R_{\chi[z/y]\varepsilon} = R_{\chi(\lambda y.\varepsilon)}$.

-Also, $[\delta/x]\chi[z/y]\varepsilon \xrightarrow{\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\varepsilon}{\nearrow}\ \textcircled{9}\ \leq n} \bar{\eta}$, where $\eta \overset{\alpha}{\Rightarrow} \lambda z.\bar{\eta}$, by 6.1.9(ii).

-By $A[\![\varepsilon]\!]$, we have :-

$$\begin{tikzcd}
{[\delta/x]\chi[z/y]\varepsilon} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\varepsilon}{\nearrow}\ \textcircled{9}\ \leq n"] \arrow[d, dashed, "R_\delta \cup R_{\chi[z/y]\varepsilon}"'] & \bar{\eta} \arrow[d, dashed, "(R_\delta \cup R_{\chi[z/y]\varepsilon})/\textcircled{9}"'] \\
{[\delta'/x]\bar{\varepsilon}'} \arrow[r, squiggly, "\overset{\delta'}{\nearrow}\ \overset{\bar{\varepsilon}'}{\nearrow}"] & \bar{\eta}',
\end{tikzcd}$$

such that it commutes.

-Hence, by prefixing every expression with "λz.", we get :-

$$\begin{tikzcd}
{[\delta/x]\chi(\lambda y.\varepsilon)} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi(\lambda y.\varepsilon)}{\nearrow}\ \textcircled{8}\ n"] \arrow[d, dashed, "R_\delta \cup R_{\chi(\lambda y.\varepsilon)}"'] & \eta \arrow[d, dashed, "(R_\delta \cup R_{\chi(\lambda y.\varepsilon)})/\textcircled{8}"'] \\
{[\delta'/x]\varepsilon'} \arrow[r, squiggly, "\overset{\delta'}{\nearrow}\ \overset{\varepsilon'}{\nearrow}"] & (\lambda z.\bar{\eta}'),
\end{tikzcd}$$

such that it commutes.

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6.6.7:LEMMA:-

Suppose $P(n)$, where $n \geq 1$. Then,

$$A[\![\omega]\!] \wedge A[\![\varepsilon]\!] \Rightarrow A[\![\omega(\varepsilon)]\!].$$

Proof:-

-Let $([\delta/x]\chi(\omega(\varepsilon)) \xrightarrow{\overset{\delta}{\nearrow}\ \overset{\chi(\omega(\varepsilon))}{\nearrow}\ \textcircled{10}\ n} \eta) \wedge (\delta \xrightarrow{R_\delta} \delta') \wedge (\chi(\omega(\varepsilon)) \xrightarrow{R_{\chi(\omega(\varepsilon))}} \sigma)$.

-There are two cases to consider.

-Case 1: $\chi(\omega(\varepsilon)) \nmid R_{\chi(\omega(\varepsilon))}$:-

-We have $R_{\chi(\omega(\varepsilon))} = R_{\chi(\omega)} \cup R_{\chi(\varepsilon)}$, where $\chi(\omega) \xrightarrow{R_{\chi(\omega)}} \omega'$, $\chi(\varepsilon) \xrightarrow{R_{\chi(\varepsilon)}} \varepsilon'$ and $\omega'(\varepsilon') = \sigma$.

-Consider the following diagram :-

$$\begin{tikzcd}[column sep=small]
([\delta/x]\chi(\omega)) \arrow[d, "{\text{⑮}}", "{(R_\delta \cup R_{\chi(\omega)})}"'] & ([\delta/x]\chi(\varepsilon)) \arrow[d, "{(R_\delta \cup R_{\chi(\varepsilon)})}"'] \arrow[rr, squiggly, "{\nearrow\delta \;\; \nearrow\chi(\omega) \;\; \nearrow\chi(\varepsilon) \;\; \text{⑪} \;\; \leq n}"] & \text{A} \quad (R_\delta \cup R_{\chi(\omega)} \cup R_{\chi(\varepsilon)})/\text{⑪} & \mu(\nu) \arrow[d, shift right] \arrow[d, shift left, "\text{⑯}"] \\
([\delta'/x]\ \omega') & ([\delta'/x]\ \varepsilon') \arrow[rr, squiggly, "{\nearrow\delta' \;\; \nearrow\omega' \;\; \nearrow\varepsilon'}", "\text{⑬}"'] & & \mu'(\nu')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mu(\nu) \arrow[d, shift left] \arrow[d, shift right, "\text{⑯}"'] \arrow[rr, squiggly, "{\nearrow\mu \;\; \nearrow\nu \;\; \text{⑫} \;\; \leq n}"] & \text{B} & \eta \arrow[d, "\text{⑰}"] \\
\mu'(\nu') \arrow[rr, squiggly, "{\nearrow\mu' \;\; \nearrow\nu'}", "\text{⑭}"'] & & \eta'.
\end{tikzcd}$$

-By 6.1.9(i), we decompose ⑩ = ⑪ ; ⑫ .

-Then, by $A[\![\omega]\!]_\wedge A[\![\varepsilon]\!]$, we construct two squares in parallel to make square A.

-Square A commutes for $[\delta/x]\chi(\omega) \cup [\delta/x]\chi(\varepsilon)$. If $[\delta/x]\chi(\omega(\varepsilon))$ is a redex it is contracted neither during ⑪ ; ⑯ nor ⑮ ; ⑬ and it has the same unique residual both ways - namely $\mu'(\nu')$. Thus, square A commutes.

-If $\mu(\nu) = \eta$, we are finished.

-Otherwise, $\mu(\nu)$ is a redex which gets contracted during ⑫ .

-Then, $\chi(\omega(\varepsilon))$ is not a redex, since $\mu(\nu)$ would be its residual relative to ⑪ and ⑪ ; ⑫ $\nearrow\{\chi(\omega(\varepsilon))\}$ - ✗ .

-∴, $\chi(\omega) \notin \lambda I.EXP$.

-Further, $\chi(\omega) \in HEAD$, since otherwise $\chi(\omega) \in NOH\backslash\lambda I.NOH$ which implies $\mu \in NOH\backslash\lambda I.NOH$, by 6.1.11 - ✗ to $\mu(\nu)$ is a redex.

-Thus, we may conclude that $\omega' \in HEAD$ and so $\omega'(\varepsilon')$ is not a redex.

-Now, construct square B, by $P(n)$.It commutes and so the combined diagram A;B commutes, by 6.4.9.

-By 6.1.9(i), ⑬ ; ⑭ is inside-out.

-Clearly, ⑬ ; ⑭ $\nearrow\{\delta',\omega',\varepsilon'\}$, and so ⑬ ; ⑭ $\nearrow\{\delta',\omega'(\varepsilon')\}$.

-Finally, ⑰ is a complete relative reduction of $((R_\delta \cup R_{\chi(\omega)} \cup R_{\chi(\varepsilon)})/$ ⑪ $)/$ ⑫ $= (R_\delta \cup R_{\chi(\omega(\varepsilon))})/$ ⑩ .

-Thus, we have established $\mathcal{A}[\![\omega(\varepsilon)]\!]$ in this case.

-Case 2: $\chi(\omega(\varepsilon)) \in R_{\chi(\omega(\varepsilon))}$:-

-This time $\omega = \lambda y.\overline{\omega}$ and $R_{\chi(\omega(\varepsilon))} = R_{\chi(\omega)} \cup R_{\chi(\varepsilon)} \cup \{\chi(\omega(\varepsilon))\}$.

-Now, the external redex, $\chi(\omega(\varepsilon))$, must be the last one contracted in an inside-out complete relative reduction of $R_{\chi(\omega(\varepsilon))}$.

-Thus, $\chi(\omega(\varepsilon)) \xrightarrow[\sim\sim\sim]{R_{\chi(\omega(\varepsilon))}} \sigma$, must be of the form :-

$$(\chi(\omega))(\chi(\varepsilon)) \xrightarrow[\sim\sim\sim]{R_{\chi(\omega)} \cup R_{\chi(\varepsilon)}} \omega'(\varepsilon') \xrightarrow{1\beta} \sigma.$$

-Choosing $z \neq x$, not in χ and not free in $\delta,\overline{\omega}$, we can analyse this further into :-

$$(\lambda z.\chi[z/y]\overline{\omega})(\chi(\varepsilon)) \xrightarrow[\sim\sim\sim]{R_{\chi[z/y]\overline{\omega}} \cup R_{\chi(\varepsilon)}} (\lambda z.\overline{\omega}')(\varepsilon') \longrightarrow [\varepsilon'/z]\overline{\omega}',$$

where $R_{\chi[z/y]\overline{\omega}} = R_{\chi(\omega)}$ and $[\varepsilon'/z]\overline{\omega}' = \sigma$.

-Consider the following diagram :-

$$\begin{tikzcd}
 & & \eta \arrow[d, equal] \\
(\lambda z.[\delta/x]\chi[z/y]\overline{\omega})([\delta/x]\chi(\varepsilon)) \arrow[rr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\overline{\omega}}{\nearrow}\ \overset{\chi(\varepsilon)}{\nearrow}\quad ⑱"] \arrow[d, squiggly, "(R_\delta \cup R_{\chi[z/y]\overline{\omega}})"', "⑲\ \overset{R_\delta}{\cup} R_{\chi(\varepsilon)})"] & & (\lambda z.\overline{\mu})(\nu) \arrow[d, squiggly, shift right=4, "(R_\delta \cup R_{\chi(\omega)} \cup R_{\chi(\delta)})/⑱"'] \arrow[d, squiggly, shift left=4, "⑳"] \\
(\lambda z.[\delta'/x]\overline{\omega}')([\delta'/x]\varepsilon') \arrow[rr, squiggly, "\overset{\delta'}{\nearrow}\ \overset{\overline{\omega}'}{\nearrow}\ \overset{\varepsilon'}{\nearrow}\quad 21"] \arrow[d, "㉒"'] & & (\lambda z.\overline{\mu}')(\nu') \arrow[d, "㉓"] \\
{[[\delta'/x]\varepsilon'/z][\delta'/x]\overline{\omega}'} \arrow[rr, squiggly, "\overset{\delta'}{\nearrow}\ \overset{\overline{\omega}'}{\nearrow}\ \overset{\varepsilon'}{\nearrow}\quad ㉔"] \arrow[d, equal] & & {[\nu'/z](\overline{\mu}')}. \\
{[\delta'/x][\varepsilon'/z]\overline{\omega}'}, \text{ by 0.3.13(xii), since } z \text{ cannot be free in } \delta'. \arrow[d, equal] & & \\
{[\delta'/x]\sigma}. & &
\arrow[from=2-1, to=3-3, phantom, "C"]
\arrow[from=3-1, to=4-3, phantom, "D"]
\end{tikzcd}$$

-By 6.1.9(i) and (ii), ⑱ = ⑩ , since $(\lambda z.\overline{\mu})(\nu)$, being a residual of $\chi(\omega(\varepsilon))$, cannot be contracted.

-Square C is constructed by $\mathcal{A}[\![\omega]\!] \wedge \mathcal{A}[\![\varepsilon]\!]$, just the same way as square A in case 1 above - i.e. $\lambda z.\overline{\omega}' = \omega'$ and $\lambda z.\overline{\mu}' = \mu'$. Again, it commutes.

-Next, construct square D by 6.4.15 - it commutes and so, by 6.4.9, the whole diagram commutes. Further, ㉑ $\nearrow\{\delta',\overline{\omega}',\varepsilon'\}$, and so, by 6.4.15, ㉔ $\nearrow\{\delta',\overline{\omega}',\varepsilon'\}$. Notice that we are using our license of 6.6.0 at two levels here.

-According to the construction of 6.4.15, the sequence ㉔ is in two parts : the first being,

$$[[\delta'/x]\varepsilon'/z][\delta'/x]\overline{\omega}' \xrightarrow{\overset{\delta'}{\nearrow}\ \overset{\varepsilon'}{\nearrow}\ \overset{\overline{\omega}'}{\nearrow}} [\nu'/z][\delta'/x]\overline{\omega}',$$

and the second being,

$$[\nu'/z][\delta'/x]\overline{\omega}' \xrightarrow{\overset{\nu'}{\nearrow}\ \overset{\delta'}{\nearrow}\ \overset{\overline{\omega}'}{\nearrow}} [\nu'/z]\overline{\mu}'.$$

-In these two parts, it is clear that no residual in $[\varepsilon'/z]\overline{\omega}'$ - i.e. σ - is being contracted and so ㉔ $\nearrow\{\delta',\sigma\}$.

-By our analysis above, ⑲ ; ㉒ is a complete relative inside-out reduction of $(R_\delta \cup R_{\chi(\omega(\varepsilon))})$ from $[\delta/x]\chi(\omega(\varepsilon))$.

-But, ⑱ × ⑳ ; ㉓ is inside-out, by 6.1.9(i), and it is a complete relative reduction of $(R_\delta \cup R_{\chi(\omega(\varepsilon))})/$ ⑩ from η.

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6.6.8:THEOREM:-

$(\forall n \geq 0)(P(n) \wedge Q(n))$.

Proof:-

-$P(n) \Rightarrow (\forall\varepsilon \in \mathrm{EXP})A[\![\varepsilon]\!]$, by 6.6.6 and 6.6.7.

$\Rightarrow Q(n)$, $\forall n \geq 1$.

-Hence, the theorem, by 6.6.3 and above.

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6.6.9:COR:-

Let $(\varepsilon(\delta) \xrightarrow[\text{㉕}]{\overset{\varepsilon}{\nearrow}\ \overset{\delta}{\nearrow}} \eta) \wedge (\varepsilon \xrightarrow{\text{㉖}} \varepsilon') \wedge (\delta \xrightarrow{\text{㉗}} \delta')$.

Then, we may construct the diagram :-

$$\begin{tikzcd}
\varepsilon(\delta) \arrow[r, squiggly, "\overset{\varepsilon}{\nearrow}\overset{\delta}{\nearrow}\ \textcircled{25}"] \arrow[d, "\textcircled{28}"'] & \eta \arrow[d, "\textcircled{29}"] \\
\varepsilon'(\delta') \arrow[r, squiggly, "\overset{\varepsilon'}{\nearrow}\overset{\delta'}{\nearrow}\ \textcircled{30}"] & \eta',
\end{tikzcd}$$

such that it commutes.

Proof:-

-Let $\varepsilon = \varepsilon_0 \xrightarrow{A_1} \varepsilon_1 \xrightarrow{A_2} \varepsilon_2 \xrightarrow{A_3} \ldots\ldots\ldots \xrightarrow{A_n} \varepsilon_n = \varepsilon'$ be ㉖.

-Let $\delta = \delta_0 \xrightarrow{B_1} \delta_1 \xrightarrow{B_2} \delta_2 \xrightarrow{B_3} \ldots\ldots\ldots \xrightarrow{B_m} \delta_m = \delta'$ be ㉗.

-Suppose n > m (it does not really matter). Then, by repeated use of theorem 6.6.8, we may construct :-

$$\begin{tikzcd}
\varepsilon(\delta) = \varepsilon_0(\delta_0) \arrow[r, squiggly, "\overset{\varepsilon_0}{\nearrow}\overset{\delta_0}{\nearrow}\ \textcircled{25}"] \arrow[d, "\{A_1,B_1\}"'] & \eta_0 = \eta \arrow[d, "\{A_1,B_1\}/\textcircled{25}"] \\
\varepsilon_1(\delta_1) \arrow[r, squiggly, "\overset{\varepsilon_1}{\nearrow}\overset{\delta_1}{\nearrow}\ \boxed{25.1}"] \arrow[d, "\{A_2,B_2\}"'] & \eta_1 \arrow[d, "\{A_2,B_2\}/\boxed{25.1}"] \\
\vdots \arrow[r, phantom, "\vdots"] \arrow[d, "\{A_n\}"'] & \vdots \arrow[d, "\{A_n\}/\boxed{25.n}"] \\
\varepsilon'(\delta') = \varepsilon_n(\delta_m) \arrow[r, squiggly, "\overset{\varepsilon_n}{/}\overset{\delta_m}{/}\ \textcircled{30}"] & \eta_m = \eta'.
\end{tikzcd}$$

-Each individual square commutes, and so the whole diagram commutes, by 6.4.10.

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6.6.10:COR:-

Let $([\delta/x]\varepsilon \overset{\delta}{\nearrow}\overset{\varepsilon}{\nearrow}\rightsquigarrow \eta) \wedge (\delta \xrightarrow{\alpha,\beta} \delta') \wedge (\varepsilon \xrightarrow{\alpha,\beta} \varepsilon')$.

Then, we may construct the diagram :-

$$\begin{tikzcd}
{[\delta/x]\varepsilon} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\ \overset{\varepsilon}{\nearrow}"] \arrow[d] & \eta \arrow[d] \\
{[\delta'/x]\varepsilon'} \arrow[r, squiggly, "\overset{\delta'}{\nearrow}\ \overset{\varepsilon'}{\nearrow}"] & \eta',
\end{tikzcd}$$

such that it commutes.

Proof:-

-Similar to 6.6.9.

⫲

6.6.11:REMARK:-

These results will be very useful in studying inside-out reductions. Notice that they are "strong" results - i.e. independen of the concept of $\overset{\sim}{\equiv}$.

6.7:Strong Serial Moves:-

6.7.0:DEF:-

$P(n) \equiv ([\delta/x]\varepsilon \xrightarrow{\overset{\delta}{\nearrow}\ \overset{\varepsilon}{\nearrow}\ \text{①}\ \leq n} \eta \xrightarrow{R_\eta\ \text{②}} \hat{\eta})$

$\wedge$(① ; ② $\nearrow\{\delta,\varepsilon\}$) $\Rightarrow$ (we can construct :-

$$\begin{tikzcd}
{[\delta/x]\varepsilon} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\ \overset{\varepsilon}{\nearrow}\ \text{①}\ \leq n"] \arrow[drr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\varepsilon}{\nearrow}\ \text{③}"'] & \eta \arrow[r, squiggly, "R_\eta\ \text{②}"] & \hat{\eta} \arrow[d, "\text{④}"] \\
 & & \hat{\eta}',
\end{tikzcd}$$

such that it commutes).

6.7.1:LEMMA:-

$P(0)$.

Proof:-

-Take $\hat{\eta}' = \hat{\eta}$ and ③ = ②.

-Then, ③ $\nearrow\{\delta,\varepsilon\}$ because ②, = ① ; ②, $\nearrow\{\delta,\varepsilon\}$.

⫲

6.7.2:DEF:-

$$A[\![\varepsilon]\!] \equiv \left([\delta/x]\chi(\varepsilon) \xrightarrow[\leadsto]{\overset{\delta}{\nearrow}\ \overset{\chi(\varepsilon)}{\nearrow}\ \textcircled{5}\ \leq n} \eta \xrightarrow[\leadsto]{R_\eta\ \textcircled{6}} \hat{\eta}\right)$$

$\wedge(\textcircled{5};\textcircled{6}\nearrow\{\delta,\chi(\varepsilon)\}) \Rightarrow$ (we can construct :-

$$\begin{tikzcd}
{[\delta/x]\chi(\varepsilon)} \arrow[rr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi(\varepsilon)}{\nearrow}\ \textcircled{5}\ \leq n"] \arrow[drrr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi(\varepsilon)}{\nearrow}", "\textcircled{7}"'] & & \eta \arrow[r, squiggly, "R_\eta\ \textcircled{6}"] & \hat{\eta} \arrow[d, "\textcircled{8}"] \\
 & & & \hat{\eta}'
\end{tikzcd}$$

such that it commutes).

6.7.3:REMARK:-

Again, as in 6.6.5, $A[\![\varepsilon]\!]$ should really be written $A[\![\varepsilon,n]\!]$. We have,

$$(\forall\varepsilon \in \text{EXP})A[\![\varepsilon]\!] \Rightarrow P(n).$$

6.7.4:LEMMA:-

(i) $A[\![y]\!]$.

(ii) $A[\![\varepsilon]\!] \Rightarrow A[\![\lambda y.\varepsilon]\!]$.

Proof:-

(i) -Similar to 6.7.1, since $\textcircled{5}$ is a trivial sequence.

(ii) -Let $[\delta/x]\chi(\lambda y.\varepsilon) \xrightarrow[\leadsto]{\overset{\delta}{\nearrow}\ \overset{\chi(\lambda y.\varepsilon)}{\nearrow}\ \textcircled{9}\ \leq n} \eta \xrightarrow[\leadsto]{R_\eta\ \textcircled{10}} \hat{\eta}$.

-Then, choosing $z \neq x$, not in χ and not free in δ,ε, we have,

$$[\delta/x]\chi[z/y]\varepsilon \xrightarrow[\leadsto]{\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\varepsilon}{\nearrow}\ \textcircled{11}\ \leq n} \bar{\eta} \xrightarrow[\leadsto]{R_{\bar{\eta}}\ \textcircled{12}} \hat{\bar{\eta}},$$

where $\eta \xrightarrow{\alpha} \lambda z.\bar{\eta}$, $\hat{\eta} \xrightarrow{\alpha} \lambda z.\hat{\bar{\eta}}$ and $R_\eta = R_{\bar{\eta}}$, by 4.1.5(i) and 6.1.9(ii).

-∴, by $A[\![\varepsilon]\!]$, we construct :-

$$\begin{tikzcd}
{[\delta/x]\chi[z/y]\varepsilon} \arrow[rr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\varepsilon}{\nearrow}\ \textcircled{11}\ \leq n"] \arrow[drrr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\chi[z/y]\varepsilon}{\nearrow}", "\textcircled{13}"'] & & \bar{\eta} \arrow[r, squiggly, "R_{\bar{\eta}}\ \textcircled{12}"] & \hat{\bar{\eta}} \arrow[d, "\textcircled{14}"] \\
 & & & \hat{\bar{\eta}}'
\end{tikzcd}$$

such that it commutes. Prefixing with "$\lambda z.$", we get $A[\![\lambda y.\varepsilon]\!]$.

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6.7.5:LEMMA:-

Suppose $P(n-1)$, for some $n \geq 1$. Then,

$$A[\![\omega]\!] \wedge A[\![\varepsilon]\!] \Rightarrow A[\![\omega(\varepsilon)]\!].$$

Proof:-

-Let $([\delta/x]\chi(\omega(\varepsilon)) \xrightarrow{\overset{\delta}{\nearrow}\ \overset{\chi(\omega(\varepsilon))}{\nearrow}\ \text{⑮}\ \leq n} \eta \xrightarrow{R_\eta\ \text{⑯}} \hat{\eta})$ and (⑮ ; ⑯ $\nearrow\{\delta, \chi(\omega(\varepsilon))\}$).

-By lemma 6.1.9(i), sequence ⑮ is of the form :-

$$([\delta/x]\chi(\omega))([\delta/x]\chi(\varepsilon)) \xrightarrow[\text{⑰}]{\overset{\delta}{\nearrow}\ \overset{\chi(\omega)}{\nearrow}\ \overset{\chi(\varepsilon)}{\nearrow}\ \leq n} \mu(\nu) \xrightarrow[\text{⑱}]{\overset{\mu}{\nearrow}\ \overset{\nu}{\nearrow}\ \leq n} \eta.$$

-Case 1: $(\mu(\nu) \neq \eta)$:-

-Consider diagram X below.

-We will construct the sequences in numerical order.

-We must have $\mu = \lambda y.\mu^*$ and ⑱ = ⑲ ; ⑳ .

-Thus, ⑮ = ⑰ ; ⑲ ; ⑳ .

-Note that $\chi(\omega(\varepsilon))$ cannot be a redex, since otherwise $(\lambda y.\mu^*)(\nu)$ would be its residual and ⑮ $\nearrow\{\chi(\omega(\varepsilon))\}$ - ✗ .

-We partition R_η into two disjoint sets :-

$$R_\eta = R_\eta^{+} \cup R_\eta^{-},$$

where $R_\eta^{+} = \{\rho \in R_\eta \mid \rho$ has an ancestor sub-redex of ν or $\mu^*\}$ and $R_\eta^{-} = R_\eta \backslash R_\eta^{+}$.

-Let $R_{\mu^*} = \{\rho \mid \rho$ is a sub-redex of μ^* and $\{\rho\}/$ ⑲ ; ⑳ $\cap R_\eta^{+} \neq \emptyset\}$ and $R_\nu = \{\rho \mid \rho$ is a sub-redex of ν and $\{\rho\}/$ ⑲ ; ⑳ $\cap R_\eta^{+} \neq \emptyset\}$.

-Then, $R_\eta^{+} \subseteq (R_\nu \cup R_{\mu^*})/$ ⑲ ; ⑳ .

-Now, ⑯ is a complete relative reduction of R_η, and so we may do a complete reduction of R_η^{-} first, ㉑ , and then the rest, by 0.4.13. Further, triangle G commutes, by 6.4.6(i).

-But, ⑳ ; ㉑ $\nearrow\{\nu, \mu^*\}$ and so, by $P(n-1)$, we construct ㉓ and ㉔ so that trapezium E commutes.

-Next, we construct ㉕ and ㉖ , by 6.4.11, so that trapezium F commutes.

-By 6.1.9(i), ⑲ ; ㉓ is inside-out and ⑲ ; ㉓ $\nearrow\{\lambda y.\mu^*, \nu\}$.

So, by the strong parallel moves theorem 6.6.8, we may construct ㉗ , ㉘ and ㉙ , again so that trapezium C commutes.

-Now, ㉙ is a complete relative reduction of :-

$(R_{\mu *} \cup R_\nu)/$ ⑲ ; ㉓ = $(R_{\mu *} \cup R_\nu)/$ ⑲ ; ⑳ ; ㉑ ; ㉔ ,

since E commutes and by 6.4.8.

-But, ㉖ is a complete relative reduction of :-

$R_\eta^{\,+}/$ ㉑ ; ㉔ $\subseteq (R_{\mu *} \cup R_\nu)/$ ⑲ ; ⑳ ; ㉑ ; ㉔ , by above.

-Hence, we may construct ㉚ so that triangle D commutes, by 6.4.6(i). (See remark 6.7.6.)

-Next, if there were a residual of a sub-redex of $\chi(\omega)$ or δ in $R_{\mu *}$, then there would be a residual of a subredex of $\chi(\omega)$ or δ in $R_\eta^{\,+} \subseteq R_\eta - \cancel{X}$, since ⑮ ; ⑯ $\nearrow\{\delta,\chi(\omega)\}$. So, $R_{\mu *}$ contains no residual sub-redexes of $\chi(\omega)$ or δ and, similarly, R_ν contains no residual sub-redexes of $\chi(\varepsilon)$ or δ - i.e. ⑰ ; ㉗ $\nearrow\{\delta,\chi(\omega),\chi(\varepsilon)\}$.

-Thus, by $\mathcal{A}[\![\omega]\!] \wedge \mathcal{A}[\![\varepsilon]\!]$, we may construct ㉛ and ㉜ so that square A commutes for $\boxed{[\delta/x]\chi(\omega)} \cup \boxed{[\delta/x]\chi(\varepsilon)}$. ∴, square A commutes, since if $[\delta/x]\chi(\omega(\varepsilon))$ is a redex, its residual relative to ㉛ or ⑰ ; ㉗ ; ㉜ is $(\lambda y.\bar{\mu}'{*})(\bar{\nu}')$ both ways.

-Finally, by the strong parallel moves corollary 6.6.9, we construct ㉝ and ㉞ so that square B commutes.

-We just have to check that the whole diagram commutes. Let ρ be a sub-redex of $[\delta/x]\chi(\omega(\varepsilon))$. Then,

$\{\rho\}/$ ㉛ ; ㉝ = $\{\rho\}/$ ⑰ ; ㉗ ; ㉜ ; ㉝
= $\{\rho\}/$ ⑰ ; ㉗ ; ㉘ ; ㉞
= $\{\rho\}/$ ⑰ ; ⑲ ; ㉓ ; ㉙ ; ㉞
= $\{\rho\}/$ ⑰ ; ⑲ ; ㉓ ; ㉖ ; ㉚ ; ㉞
= $\{\rho\}/$ ⑰ ; ⑲ ; ⑳ ; ㉑ ; ㉔ ; ㉖ ; ㉚ ; ㉞
= $\{\rho\}/$ ⑰ ; ⑲ ; ⑳ ; ㉑ ; ㉒ ; ㉕ ; ㉚ ; ㉞
= $\{\rho\}/$ ⑰ ; ⑲ ; ⑳ ; ⑯ ; ㉕ ; ㉚ ; ㉞ ,

using 6.4.8, since each sub-diagram commutes. ∴, diagram X commutes.

$$\begin{tikzcd}
([\delta/x]\chi(\omega)) \arrow[d, squiggly, "\delta\ \ \chi(\omega)"] & ([\delta/x]\chi(\varepsilon)) \arrow[r, squiggly, "\delta\ \ \chi(\omega)\ \ \chi(\varepsilon)\ \ \leq n", "⑰"'] \arrow[d, squiggly, "\delta\ \ \chi(\varepsilon)"] \arrow[l, phantom, "㉛"'] \arrow[dr, phantom, "A"] & (\lambda y.\mu^*) \arrow[d, squiggly, "R_{\mu^*}"'] \arrow[r, phantom, "㉗"'] & (\nu) \arrow[r, "⑲"'] \arrow[d, squiggly, "R_\nu"] & {[\nu/y]\mu^*} \arrow[rr, squiggly, "\nu\ \ \mu^*\ \ \leq n-1", "⑳"'] \arrow[d, squiggly, "\nu\ \ \mu^*", "㉓"'] \arrow[dl, phantom, "C"] \arrow[dr, phantom, "E"] & & \eta \arrow[dl, squiggly, "R_\eta^-"', "㉑"] \arrow[dd, squiggly, "R_\eta", "⑯"'] \\
(\lambda y.\bar\mu^{*\prime}) & (\bar\nu') \arrow[d, squiggly, "\bar\nu'\ \ \bar\mu^{*\prime}", "㉝"'] \arrow[dr, phantom, "B"] & (\lambda y.\bar\mu^*) \arrow[l, "㉜"'] & (\bar\nu) \arrow[d, squiggly, "\bar\nu\ \ \bar\mu^*", "㉘"'] & \bar\eta' \arrow[dl, "㉙"'] \arrow[d, squiggly, "R_\eta^+/㉑\ ;\ ㉔"', "㉖"] \arrow[dl, phantom, "D"', shift left=4] \arrow[dr, phantom, "F"] & \bar\eta \arrow[l, "㉔"'] \arrow[dr, squiggly, "R_\eta^+/㉑", "22"'] \arrow[r, phantom, "G"] & \\
& \hat\eta' & & \eta'' \arrow[ll, "㉞"'] & \eta' \arrow[l, squiggly, "㉚"'] & & \hat\eta \arrow[ll, "㉕"']
\end{tikzcd}$$

DIAGRAM X.

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6.7.8:REMARK:-

We shall use the following corollary of the above theorem to establish the "strong" property E. It combines the ideas of both serial and parallel moves, thus enabling us to move in an inside-out manner over "two dimensions".

6.7.9:COR:- (Strong Serial and Parallel Moves)

Let $([\nu/y]\mu^* \xrightarrow{\nu\nearrow\ \mu^*\nearrow\ \text{⑳}} \eta)$ and $(\eta \xrightarrow{R_\eta\ \text{⑯}} \hat{\eta})$. Then, there exist sets of sub-redexes R_{μ^*} and R_ν of μ^* and ν respectively, so that we may construct the following diagram :-

$$\begin{tikzcd}
 & & {[\nu/y]\mu^*} \arrow[rr, squiggly, "\nu\nearrow\ \mu^*\nearrow\ \text{⑳}"] \arrow[dll, "(R_{\mu^*} \cup R_\nu)\ \text{㉗}'"'] \arrow[d, squiggly, "\nu\nearrow\ \mu^*\nearrow"', "\text{㉓}"] & & \eta \arrow[dd, "\text{⑯}", "R_\eta"'] \\
{[\bar{\nu}/y]\bar{\mu}^*} \arrow[dd, squiggly, "\bar{\nu}\nearrow\ \bar{\mu}^*\nearrow", "\text{㉘}'"'] & C' & \bar{\eta}' \arrow[ddll, "\text{㉙}"] & EFG & \\
 & & & & \\
\eta'' & & & & \hat{\eta}, \arrow[llll, "\text{㉕}\ ;\ \text{㉚}"']
\end{tikzcd}$$

such that it commutes.

Proof:-

-Look at diagram X from the proof of 6.7.5.

-Define R_{μ^*} and R_ν as done there and construct the sub-diagram EFG similarly, using the full theorem 6.7.7 instead of $P(n-1)$.

-Also, construct the parallelogram C′ as done in 6.7.5, except that we use the $\mathcal{Q}(n)$ part of 6.6.8 instead of the $P(n)$. Thus, ㉗′ is the sequence parallel to ㉗ relative to the contraction of the external redex; ㉘′ is the "tail" of ㉘ after the first head reduction; while ㉙ is the same as in 6.7.5.

-Commutativity is established as in 6.7.5.

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6.8:Strong Property E:-

6.8.0:THEOREM:-

Let $([\delta/x]\omega(\varepsilon) \xrightarrow{\overset{\delta}{\nearrow}\;\overset{\omega}{\nearrow}\;\overset{\varepsilon}{\nearrow}} \eta)$. Then, $(\omega(\varepsilon) \xrightarrow{\overset{\omega}{\nearrow}\;\overset{\varepsilon}{\nearrow}} \rho)$ and we may construct the following diagram :-

$$\begin{tikzcd}
(\lambda x.\omega(\varepsilon))(\delta) \arrow[r] \arrow[d, squiggly, "\overset{\omega}{\nearrow}\;\overset{\varepsilon}{\nearrow}"] & {[\delta/x]\omega(\varepsilon)} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\;\overset{\omega}{\nearrow}\;\overset{\varepsilon}{\nearrow}"] \arrow[d] & \eta \arrow[d] \\
(\lambda x.\rho)(\delta) \arrow[r] & {[\delta/x]\rho} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\;\overset{\rho}{\nearrow}"] & \eta',
\end{tikzcd}$$

such that it commutes.

Proof:-

-If $\omega(\varepsilon)$ is not a redex, then the result is trivial with $\rho := \omega(\varepsilon)$ and $\eta' := \eta$.

-Suppose $\omega = \lambda y.\overline{\omega}$.

-Let $([\delta/x](\lambda y.\overline{\omega})(\varepsilon) \xrightarrow{\overset{\delta}{\nearrow}\;\overset{\overline{\omega}}{\nearrow}\;\overset{\varepsilon}{\nearrow}\;①} \eta)$.

-Then, for some $z \neq x$ and not free in $\delta, \overline{\omega}$:-

$$(\lambda z.[\delta/x][z/y]\overline{\omega})([\delta/x]\varepsilon) \xrightarrow{\overset{\delta}{\nearrow}\;\overset{[z/y]\overline{\omega}}{\nearrow}\qquad\overset{\varepsilon}{\nearrow}\;①} \eta.$$

-By lemma 6.1.9(i) and (ii), we have :-

$$[\delta/x][z/y]\overline{\omega} \xrightarrow{\overset{\delta}{\nearrow}\;\overset{[z/y]\overline{\omega}}{\nearrow}\quad \overline{\textcircled{2}}} \overline{\mu},$$

$$[\delta/x]\varepsilon \xrightarrow{\overset{\delta}{\nearrow}\;\overset{\varepsilon}{\nearrow}\quad ③} \nu$$

and $$(\lambda z.\overline{\mu})(\nu) \xrightarrow{\overset{\overline{\mu}}{\nearrow}\;\overset{\nu}{\nearrow}\quad ④} \eta.$$

-If ② is sequence $\overline{\textcircled{2}}$ prefixed by "λz.", we have ① = ⑤;④, where ⑤ := ② × ③. Note that ⑤ $\nearrow\{\delta, (\lambda y.\overline{\omega})(\varepsilon)\}$.

-Suppose sequence ④ is of the form :-

$$(\lambda z.\overline{\mu})(\nu) = \Psi_0 \xrightarrow{\rho_1} \Psi_1 \xrightarrow{\rho_2} \ldots\ldots \xrightarrow{\rho_n} \Psi_n = \eta \quad \ldots\ldots\ldots ④$$

-Let $\Phi_0 := (\lambda y.\bar{\omega})(\varepsilon)$.

-Let $P(i) \equiv$ (there exist $\Phi_i, \Psi'_i \in$ EXP and paths ⑦, ⑧, ⑨, ⑩ and ⑫ such that $\Phi_0 \xrightarrow{\nearrow^{\bar{\omega}}\ \nearrow^{\varepsilon}\ ⑩} \Phi_i$ and we can construct :-

$$\begin{tikzcd}
(\lambda x.\Phi_0)(\delta) \arrow[r, "⑪"] \arrow[d, squiggly, "⑩"', "\nearrow^{\delta}\nearrow^{\bar{\omega}}\nearrow^{\varepsilon}"] \arrow[rd, phantom, "A"] & {[\delta/x]\Phi_0} \arrow[rr, squiggly, "\nearrow^{\delta}\ \nearrow^{\Phi_0}\ ⑤"] \arrow[d, squiggly, "⑦"', "\nearrow^{\delta}\nearrow^{\bar{\omega}}\nearrow^{\varepsilon}"] \arrow[rrd, phantom, "B"] & & \Psi_0 \arrow[d, squiggly, "⑥"] \\
(\lambda x.\Phi_i)(\delta) \arrow[r, "⑫"] & {[\delta/x]\Phi_i} \arrow[r, dashed, squiggly, "\nearrow^{\delta}\ \nearrow^{\Phi_i}\ ⑨"] & \Psi'_i & \Psi_i, \arrow[l, "⑧"']
\end{tikzcd}$$

such that it commutes and where $\widehat{⑩}$ = ⑩ × (null) and square A is constructed as in 6.4.15).

-In the rest of this proof we will write $\widehat{ⓘ}$ for ⓘ × (null) .

-Clearly, we have $P(0)$, since we already have ⑤ and ⑪ .

-<u>Claim:</u> $P(1)$:-

-By 6.7.9, construct ⑦, ⑧ and ⑨ so that square B commutes.

-In this case, $R_\eta = \{\Psi_0\} = \{\rho_1\}$, $R_{\mu*} = \{\Phi_0\}$ and $R_\nu = \emptyset$ (see notation of 6.7.9).

-Then, ⑦ is just the contraction of the residual of the redex Φ_0 relative to ⑪ .

-Clearly, we can construct $\widehat{⑩}$ and ⑫ by 6.4.15 so that square A commutes. Hence, by 6.4.9, the whole diagram commutes.

-<u>Claim:</u> $P(i) \Rightarrow P(i+1)$, $\forall\ 1 \le i < n$:-

-Consider diagram Y.

-By $P(i)$, construct ⑦, ⑧, ⑨, $\widehat{⑩}$ and ⑫ so that the sub-diagram AB commutes.

-Now, ⑬ is just the contraction of the redex ρ_{i+1}.

-So, by 6.4.11, we may construct ⑭ and ⑮ so that square

C commutes.

-But, $\rho_{i+1} \nmid (\textcircled{\delta} \cup \textcircled{\overline{\omega}} \cup \textcircled{\varepsilon})/⑤;⑥$, since ⑤;⑥;⑬ is an initial part of ① and $①\nearrow\{\delta,\overline{\omega},\varepsilon\}$.

-∴, $\{\rho_{i+1}\}/⑧ \cap (\textcircled{\delta} \cup \textcircled{\overline{\omega}} \cup \textcircled{\varepsilon})/⑤;⑥;⑧ = \emptyset$, since otherwise there would be a redex in $\{\rho_{i+1}\}/⑧$ with two distinct ancestors in Ψ_i, namely ρ_{i+1} and one from $(\textcircled{\delta} \cup \textcircled{\overline{\omega}} \cup \textcircled{\varepsilon})/⑤;⑥$, which contradicts 6.4.2 - $\divideontimes$.

-Since AB commutes, we have, using 6.6.0,

$\{\rho_{i+1}\}/⑧ \cap (\textcircled{\delta} \cup \textcircled{\overline{\omega}} \cup \textcircled{\varepsilon})/⑦;⑨ = \emptyset$.

-Next, construct ⑯ , ⑰ , ⑱ and ⑲ , by 6.7.9, so that square D commutes.

-In this case, $R_\nu = \emptyset$, since $\{\rho_{i+1}\}/⑧$ contains no residuals of δ. Also, $R_{\mu*}$ contains no residuals of any sub-redexes of $\overline{\omega}$ or ε, since otherwise so would $\{\rho_{i+1}\}\nearrow⑧$ - $\divideontimes$. Thus, $⑩ ; ⑲ \nearrow\{\overline{\omega},\varepsilon\}$.

-Construct ⑳ and we have, by 6.4.15, that square E commutes.

-Now, sequence ⑩ is of the form :-

$$\Phi_0 \xrightarrow{(10.1)} \Phi_1 = [\varepsilon/y]\overline{\omega} \overset{\overset{\varepsilon}{\nearrow}\ \overset{\overline{\omega}}{\nearrow}\quad (10.2)}{\rightsquigarrow} \Phi_i \ \ldots\ldots\ \textcircled{10},$$

and $(10.2);⑲ /\{\overline{\omega},\varepsilon\}$.

-∴, by 6.7.7, we may construct :-

$$\begin{tikzcd}
{[\varepsilon/y]\overline{\omega}} \arrow[r, squiggly, "\overset{\varepsilon}{\nearrow}\ \overset{\overline{\omega}}{\nearrow}\ (10.2)"] \arrow[rrd, squiggly, "\overset{\varepsilon}{\nearrow}\ \overset{\overline{\omega}}{\nearrow}\ (21.2)"'] & \Phi_i \arrow[r, squiggly, "R_{\mu*}\ \textcircled{19}"] & \Phi_i' \arrow[d, "\textcircled{22}"] \\
 & & \Phi_{i+1}',
\end{tikzcd}$$

such that it commutes.

-Then, letting $㉑ = (10.1);(21.2)$ and constructing $\widehat{㉑}$ and $\widehat{㉒}$, we have that segment F commutes.

-Again, by 6.4.15, construct ㉓ and ㉔ so that square G commutes.

-Also, by 6.4.15, construct ㉕ so that the sub-diagram consisting of ⑪ , $\widehat{\text{㉑}}$, ㉔ and ㉕ commutes.

-Finally, by the strong parallel moves corallary 6.6.10, we construct ㉖ and ㉗ so that square H commutes.

-Now, we have constructed the sequences required for *P*(i+1), namely :-

$$\begin{tikzcd}
(\Phi_0 \arrow[r, squiggly, "\overset{\bar{\omega}}{\nearrow}\ \overset{\varepsilon}{\nearrow}\ \text{㉑}"] & \Phi_{i+1})
\end{tikzcd}$$

and

$$\begin{tikzcd}
(\lambda x.\Phi_0)(\delta) \arrow[r, "\text{⑪}"] \arrow[d, squiggly, "\overset{\delta}{\nearrow}\ \overset{\bar{\omega}}{\nearrow}\ \overset{\varepsilon}{\nearrow}", "\widehat{\text{㉑}}"'] & {[\delta/x]\Phi_0} \arrow[rr, squiggly, "\overset{\delta}{\nearrow}\ \overset{\Phi_0}{\nearrow}\ \text{⑤}"] \arrow[d, squiggly, "\overset{\delta}{\nearrow}\ \overset{\bar{\omega}}{\nearrow}\ \overset{\varepsilon}{\nearrow}", "\text{㉕}"'] \arrow[l, phantom, "A'"] & \arrow[d, phantom, "B'"] & \Psi_0 \arrow[d, squiggly, "\text{⑥}\,;\,\text{⑬}"'] \\
(\lambda x.\Phi_{i+1})(\delta) \arrow[r, "\text{㉔}"] & {[\delta/x]\Phi_{i+1}} \arrow[r, squiggly, "\overset{\delta}{\nearrow}\ \overset{\Phi_i}{\nearrow}", "\text{㉗}"'] & \Psi'_{i+1} & \Psi_{i+1} \arrow[l, "\text{⑭}\,;\,\text{⑱}\,;\,\text{㉖}"]
\end{tikzcd}$$

-Square A is constructed by 6.4.15, so all that remains is to check commutativity.

-Let π be a sub-redex of $(\lambda x.\Phi_0)(\delta)$. Then,

$\{\pi\}/$ ⑪ ; ㉕ ; ㉗ $= \{\pi\}/$ $\widehat{\text{㉑}}$; ㉔ ; ㉗

$= \{\pi\}/$ $\widehat{\text{⑩}}$; $\widehat{\text{⑲}}$; $\widehat{\text{㉒}}$; ㉔ ; ㉗

$= \{\pi\}/$ $\widehat{\text{⑩}}$; $\widehat{\text{⑲}}$; ⑳ ; ㉓ ; ㉗

$= \{\pi\}/$ $\widehat{\text{⑩}}$; ⑫ ; ⑯ ; ㉓ ; ㉗

$= \{\pi\}/$ ⑪ ; ⑦ ; ⑯ ; ㉓ ; ㉗

$= \{\pi\}/$ ⑪ ; ⑦ ; ⑯ ; ⑰ ; ㉖

$= \{\pi\}/$ ⑪ ; ⑦ ; ⑨ ; ⑮ ; ⑱ ; ㉖

$= \{\pi\}/$ ⑪ ; ⑤ ; ⑥ ; ⑧ ; ⑮ ; ⑱ ; ㉖

$= \{\pi\}/$ ⑪ ; ⑤ ; ⑥ ; ⑬ ; ⑭ ; ⑱ ; ㉖ ,

using 6.4.8, since each sub-diagram commutes.

-Hence, diagram Y commutes and we have *P*(i+1).

$$\begin{tikzcd}[column sep=large, row sep=large]
(\lambda x.\Phi_0)(\delta) \arrow[r, "⑪"] \arrow[d, squiggly, "\delta\ \overline{\omega}\ \varepsilon"', "\widehat{⑩}"] \arrow[ddd, squiggly, bend right=50, "\delta\ \overline{\omega}\ \varepsilon", "\widehat{㉑}"'] \arrow[rd, phantom, "A"] \arrow[ddd, phantom, bend right=25, "F"] & {[\delta/x]\Phi_0} \arrow[rrr, squiggly, "\delta\quad \Phi_0 = (\lambda y.\overline{\omega})(\varepsilon)", "⑤"'] \arrow[d, squiggly, "\delta\ \overline{\omega}\ \varepsilon", "⑦"'] \arrow[ddd, squiggly, bend left=50, "\delta\ \overline{\omega}\ \varepsilon", "㉕"'] \arrow[rrrd, phantom, "B"] & & & \Psi_0 \arrow[d, squiggly, "⑥"] \\
(\lambda x.\Phi_i)(\delta) \arrow[r, "⑫"] \arrow[d, squiggly, "R_{\mu *}", "\widehat{⑲}"'] \arrow[rd, phantom, "E"] & {[\delta/x]\Phi_i} \arrow[rr, squiggly, "\delta\quad \Phi_i", "⑨"'] \arrow[d, squiggly, "R_{\mu *}", "⑯"'] \arrow[rrd, phantom, "D"] & & \Psi'_i \arrow[d, "\{\rho_{i+1}\}/⑧"', "⑮"] \arrow[rd, phantom, "C"] & \Psi_i \arrow[l, "⑧"'] \arrow[d, "\{\rho_{i+1}\}", "⑬"'] \\
(\lambda x.\Phi'_i)(\delta) \arrow[r, "⑳"'] \arrow[d, "\widehat{㉒}"'] \arrow[rd, phantom, "G"] & {[\delta/x]\Phi'_i} \arrow[r, squiggly, "\delta\quad \Phi'_i", "⑰"'] \arrow[d, "㉓"'] \arrow[rd, phantom, "H"] & \Psi'''_{i+1} \arrow[d, "㉖"] & \Psi''_{i+1} \arrow[l, "⑱"] & \Psi_{i+1} \arrow[l, "⑭"] \\
(\lambda x.\Phi_{i+1})(\delta) \arrow[r, "㉔"'] & {[\delta/x]\Phi_{i+1}} \arrow[r, squiggly, "\delta\quad \Phi_{i+1}", "㉗"'] & \Psi_{i+1} & &
\end{tikzcd}$$

DIAGRAM Y.

-$\therefore$, by ordinary induction, we have $P(n)$. But, this is the theorem with $\rho := \Phi_n$ and $\eta' := \Psi'_n$, since $\Phi_0 = \omega(\varepsilon)$ and $\Psi_n = \eta$.

ǂ

6.8.1:COR:-

Properties E,D,C,B,M,X,Z and $\langle \mathrm{E}_\infty, E\rangle = \langle \mathrm{E}_\infty, V\rangle$ is a normal solveable β-model of the λ-K-β-calculus.

Proof:-

-Property E, by 6.8.0, since $\xrightarrow{\alpha,\beta} \Rightarrow \overset{\sim}{\sqsubseteq}$, by 4.4.9.

ǂ

6.8.2:COR:-

(i) If an expression has a normal form, then it can be reached by an inside-out reduction.

(ii) If an expression has a normal form, then it is an i'th reduction for a suitably large i.

(iii) If $\varepsilon \in$ SOL, then $\varepsilon \dashrightarrow \delta' \in$ HNF.

(iv) If $\varepsilon \in$ SOL, then $i\langle\varepsilon\rangle \in$ HNF, for a suitably large i.

Proof:-

(i) -Let $\varepsilon \xrightarrow{\alpha,\beta} \delta \in$ NF.

-Then, by the weak completeness of inside-out reductions,

$\varepsilon \dashrightarrow \delta'$ and $\delta \overset{\sim}{\sqsubseteq} \delta'$.

-But, since normal forms are maximal under $\overset{\sim}{\sqsubseteq}$ (4.8.6), $\delta' \xrightarrow{\alpha} \delta$.

(ii) -Same as part (i), using the weak completeness of i'th reductions.

(iii) -Let $\varepsilon \in$ SOL. Then, $\varepsilon \xrightarrow{\alpha,\beta} \delta \in$ HNF.

-Again, by property B, $\varepsilon \dashrightarrow \delta'$ and $\delta \overset{\sim}{\sqsubseteq} \delta'$.

-But, by the characterisation of HNF in 4.5.2(ii), $\delta' \in$ HNF.

(iv) -Same as part (iii), using property M.

ǂ

6.8.3:COR:-

$<\mathrm{EXP}/\S,[\S]> \;\leq\; <\mathrm{E}_\infty,\mathcal{E}> \;\leq\; <\mathrm{EXP}/\simeq,[\simeq]>$.

Proof:-

-$<\mathrm{EXP}/\underline{\mathrm{cnv}}_{\mathrm{V}}\mathrm{INSOL},[\underline{\mathrm{cnv}}_{\mathrm{V}}\mathrm{INSOL}]> \;\leq\; <\mathrm{E}_\infty,\mathcal{E}>$, by 4.6.12 and since $<\mathrm{E}_\infty,\mathcal{E}>$ is a model by 6.8.1.

-Hence, the first inequality, since $<\mathrm{E}_\infty,\mathcal{E}>$ is substitutive.

-Let ε not$(\underline{\mathrm{cnv}}_{\mathrm{V}}\sim)$ δ.

-*Either* ε has a normal form and δ does not (or vice-versa) : in this case, $\varepsilon \not\equiv \delta$, since $<\mathrm{E}_\infty,\mathcal{E}>$ is normal by 6.8.1 and 4.8.9.

-*Or* ε and δ both have normal form, but $\varepsilon \;\not\underline{\mathrm{cnv}}\; \delta$: in this case, if $\varepsilon \equiv \delta$, then $\varepsilon \xrightarrow{\beta} \varepsilon' \in \mathrm{NF}$ and, by 6.8.1, $\varepsilon' \equiv \delta$. Thus, by 4.8.8, $\delta \xrightarrow{\beta} \varepsilon'$ - i.e. $\varepsilon \;\underline{\mathrm{cnv}}\; \delta$ - $\divideontimes$. Hence, $\varepsilon \not\equiv \delta$.

-$\therefore$, $<\mathrm{E}_\infty,\mathcal{E}> \;\leq\; <\mathrm{EXP}/\underline{\mathrm{cnv}}_{\mathrm{V}}\sim,[\underline{\mathrm{cnv}}_{\mathrm{V}}\sim]>$, and so we get the second inequality, since, again, $<\mathrm{E}_\infty,\mathcal{E}>$ is substitutive.

ǂ

6.8.4:COR:-

(i) $(\varepsilon \in \mathrm{INSOL}) \Rightarrow (\forall\delta \in \mathrm{EXP})(\varepsilon << \delta)$, (see 0.7.9(vii))

(ii) $(\varepsilon \in \mathrm{INSOL})_{\wedge}(\phi\varepsilon$ has normal form$)$
$\Rightarrow$ (ϕ is a "constant" function). (see Barendregt[78])

Proof:-

(i) -We use the notation of 0.7.9 in this proof.

-Let $C[\varepsilon] \xrightarrow{\beta} \nu \in \mathrm{NF}$.

-Then, $\nu \equiv C[\varepsilon] \equiv C[\delta]$, since $<\mathrm{E}_\infty,\mathcal{E}>$ is a substitutive model.

-Hence, $C[\delta] \xrightarrow{\beta} \nu$, by 4.8.8.

-$\therefore$, $C[\varepsilon]\ \rho\ C[\delta]$, for all contexts $C[\ ]$, and, so, $\varepsilon << \delta$.

(ii) -By part (i), $\varepsilon << \delta$, for all $\delta \in \mathrm{EXP}$.

-$\therefore$, $\phi\varepsilon\ \rho\ \phi\delta$, for all $\delta \in \mathrm{EXP}$.

-$\therefore$, $\phi\delta$ reaches the same normal form as $\phi\varepsilon$ for all $\delta \in \mathrm{EXP}$.

ǂ

6.9:Alternative Proof:-

Whilst preparing this thesis, a proof of the strong completeness of inside-out reductions has come to my attention. The proof is due to J.J.Lévy - [66].

Briefly, his proof is as follows. He describes a version of Wadsworth's typed λ-calculus - [67] -, the difference being that you are not allowed to contract β-redexes of degree zero. Thus, typed β-normal forms are typed expressions whose only redexes have degree zero. Any innermost sequence of typed β-reductions leads to a typed normal form and this sequence is clearly (typed) inside-out. Further, the typed λ-calculus has a Church-Rosser property and so the typed normal form is unique up to typed α-conversions. So, if $E \longrightarrow D$ is a typed reduction, then there is a typed expression, D', which is the typed normal form of both E and D, such that :-

$$\begin{tikzcd} E \arrow[rr, "\text{typed}"] \arrow[dr, "\text{typed}"'] & & D \arrow[dl, "\text{typed}"] \\ & D' & \end{tikzcd}$$

(*)

Now, any ordinary reduction sequence can be simulated by a typed reduction - i.e. if $\varepsilon \longrightarrow \delta$ then there are typed expressions E and D such that $E \xrightarrow{\text{typed}} D$ and det(E) = ε and det(D) = δ, where det is just the function from typed expressions to ordinary expressions that removes the type assignments.

Further, any typed reduction has a corresponding "isomorphic" ordinary reduction - i.e. if $E \xrightarrow{\text{typed}} D$ then det(E) $\longrightarrow$ det(D) and properties of the reduction like "$\rightsquigarrow$" or "$\nearrow\{\gamma\}$" are preserved.

So, if $\varepsilon \longrightarrow \delta$, then $E \xrightarrow{\text{typed}} D$ where det(E) = ε and det(D) = δ. Then, we get the diagram (*) above together with the corresponding type-free diagram :-

$$\begin{tikzcd}
\varepsilon \arrow[rr] \arrow[dr, squiggly] & & \delta \arrow[dl, squiggly] \\
& \delta', &
\end{tikzcd}$$

where $\delta' = \underline{\mathrm{det}}(D')$. Hence, property A - inside-out reductions are strongly complete.

7:EPILOGUE.

7.0:Review of $<E_\infty, \mathcal{E}>$:-

E_∞ is the inverse-limit of a sequence of countable finite-dept lattices, $<E_i>_{i=0}^{\infty}$. It is therefore a complete continuous lattice which is countably based by the direct-limit, $\bigcup_{i=0}^{\infty} \phi_{i,\infty}(E_i)$. There is another inverse-limit structure, A_∞, constructed in parallel with E_∞, so that the following decompositions hold :-

$$\left\{\begin{array}{l} E_\infty \doteq \lambda I.E_\infty + A_\infty \\ A_\infty \doteq I + A_\infty(E_\infty) \end{array}\right\} \ldots\ldots (2.2.20).$$

Notice how this decomposition matches the context-free ~~descrip~~ description of λ-expressions in β-normal form :-

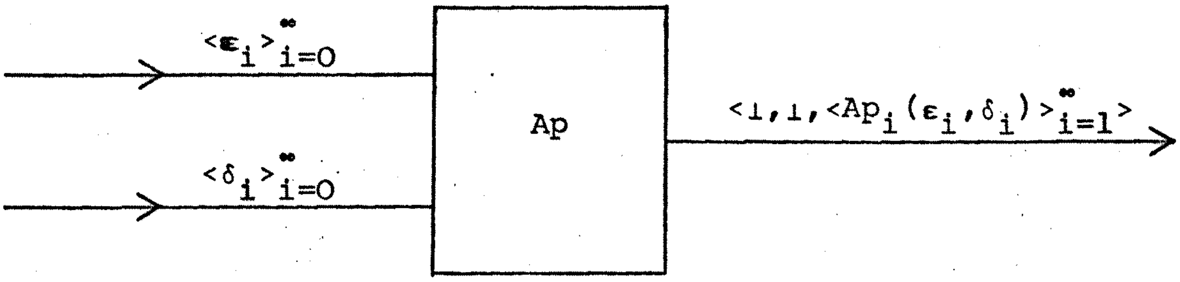
$$\left\{\begin{array}{l} NF ::= \lambda I.NF \mid HD \\ HD ::= I \mid HD(NF) \end{array}\right\} \ldots (0.4.3).$$

Indeed, each E_i contains finitely long syntactic objects which closely resemble λ-expressions in β-normal form, modulo $\xrightarrow{\alpha}$, except that they allow the symbols $\top$ and $\bot$ as well as ordinary variables. For this reason, we call E_∞ a syntactic lattice.

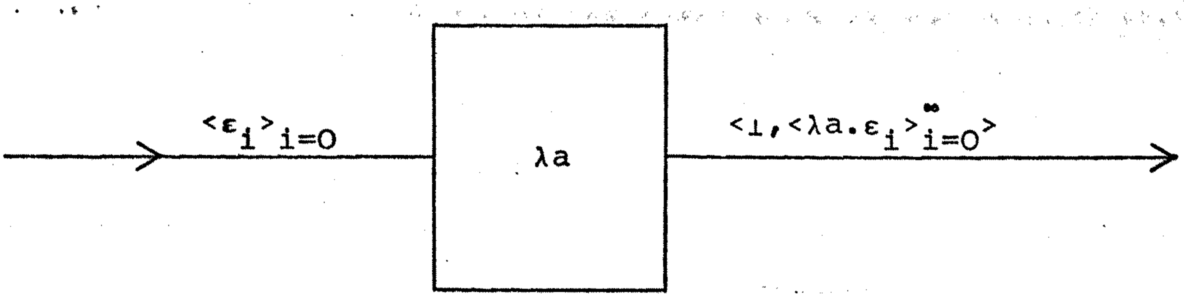
Using the rules of β-conversion as a guide, we define an application, Ap $\in [E_\infty \times E_\infty \to E_\infty]$, as the limit of i'th coordinate applications. This turns out to be continuous.

Then, using the decompositions of 2.2.20 to handle the variable and abstraction cases and Ap to handle the combinations, we find it very natural to define a semantic function, $\mathcal{E}$, from λ-expressions to E_∞. This definition is constructive in the sense that, given any λ-expression, we do not need to know anything about λ-calculus to compute its value in E_∞.

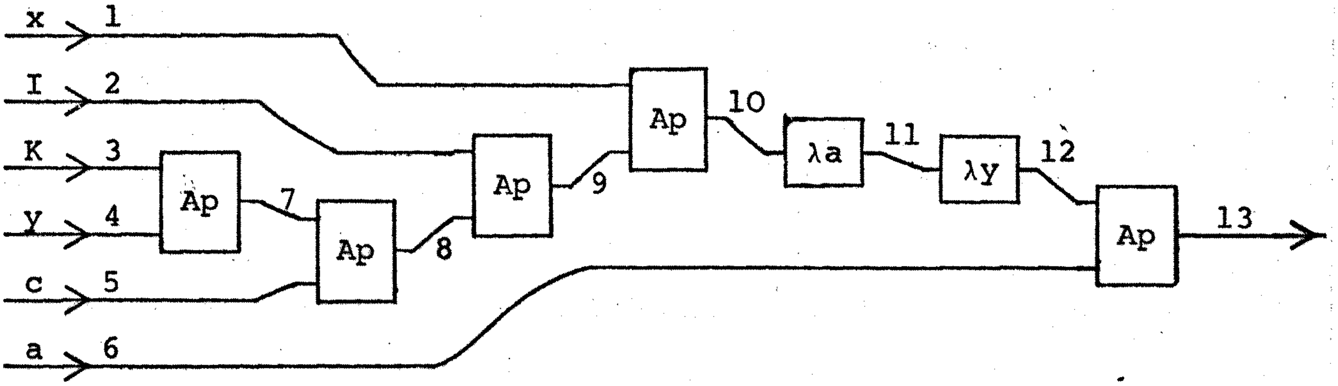
We can visualise the situation as follows. Let "Ap" be a black box with two input lines and one output. On each input line, it receives an infinite increasing sequence of approximate information about its arguments and it outputs a similar sequence of information about its answer :-



We also have various trivial boxes with labels like "λa" that have the following behaviour :-



Then, for example, to evaluate $\mathcal{E}[\![(\lambda y.\lambda a.x(I(Kyc)))(a)]\!]$, we set up the network :-



We break up the expression into its normal-form sub-expressions and give one input line for each chunk. The rest of the connections are dictated by the syntax of the expression. In the above example, we would push the following sequences into the input lines :-

line 1 carries <x,x,x,x,..................>,

line 2 carries <⊥,λc.c,λc.c,..............>,

line 3 carries <⊥,⊥,λp.λq.p,λp.λq.p,......>, line 4 carries <y,y,y,y,..................>,

line 5 carries <c,c,c,c,..................>

and line 6 carries <a,a,a,a,..................>.

The system then causes :-

line 7 to have <⊥,⊥,⊥,λq.y,λq.y,.............................>,

line 8 to have <⊥,⊥,⊥,⊥,y,y,y,y,...........................>,

line 9 to have <⊥,⊥,⊥,⊥,⊥,y,y,y,...........................>,

line 10 to have <⊥,⊥,x⊥,x⊥,x⊥,x⊥,xy,xy,.....................>,

line 11 to have <⊥,⊥,⊥,λa.x⊥,λa.x⊥,λa.x⊥,λa.x⊥,λa.xy,.......>,

line 12 to have <⊥,⊥,⊥,⊥,λy.λa.x⊥,λy.λa.x⊥,λy.λa.x⊥,
λy.λa.x⊥,λy.λa.xy,............>

and line 13 to have <⊥,⊥,⊥,⊥,⊥,λb.x⊥,λb.x⊥,λb.x⊥,λb.x⊥,λb.xa,...>

We see that the final output line, 13 , starts by producing undefined symbols, but then gives increasing amounts of information and finally settles on the normal form of the input expression.

But, does $<E_\infty,\mathcal{E}>$ always model β-conversion? Computing several examples encourages this speculation (4.0.3), and, in particular, we have I(ε) ≡ ε and K(ε)(δ) ≡ ε (4.2.4). However, we soon see that the reverse is not true, since expressions in INSOL - e.g. ΔΔ and ΔΔΔ - seem to be all mapped to ⊥;but ΔΔ $\underline{c\not{n}v}$ ΔΔΔ. Thus, the semantic given by $<E_\infty,\mathcal{E}>$ agrees with our intuition which lumps expressions like ΔΔ and ΔΔΔ together as "equally bad" in that "no useful information" can be extracted from them, no matter how much they are reduced.

7.1:The Y Combinator:-

The fixed-point combinators, $\{Y_i | i \geq 0\}$, form another counter-example to ≡ => $\underline{cnv}$. We know that Y_0 $\underline{c\not{n}v}$ Y_1 and yet they are all equivalent under ≡ (4.0.5).

Let us examine Y, = Y_0, further. In any decent semantics, it should have something to do with fixed-point operators. We may continuously map E_∞ into its continuous function space by "Currying" the application function - i.e. :-

$$Ap_C(\varepsilon)(\delta) := Ap(\varepsilon, \delta).$$

Thus, for any $\varepsilon \in E_\infty$, $Ap_C(\varepsilon)$ is a continuous function in $[E_\infty \to E_\infty]$. As such, it has a minimal fixed point $\mu Ap_C(\varepsilon)$, which we shall write as just $\mu\varepsilon$, and it is given by :-

$$\begin{aligned}\mu\varepsilon := \mu Ap_C(\varepsilon) &= \bigsqcup_{i=0}^{\infty} (Ap_C(\varepsilon))^i(\bot), \text{ by } 0.6.5(iv).\\ &= \bigsqcup_{i=0}^{\infty} Ap(\varepsilon, Ap(\varepsilon, \ldots\ldots, Ap(\varepsilon, \bot)\ldots\ldots))\\ &= \bigsqcup_{i=0}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\varepsilon_i, Ap_{i-1}(\varepsilon_{i-1}, \ldots\ldots, Ap_1(\varepsilon_1, \bot)\ldots\ldots)),\end{aligned}$$

by similar reasoning to 5.2.5.

On the other hand, if $\varepsilon = \mathcal{E}[\![\overline{\varepsilon}]\!]$, we also have :-

$$\begin{aligned}\mathcal{E}[\![Y(\overline{\varepsilon})]\!] &= Ap(<\bot, \bot, \bot, \lambda f.f(\bot), \lambda f.f^2(\bot), \ldots\ldots>, \varepsilon), \text{ by } 4.0.3.\\ &= \bigsqcup_{i=2}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\lambda f.f^{i-2}(\bot), \varepsilon_i)\\ &= \bigsqcup_{i=2}^{\infty} \phi_{i,\infty} \circ [\varepsilon_i/f]_{i-1} f^{i-2}(\bot)\\ &= \bigsqcup_{i=0}^{\infty} \phi_{i+1,\infty} \circ Ap_i(\varepsilon_i, Ap_{i-1}(\varepsilon_{i-1}, \ldots\ldots, Ap_1(\varepsilon_1, \bot)\ldots\ldots)), \text{ by}\end{aligned}$$

suitable juggling of the indices.

So, we have the following theorem relating Y and μ :-

$$\mathcal{E}[\![Y(\varepsilon)]\!] = \mu\mathcal{E}[\![\varepsilon]\!],$$

or, alternatively, the (Curry) $<E_\infty, \mathcal{E}>$-semantics of Y is the same as the minimal fixed-point operator :-

$$Ap_C \circ \mathcal{E}[\![Y]\!] \equiv \mu.$$

7.2:Review of $\tilde{\mathcal{E}}$ and $\mathcal{V}$:-

By using just the decompositions of 2.2.20, we define an "approximate" semantic function $\tilde{\mathcal{E}}$, $\sqsubseteq \mathcal{E}$. (In fact, we first defined an approximate application on E_∞, $\widetilde{Ap}$, which used these decompositions.)

An alternative method to have defined $\tilde{E}$, and one that is more in common with other approaches - see Wadsworth [68] and Lévy [69], is to define first of all a λ-Ω-calculus as is described in 0.7.17. Briefly, the symbol Ω is added to the list of variables, but it is not allowed to be bound. This Ω stands for the "undefined" symbol and so we naturally invent the ω-rules :-

$$\lambda x.\Omega \xrightarrow{\omega} \Omega$$

$$\text{and } \Omega(\delta) \xrightarrow{\omega} \Omega,$$

to agree with our intuition of "undefinedness". Then, E is extended to a semantic function E' on the λ-Ω-calculus by extending the definition in 4.0.1 with :-

$$(\text{SO}') \qquad E'[\![\Omega]\!] = \bot.$$

It had better be true that $\langle E_\infty, E'\rangle$ models the ω-rules and this is trivial to check. Next, we should define an approximation map, $\sim$: EXP → Ω-EXP, by :-

$$\tilde{x} \quad := x,$$

$$\widetilde{\lambda x.\varepsilon} := \lambda x.\tilde{\varepsilon}$$

$$\text{and } \widetilde{\varepsilon(\delta)} := \begin{Bmatrix} \Omega, \text{ if } \varepsilon \in \lambda I.\text{EXP} \\ \tilde{\varepsilon}(\tilde{\delta}), \text{ if not} \end{Bmatrix}.$$

Then, we define our approximate semantics as $\langle E_\infty, E'\circ\sim\rangle$. We claim, however, that this is the same as $\langle E_\infty, \tilde{E}\rangle$. Note that if $\varepsilon \in$ NOH, then $\tilde{\varepsilon} \xrightarrow{\omega} \Omega$, by a trivial induction on the formation of NOH. Recall the following characterisations of NOH, HNF and HEAD by $\tilde{E}$:-

$$\left\{ \begin{array}{lll} \varepsilon \in \text{NOH} & \iff & \tilde{E}[\![\varepsilon]\!] = \bot \\ \varepsilon \in \text{HNF} & \iff & \tilde{E}[\![\varepsilon]\!] \neq \bot \\ \varepsilon \in \text{HEAD} & \iff & \pi_2 \circ \tilde{E}[\![\varepsilon]\!] \neq \bot \end{array} \right\} \ldots\ldots (4.5.2)$$

Now, do a structural induction on $A[\![\varepsilon]\!] \equiv (\tilde{E}[\![\varepsilon]\!] = E'[\![\tilde{\varepsilon}]\!])$. The cases $A\llbracket x\rrbracket$ and $(A\llbracket \varepsilon\rrbracket \Rightarrow A\llbracket \lambda y.\varepsilon\rrbracket)$ are trivial. <u>Claim:</u> $(A\llbracket \omega\rrbracket \wedge A\llbracket \delta\rrbracket \Rightarrow A\llbracket \omega(\delta)\rrbracket)$:-

$$-\omega(\delta) \in \text{NOH} \Rightarrow \tilde{E}\llbracket \omega(\delta)\rrbracket = \bot = E'\llbracket \widetilde{\omega(\delta)}\rrbracket.$$

$$\begin{aligned} -\omega(\delta) \notin \text{NOH} \Rightarrow \omega \in \text{HEAD} &\Rightarrow \pi_2 \circ \tilde{E}\llbracket \omega\rrbracket \neq \bot \\ &\Rightarrow \tilde{E}\llbracket \omega(\delta)\rrbracket = \widetilde{\text{Ap}}(\tilde{E}\llbracket \omega\rrbracket, \tilde{E}\llbracket \delta\rrbracket) \\ &= \text{Ap}(\tilde{E}\llbracket \omega\rrbracket, \tilde{E}\llbracket \delta\rrbracket), \text{ by } 4.3.3(\text{xii}). \\ &= \text{Ap}(E'\llbracket \tilde{\omega}\rrbracket, E'\llbracket \tilde{\delta}\rrbracket), \text{ by } A\llbracket \omega\rrbracket \wedge A\llbracket \delta\rrbracket. \\ &= E'\llbracket \tilde{\omega}(\tilde{\delta})\rrbracket = E'\llbracket \widetilde{\omega(\delta)}\rrbracket. \end{aligned}$$

Thus, we have our claim that the two ways of defining approximate semantics are the same.

From $\tilde{E}$, we derive the substitutive relations $\tilde{\equiv}$ and $\tilde{\sqsubseteq}$. The relation $\tilde{\sqsubseteq}$ is of considerable use later and we give a purely syntactic characterisation (4.8.4). We see that $\xrightarrow{\beta} \Rightarrow \tilde{\sqsubseteq}$ (4.4.9) and deduce, using the Church-Rosser theorem, that the set of β-reductions of an expression form a directed set under $\tilde{\sqsubseteq}$.

Next, we define our alternative semantic function, V, as the (directed) limit of the approximate semantics of the β-reductions. By the equivalence described above, this is the same as Wadsworth's notion of the limit of the extended semantics of the approximated β-reductions - see Wadsworth [70]. We have :-

$$\begin{aligned} V\llbracket \varepsilon\rrbracket &:= \bigsqcup\{\tilde{E}\llbracket \delta\rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \delta\} \\ &= \bigsqcup\{E'\llbracket \tilde{\delta}\rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \delta\}. \end{aligned}$$

Now, V is not constructive in the way that E is. We need to know about λ-calculus in order to compute $V\llbracket \varepsilon\rrbracket$. Given the semantics of two expressions under V, we do not know how to combine them in E_∞ so as to remain consistent with V. Of course, the answer is to use Ap, but it is not easy to prove :-

$$V\llbracket \varepsilon(\delta)\rrbracket = \text{Ap}(V\llbracket \varepsilon\rrbracket, V\llbracket \delta\rrbracket).$$

For the same reason, we cannot easily deduce that the semantic equivalence induced by V is substitutive. However, it is easy to show that $\langle E_\infty, V\rangle$ is a solvable (4.6.2) model (4.6.1) in which :-

$$\varepsilon \in \text{SOL} \iff V[\![\varepsilon]\!] \neq \bot.$$

Further, we show that $E \sqsubseteq V$ (4.6.11) and, so, we can deduce that if $\varepsilon \in$ INSOL, then $E[\![\varepsilon]\!] = \bot$, which is a bonus for the $\langle E_\infty, E\rangle$-semantics.

Clearly, we want to show that $E \equiv V$, so that the (then coincident) semantics has both sets of properties. Initially, we can at least observe that they are the same on elements in normal form (4.7.4).

We investigate the relation $\prec$ in E_∞. We call elements in the direct limit finite. We find that if $\varepsilon \sqsubseteq \delta$ and ε is finite, then $\varepsilon \prec \delta$ (4.7.9). If we remain in the image of E, then the converse also is true (4.7.17). Also, finite and isolated are synonymous (4.7.14). Moreover, the $\langle E_\infty, E\rangle$-semantics of normal forms are isolated and this suggests that the semantics are normal. Anyway, we see that the $\langle E_\infty, V\rangle$-semantics are normal (4.8.9).

7.3:Review of I'th and Inside-Out Reductions:-

The motivation for introducing these reduction rules is to prove that the semantic functions E and V are the same. However, it is possible that inside-out reductions, in particular, will have applications elsewhere as a tool for analysing reductions in general (see 0.5.14).

I'th reductions define constructively a sequence of expressions which form a chain under $\xrightarrow{\alpha,\ \beta}$ (5.3.3). They were derived by examining the way in which E works and, so, it is not surprising that,

$$E[\![\varepsilon]\!] = \bigsqcup_{i=1}^{\infty} \tilde{E}[\![i\langle\varepsilon\rangle]\!] \ldots\ldots\ldots\ldots (5.3.10).$$

Now, i'th reductions are not strongly complete (5.3.12 and 14), but weak completeness is all that is needed to prove $E = V$ (5.3.13). The proof of weak completeness was still elusive and so i'th reductions were generalised to the non-constructive rule of inside-outness (6.2.1). Now, i'th reductions are not strongly complete with respect to inside-out reductions (6.1.4), but they do turn out to be weakly equivalent (6.2.6). Thus, the proof of $\mathcal{E} = \mathcal{V}$ now rests upon the weak completeness of inside-out reductions (property **B**), to which the rest of the chapter is devoted and where it is finally proved (6.8.1). In fact, Lévy has since proved that inside-out reductions are strongly complete (see section 6.9 and [71]). It is interesting to note that, therefore, inside-out reductions have the same completeness properties as standard ones : they are both strongly complete (5.3.14) and their "normal" versions - i.e. where we are not allowed to skip any redexes - are not even weakly complete, by 5.3.14 again and the fact that :-

$$(\lambda x.y)(\Delta\Delta) \longrightarrow y$$

but $(\lambda x.y)(\Delta\Delta) \xrightarrow{\text{"normal"}} (\lambda x.y)(\Delta\Delta)$ only

and $y \not\geq (\lambda x.y)(\Delta\Delta)$.

The weak completeness of inside-out reductions has an interpretation in ordinary programming terms. This is because of the following very nice property of inside-out reductions :

$$(C[\varepsilon] \longrightarrow \eta) \iff (\varepsilon \longrightarrow \varepsilon') \wedge (C[\varepsilon'] \xrightarrow{\overset{\varepsilon'}{\wedge}} \eta).$$

This is a simple generalisation of 6.1.9 and it is self-evident (simple induction on the complexity of contexts : (a) prove true for the trivial context [ ] and, then, (b) assuming true for C[ ], prove true for $\lambda x.C[\ ]$, $\delta(C[\ ])$ and $(C[\ ])(\delta)$, using 6.1.9). Really, it is just a restatement of the idea of inside-outness, namely that sub-expressions are evaluated first (though not neccessarily fully) and then their <u>values</u> are passed to the rest of the expression - i.e. no further computation on them is permitted. The weak completeness theorem, saying that we can always compute like this and get all the answers possible by any other mechanism, lets us instantly deduce the following (stronger) theorem of Wadsworth - [72] :-

"if $C[\varepsilon] \xrightarrow{\alpha,\beta \quad \text{①}} \eta$, then we can construct :-

$$\begin{tikzcd}
C[\varepsilon] \arrow[r, "\alpha{,}\beta \quad \text{①}"] \arrow[d, "\text{②}"'] & \eta \arrow[d, phantom, "\tilde{\sqsubseteq}\ \text{④}"] \\
C[\varepsilon'] \arrow[r, "\varepsilon' \quad \text{③}"] & \eta'
\end{tikzcd}$$

".

Of course, if η were in normal form, then $\eta = \eta'$ (6.8.2) and if $\eta \in$ SOL, then so is η'. We interpret $\eta \mathrel{\tilde{\sqsubseteq}} \eta'$ as "η' tells us more, but consistent, information than η". We note that since we have now got strong completeness (property A), then ④ can be replaced with an $\xrightarrow{\alpha,\beta}$: however, if all we are looking for is the "information content" of the reduced expressions, then $\tilde{\sqsubseteq}$ is a perfectly good measure.

7.4:Embedding of E_∞ in $P(\omega)$:-

7.4.0:REMARK:-

Since E_∞ is a complete continuous countably-based lattice, it is embedded as a retraction in $P(\omega)$ - see section 0.7.25. It would be interesting to see whether the $P(\omega)$-semantics obtained from this embedding was equivalent to the Plotkin/Scott $P(\omega)$-model described in the same section,i.e. whether $\langle E_\infty, \mathcal{E}\rangle \equiv \langle[ENV \to P(\omega)], \mathcal{P}\rangle$. We are inclined to think not, simply because of the very different images of λ-expressions in $P(\omega)$; for instance, finite λ-expressions (e.g. I) are represented by finite subsets in our interpretation (7.5.7), but by infinite subsets in theirs (0.7.26(vii)).

In this section, we give the embedding of E_∞ into the set of subsets of the countable basis, the direct limit. This can be transfered, trivially, to $P(\omega)$ by some ennumeration of that basis.

7.4.1:DEF:-

$$\mathrm{emb}_i \;:\; E_i \longrightarrow P(\bigsqcup_{i=0}^{\infty} \phi_{i,\infty}(E_i))$$

$$\varepsilon_i \longmapsto \{\phi_{i,\infty}(\varepsilon_i') \mid \varepsilon_i' \sqsubseteq \varepsilon_i\}.$$

$$\mathrm{emb} \;:\; E_\infty \longrightarrow P(\bigsqcup_{i=0}^{\infty} \phi_{i,\infty}(E_i))$$

$$\langle \varepsilon_i \rangle_{i=0}^{\infty} \longmapsto \bigcup_{i=0}^{\infty} \mathrm{emb}_i(\varepsilon_i).$$

7.4.2:LEMMA:-

(i) $\mathrm{emb}(\bot) = \{\phi_{0,\infty}(\bot_{E_0})\}$ and $\mathrm{emb}(\top) = \bigcup_{i=0}^{\infty} \phi_{i,\infty}(E_i)$.

(ii) emb_i and emb are continuous.

(iii) If $\langle \varepsilon_i \rangle_{i=0}^{\infty} \in E_\infty$, then $\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_{i+1}(\varepsilon_{i+1})$.

(iv) $\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_i(\delta_i) \Rightarrow \varepsilon_i \sqsubseteq \delta_i$.

(v) $\mathrm{emb}(\varepsilon) \subseteq \mathrm{emb}(\delta) \Rightarrow \varepsilon \sqsubseteq \delta$.

(vi) emb_i and emb are injective.

(vii) E_∞ and $\mathrm{emb}(E_\infty)$ are homeomorphic.

Proof:-

(i) -Clear.

(ii) -If $\varepsilon_i \sqsubseteq \delta_i$, then, clearly, $\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_i(\delta_i)$.

-Hence, emb_i is monotonic.

-Hence, emb_i is continuous, since E_i has finite depth.

-Hence, emb is continuous, since it is the limit of the continuous maps,

$$\mathrm{emb} = \bigcup_{i=0}^{\infty} \mathrm{emb}_i \circ \phi_{\infty,i}.$$

(iii) -Let $\langle \varepsilon_i \rangle_{i=0}^{\infty} \in E_\infty$.

-Let $\phi_{i,\infty}(\varepsilon_i') \in \mathrm{emb}_i(\varepsilon_i)$. Then, $\varepsilon_i' \sqsubseteq \varepsilon_i$.

-∴, $\phi_{i,i+1}(\varepsilon_i') \sqsubseteq \phi_{i,i+1}(\varepsilon_i) \sqsubseteq \varepsilon_{i+1}$.

-∴, $\phi_{i+1,\infty} \circ \phi_{i,i+1}(\varepsilon_i') \in \mathrm{emb}_{i+1}(\varepsilon_{i+1})$.

-i.e. $\phi_{i,\infty}(\varepsilon_i') \in \mathrm{emb}_{i+1}(\varepsilon_{i+1})$.

-∴, $\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_{i+1}(\varepsilon_{i+1})$.

(iv) -Let $\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_i(\delta_i)$.

-Now, $\phi_{i,\infty}(\varepsilon_i) \in \mathrm{emb}_i(\varepsilon_i)$.

-∴, $\phi_{i,\infty}(\varepsilon_i) = \phi_{i,\infty}(\delta_i')$, where $\delta_i' \sqsubseteq \delta_i$.

-∴, $\phi_{\infty,i} \circ \phi_{i,\infty}(\varepsilon_i) = \phi_{\infty,i} \circ \phi_{i,\infty}(\delta_i')$.

-∴, $\varepsilon_i = \delta_i'$ - i.e. $\varepsilon_i \sqsubseteq \delta_i$.

(v) -Let $\mathrm{emb}(\varepsilon) \subseteq \mathrm{emb}(\delta)$. Consider ε_i.

-Now, $\phi_{i,\infty}(\varepsilon_i) \in \mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}(\varepsilon)$.

-∴, $\phi_{i,\infty}(\varepsilon_i) = \phi_{j,\infty}(\delta_j')$, for some j and where $\delta_j' \sqsubseteq \delta_j$.

-∴, $\phi_{\infty,i} \circ \phi_{i,\infty}(\varepsilon_i) = \phi_{\infty,i} \circ \phi_{j,\infty}(\delta_j')$.

-∴, $\varepsilon_i = \phi_{j,i}(\delta_j')$.

-But, $\phi_{j,i}(\delta_j') \sqsubseteq \phi_{j,i}(\delta_j) \sqsubseteq \delta_i$.

-So, for all $i \geq 0$, $\varepsilon_i \sqsubseteq \delta_i$ - i.e. $\varepsilon \sqsubseteq \delta$.

(vi) -$\mathrm{emb}_i(\varepsilon_i) \subseteq \mathrm{emb}_i(\delta_i) \iff \varepsilon_i \sqsubseteq \delta_i$, by parts (ii) and (iv).

-∴, $\mathrm{emb}_i(\varepsilon_i) = \mathrm{emb}_i(\delta_i) \iff \varepsilon_i = \delta_i$.

-Similarly, $\mathrm{emb}(\varepsilon) \subseteq \mathrm{emb}(\delta) \iff \varepsilon \sqsubseteq \delta$, by parts (ii) and (v).

-∴, $\mathrm{emb}(\varepsilon) = \mathrm{emb}(\delta) \iff \varepsilon = \delta$.

-∴, emb_i and emb are injective.

(vii) -By parts (ii), (v) and (vi), emb is continuous and injective and emb^{-1} exists on $\mathrm{emb}(E_\infty)$ and is monotonic.

-Let X be directed in $\mathrm{emb}(E_\infty)$.

-Then, $\mathrm{emb}^{-1}(X)$ is directed in E_∞.and, so,

-But, $\bigsqcup \mathrm{emb}^{-1}(X) = \mathrm{emb}^{-1} \circ \mathrm{emb}(\bigsqcup \mathrm{emb}^{-1}(X))$

$$= \mathrm{emb}^{-1}(\cup \mathrm{emb} \circ \mathrm{emb}^{-1}(X)) = \mathrm{emb}^{-1}(\cup X).$$

-Hence, emb^{-1} is continuous and emb is a lattice homeomorphism.

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7.4.3: REMARK:-

Thus, we have an isomorphic image of E_∞ in $P(\omega)$ by $\gamma \circ \mathrm{emb}$, where γ is any ennumeration of the direct limit. To show that it is a retraction, we quote the following theorem of Scott - [73]. We also use it to generalise the application function on E_∞ to an application over the whole of $P(\omega)$.

7.4.4:THEOREM:- (Scott)

If D is a complete continuous lattice and $e : X \to Y$ is some subspace embedding, then for each $f \in [X \to D]$, there is an extension $\overline{f} \in [Y \to D]$ so that the triangle,

$$\begin{tikzcd}
X \arrow[rr, "e"] \arrow[dr, "f"'] & & Y \arrow[dl, "\overline{f}"] \\
& D &
\end{tikzcd}$$

commutes. Furthermore, the maximal extension in the partially ordered set $[Y \to D]$ is given by :-

$$\overline{f}(y) = \bigsqcup\{\bigsqcap\{f(x) \mid e(x) \in U\} \mid U \text{ is open and } y \in U\}.$$

Proof:-

-See Scott [73].

⊣

7.4.5:COR:-

E_∞ is a retraction of $P(\omega)$.

Proof:-

-We have $\gamma \circ emb : E_\infty \longrightarrow P(\omega)$ is a subspace embedding, by 7.4.2.

-∴, by 7.4.4, we can construct the continuous map ρ so that the following triangle commutes :-

$$\begin{tikzcd}
E_\infty \arrow[rr, "emb"] \arrow[dr, "identity"'] & & P(\omega) \arrow[dl, "\rho"] \\
& E_\infty &
\end{tikzcd}$$

-Hence, E_∞ is a retraction of $P(\omega)$.

⊣

7.4.6:COR:-

We can define a continuous application function, Ap^ω, on $P(\omega)$ so that it is well-behaved with respect to the λ-calculus model, $\langle P(\omega), \gamma \circ emb \circ \mathcal{E} \rangle$, in the sense that :-

$$Ap^{\omega}(\gamma\circ emb\circ \mathcal{E}[\![\varepsilon]\!], \gamma\circ emb\circ \mathcal{E}[\![\delta]\!]) = \gamma\circ emb\circ \mathcal{E}[\![\varepsilon(\delta)]\!].$$

Proof:-

-Clearly, $(\gamma\circ emb \times \gamma\circ emb) \in [E_\infty \times E_\infty \to P(\omega) \times P(\omega)]$ is a subspace embedding.

-$\therefore$, by 7.4.4, we may construct Ap^{ω} so that the following diagram commutes :-

$$\begin{tikzcd}
E_\infty \times E_\infty \arrow[rr, "(\gamma\circ emb \times \gamma\circ emb)"] \arrow[dr, "\gamma\circ emb\circ Ap"'] & & P(\omega) \times P(\omega) \arrow[dl, "Ap^{\omega}"] \\
& P(\omega) &
\end{tikzcd}$$

-$\therefore$, $Ap^{\omega}(\gamma\circ emb\circ \mathcal{E}[\![\varepsilon]\!], \gamma\circ emb\circ \mathcal{E}[\![\delta]\!]) = \gamma\circ emb\circ Ap(\mathcal{E}[\![\varepsilon]\!], \mathcal{E}[\![\delta]\!])$

$= \gamma\circ emb\circ \mathcal{E}[\![\varepsilon(\delta)]\!]$.

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7.5:Pruning the Lattice:-

7.5.0:REMARK:-

In this section, we remove the top half of the lattice E_∞, giving ourselves a directedly complete continuous and countably based sub-semi-lattice, $Low(E_\infty)$, which still contains the image of the semantic function $\mathcal{E}$. The need for the top half of the lattice is, therefore, debatable since it plays no part in the modelling of the λ-calculus (λ-calculus throws up no "inconsistencies" which would require $\top$ elements for their representation).

7.5.1:DEF:-

$Low(I^{\wedge}) := I^{\wedge}\backslash\{\top\}$.

$Low(E_{i+1}) := \{\lambda x.\, \varepsilon_i \mid x \in I \text{ and } \varepsilon_i \in Low(E_i)\}$
$\cup \{\alpha_i \mid \alpha_i \in Low(A_i)\}$.

$Low(A_{i+1}) := Low(I^{\wedge}) \cup \{\alpha_i(\varepsilon_{i+1}) \mid \alpha_i \in Low(A_i)$
and $\varepsilon_{i+1} \in Low(E_{i+1})\}$.

7.5.2:LEMMA:-

(i) Let $P(i) \equiv$ (Low(E_i) and Low(A_i) are well-defined subsets of E_i and A_i rexpectively)

$$\& \ (\varepsilon_i \in \mathrm{Low}(E_i) \Rightarrow [x/y]\varepsilon_i \in \mathrm{Low}(E_i))$$
$$\& \ (\alpha_i \in \mathrm{Low}(A_i) \Rightarrow [x/y]\alpha_i \in \mathrm{Low}(A_i))$$
$$\& \ (\perp \in \mathrm{Low}(E_i) \text{ and } \mathrm{Low}(A_i))$$
$$\& \ (\top \notin \mathrm{Low}(E_i) \text{ or } \mathrm{Low}(A_i)).$$

Then, $(\forall i \geq 0)P(i)$.

(ii) Let $P(i) \equiv (\varepsilon_i \in \mathrm{Low}(E_i)) \wedge (\varepsilon_i' \sqsubseteq \varepsilon_i) \Rightarrow (\varepsilon_i' \in \mathrm{Low}(E_i))$

$$\& \ (\alpha_i \in \mathrm{Low}(A_i)) \wedge (\alpha_i' \sqsubseteq \alpha_i) \Rightarrow (\alpha_i' \in \mathrm{Low}(A_i)).$$

Then, $(\forall i \geq 0)P(i)$.

(iii) $(\forall i \geq 0)$(Low(E_i) and Low(A_i) are sub-semi-lattices of E_i and A_i respectively).

(iv) Let $P(i) \equiv (\varepsilon_i \in \mathrm{Low}(E_i)) \Leftrightarrow (\{\varepsilon_i' | \varepsilon_i' \sqsubseteq \varepsilon_i\}$ is finite)

$$\& \ (\alpha_i \in \mathrm{Low}(A_i)) \Leftrightarrow (\{\alpha_i' | \alpha_i' \sqsubseteq \alpha_i\} \text{ is finite}).$$

Then, $(\forall i \geq 0)P(i)$.

(v) Let $P(i) \equiv (\varepsilon_i \in \mathrm{Low}(E_i)) \wedge (\varepsilon_{i+1}' \sqsubseteq \phi_{i,i+1}(\varepsilon_i))$

$$\Rightarrow (\varepsilon_{i+1}' = \phi_{i,i+1}(\varepsilon_i'))$$
$$\& \ (\alpha_i \in \mathrm{Low}(A_i)) \wedge (\alpha_{i+1}' \sqsubseteq \theta_{i,i+1}(\alpha_i))$$
$$\Rightarrow (\alpha_{i+1}' = \theta_{i,i+1}(\alpha_i')).$$

Then, $(\forall i \geq 0)P(i)$.

(vi) Let $P(i) \equiv (\mathrm{Low}(E_i) \lhd \mathrm{Low}(E_{i+1}))$ & $(\mathrm{Low}(A_i) \lhd \mathrm{Low}(A_{i+1}))$.

Then, $(\forall i \geq 0)P(i)$.

(vii) Let $P(i) \equiv (Ap_i \in [\mathrm{Low}(E_i) \times \mathrm{Low}(E_i) \rightarrow \mathrm{Low}(E_{i+1})])$.

Let $S(i) \equiv (\varepsilon_i \in \mathrm{Low}(E_i)) \Rightarrow ([\varepsilon_i/x]_I \in [\mathrm{Low}(I) - \mathrm{Low}(E_i)])$.

Let $Q(i) \equiv (\varepsilon_i' \in \mathrm{Low}(E_i)) \Rightarrow ([\varepsilon_i'/x]_{E_{i-1}} \in [\mathrm{Low}(E_{i-1}) \rightarrow \mathrm{Low}(E_i)])$.

Let $R(i) \equiv (\varepsilon_i' \in \mathrm{Low}(E_i)) \Rightarrow ([\varepsilon_i'/x]_{A_{i-2}} \in [\mathrm{Low}(A_{i-2}) \rightarrow \mathrm{Low}(E_i)])$.

Then, $(\forall i \geq 0)S(i)$ and $(\forall i \geq 1)(P(i) \wedge Q(i))$ and $(\forall i \geq 2)R(i)$.

Proof:-

(i) -Straightforward but tedious : what we are after here is the well-definedness.

(ii) -Trivial induction.

(iii) -Consequence of part (ii), by 0.6.29(i).

(iv), (v), (vi) and (vii) -Trivial.

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7.5.3:COR:-

(i) $(\forall i \geq 0)(\varepsilon_i \in Low(E_i)) \Longleftrightarrow (emb_i(\varepsilon_i)$ is a finite subset).

(ii) $(\forall i \geq 0)(\varepsilon_i \in Low(E_i)) \Rightarrow (emb_{i+1}\circ\phi_{i,i+1}(\varepsilon_i) = emb_i(\varepsilon_i))$.

Proof:-

(i) -By 7.5.2(iv), since $\phi_{i,\infty}$ is injective.

(ii) -By 7.4.2(iii), $emb_i(\varepsilon_i) \subseteq emb_{i+1}\circ\phi_{i,i+1}(\varepsilon_i)$.

-Let $\varepsilon_i \in Low(E_i)$.

-Then, $\phi_{i+1,\infty}(\varepsilon'_{i+1}) \in emb_{i+1}\circ\phi_{i,i+1}(\varepsilon_i)$

$\Longleftrightarrow \varepsilon'_{i+1} \sqsubseteq \phi_{i,i+1}(\varepsilon_i)$

$\Rightarrow \varepsilon'_{i+1} = \phi_{i,i+1}(\varepsilon'_i)$, by 7.5.2(v).

$\Rightarrow \phi_{i+1,\infty}(\varepsilon'_{i+1}) = \phi_{i,\infty}(\varepsilon'_i)$.

-But, $\varepsilon'_i = \phi_{i+1,i}(\varepsilon'_{i+1})$, by notation.

$\sqsubseteq \phi_{i+1,i}\circ\phi_{i,i+1}(\varepsilon_i) = \varepsilon_i$.

-∴, $\phi_{i+1,\infty}(\varepsilon'_{i+1}) \in emb_i(\varepsilon_i)$.

-∴, $emb_{i+1}\circ\phi_{i,i+1}(\varepsilon_i) \subseteq emb_i(\varepsilon_i)$ and, so, they are equal.

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7.5.4:THEOREM:-

$$\varepsilon \in \bigcup_{i=0}^{\infty} \phi_{i,\infty}(Low(E_i)) \Longleftrightarrow emb(\varepsilon) \text{ is a finite subset.}$$

Proof:-

(⇒) -Let $\varepsilon = \phi_{k,\infty}(\varepsilon_k)$, where $\varepsilon_k \in Low(E_k)$.

-Then, $emb(\varepsilon) = emb_k(\varepsilon_k)$, by 7.4.2(iii) and 7.5.3(ii).

-Thence, by 7.5.3(i), $emb(\varepsilon)$ is finite.

(⇐) -Let $\varepsilon \in E_\infty \setminus \bigcup_{i=0}^{\infty} \phi_{i,\infty}(E_i)$.

-Then, $\{\phi_{i,\infty}(\varepsilon_i) \mid i \geq 0\}$ is an infinite subset of $emb(\varepsilon)$.

-Thus, $emb(\varepsilon)$ is an infinite set.

-Now, let $\varepsilon \in \bigcup_{i=0}^{\infty} \phi_{i,\infty}(E_i) \setminus \bigcup_{i=0}^{\infty} \phi_{i,\infty}(\mathrm{Low}(E_i))$.

-Then, $\varepsilon = \phi_{k,\infty}(\varepsilon_k)$ and $\varepsilon_k \in E_k \setminus \mathrm{Low}(E_k)$.

-But, $\mathrm{emb}_k(\varepsilon_k)$ is infinite, by 7.5.3(i), and, since $\mathrm{emb}_k(\varepsilon_k) \subseteq \mathrm{emb}(\varepsilon)$, we have that $\mathrm{emb}(\varepsilon)$ is infinite.

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7.5.5:LEMMA:-

(i) Let $A[\![\varepsilon]\!] \equiv (\tilde{\mathcal{E}}[\![\varepsilon]\!] \in \bigcup_{i=0}^{\infty} \phi_{i,\infty}(\mathrm{Low}(E_i)))$. Then, $(\forall \varepsilon \in \mathrm{EXP}) A[\![\varepsilon]\!]$.

(ii) Let $P(i) \equiv (\varepsilon_i \in \mathrm{Low}(E_i) \Rightarrow \phi_{i,\infty}(\varepsilon_i) = \tilde{\mathcal{E}}[\![\varepsilon]\!]$, some $\varepsilon \in \mathrm{EXP})$ Then, $(\forall i \geq 0) P(i)$.

Proof:-

(i) and (ii) -Both inductions are straightforward.

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7.5.6:COR:-

$\{\tilde{\mathcal{E}}[\![\varepsilon]\!] \mid \varepsilon \in \mathrm{EXP}\} = \bigcup_{i=0}^{\infty} \phi_{i,\infty}(\mathrm{Low}(E_i))$.

Proof:-

-By 7.5.5.

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7.5.7:THEOREM:-

ε is a finite λ-expression $\Leftrightarrow$ $\mathrm{emb} \circ \mathcal{E}[\![\varepsilon]\!]$ is a finite subset.

Proof:-

-Claim: ε is a finite λ-expression $\Leftrightarrow$ $\mathcal{E}[\![\varepsilon]\!] = \tilde{\mathcal{E}}[\![\varepsilon']\!]$, some ε' :-

($\Leftarrow$) -Clear, by 4.7.3.

($\Rightarrow$) $-\mathcal{E}[\![\varepsilon]\!] \sqsubseteq \bigsqcup\{\tilde{\mathcal{E}}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$.

-∴, $\mathcal{E}[\![\varepsilon]\!] \prec \bigsqcup\{\tilde{\mathcal{E}}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$, by 4.7.9.

-∴, $\mathcal{E}[\![\varepsilon]\!] \sqsubseteq \tilde{\mathcal{E}}[\![\varepsilon']\!]$.

-But, $\tilde{\mathcal{E}}[\![\varepsilon']\!] \sqsubseteq \mathcal{E}[\![\varepsilon']\!] = \mathcal{E}[\![\varepsilon]\!]$, since $\langle E_\infty, \mathcal{E} \rangle$ is a model.

-Hence, $\mathcal{E}[\![\varepsilon]\!] = \tilde{\mathcal{E}}[\![\varepsilon']\!]$ and the claim is established.

-∴, we have the result, by 7.5.6 and 7.5.4.

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7.5.8:DEF:-

$$\mathrm{Low}(E_\infty) := \text{Inverse limit of } \langle \mathrm{Low}(E_i), \phi_{i+1,i} \rangle_{i=0}^{\infty}$$
$$= \{\varepsilon \in E_\infty \mid \varepsilon_i \in \mathrm{Low}(E_i), \forall i \geq 0\}.$$

$$\mathrm{Low}(A_\infty) := \text{Inverse limit of } \langle \mathrm{Low}(A_i), \theta_{i+1,i} \rangle_{i=0}^{\infty}$$
$$= \{\alpha \in A_\infty \mid \alpha_i \in \mathrm{Low}(A_i), \forall i \geq 0\}.$$

7.5.9:LEMMA:-

(i) $\bot \in \mathrm{Low}(E_\infty)$ and $\mathrm{Low}(A_\infty)$.

(ii) $\top \notin \mathrm{Low}(E_\infty)$ or $\mathrm{Low}(A_\infty)$.

(iii) $(\varepsilon \in \mathrm{Low}(E_\infty)) \wedge (\varepsilon' \sqsubseteq \varepsilon) \Rightarrow (\varepsilon' \in \mathrm{Low}(E_\infty))$.

(iv) $(\alpha \in \mathrm{Low}(A_\infty)) \wedge (\alpha' \sqsubseteq \alpha) \Rightarrow (\alpha' \in \mathrm{Low}(A_\infty))$.

Proof:-

-Trivial, by 7.5.2(i) and (ii).

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7.5.10:THEOREM:-

(i) $\mathrm{Low}(E_\infty)$ and $\mathrm{Low}(A_\infty)$ are sub-semi-lattices of E_∞ and A_∞.

(ii) $\mathrm{Low}(E_\infty)$ and $\mathrm{Low}(A_\infty)$ are directedly complete.

(iii) $\mathrm{Low}(E_\infty)$ and $\mathrm{Low}(A_\infty)$ are continuous.

(iv) Let $\varepsilon, \delta \in \mathrm{Low}(E_\infty)$. Then, ($\varepsilon \prec \delta$ in $\mathrm{Low}(E_\infty)$) <=> ($\varepsilon \prec \delta$ in E_∞). Same for $\mathrm{Low}(A_\infty)$.

Proof:-

(i) -By 7.5.9(iii) and (iv) and 0.6.29(i).

(ii) -Let D be directed in $\mathrm{Low}(E_\infty)$.

-Then, D is directed in E_∞.

-Taking the least upper bound in E_∞, we get :-

$$\bigsqcup D = \langle \bigsqcup D_i \rangle_{i=0}^{\infty}, \text{ where } D_i = \phi_{\infty,i}(D), \text{ by 0.6.22(ii).}$$

-But, since E_i has finite depth, $\bigsqcup D_i \in D_i$.

-∴, $\bigsqcup D_i = \delta_i$, for some $\delta \in D \subseteq \mathrm{Low}(E_\infty)$.

-∴, $\bigsqcup D_i \in \mathrm{Low}(E_i)$, for all $i \geq 0$, and, so, $\bigsqcup D \in \mathrm{Low}(E_\infty)$.

-Similarly, $\mathrm{Low}(A_\infty)$ is directedly complete.

(iii) and (iv) -By 0.6.29(iv) and (vi) respectively.

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7.5.11:LEMMA:-

(i) $\varepsilon, \varepsilon' \in \text{Low}(E_\infty) \Rightarrow \text{Ap}(\varepsilon, \varepsilon') \in \text{Low}(E_\infty)$.

(ii) $\varepsilon, \varepsilon' \in \text{Low}(E_\infty) \Rightarrow [\varepsilon'/x]\varepsilon \in \text{Low}(E_\infty)$.

(iii) $\varepsilon' \in \text{Low}(E_\infty)$ and $\alpha \in \text{Low}(A_\infty) \Rightarrow [\varepsilon'/x]\alpha \in \text{Low}(E_\infty)$.

Proof:-

-By 7.5.2(vii) and 7.5.10(ii).

ǂ

7.5.12:COR:-

$(\forall\varepsilon \in \text{EXP})(\mathcal{E}[\![\varepsilon]\!] \in \text{Low}(E_\infty))$.

Proof:-

-Trivial structural induction, using 7.5.11(i).

ǂ

7.5.13:COR:-

$$\bigcup_{i=0}^{\infty} \phi_{i,\infty}(\text{Low}(E_i)) \subset \{\mathcal{E}[\![\varepsilon]\!] \mid \varepsilon \in \text{EXP}\} \subset \text{Low}(E_\infty).$$

Proof:-

-By a trivial structural induction on :-

$$A[\![\varepsilon]\!] \equiv (\text{there exists } \varepsilon' \in \text{EXP})(\tilde{\mathcal{E}}[\![\varepsilon]\!] = \mathcal{E}[\![\varepsilon']\!]),$$

we get $\{\tilde{\mathcal{E}}[\![\varepsilon]\!] \mid \varepsilon \in \text{EXP}\} \subseteq \{\mathcal{E}[\![\varepsilon]\!] \mid \varepsilon \in \text{EXP}\}$.

-Hence, the inequalities, by 7.5.6 and 7.5.12.

-They are strict because, for instance, there does not exist an $\varepsilon \in \text{EXP}$ such that $\tilde{\mathcal{E}}[\![\varepsilon]\!] = \mathcal{E}[\![Y]\!]$. Also, $\text{Low}(E_\infty)$ contains elements with infinitely many free variables, as remarked upon in 2.2.9 :-

e.g. $\langle \bot, \bot, x_0(\bot), x_0(x_1(\bot)), x_0(x_1(x_2(\bot))), \ldots\ldots \rangle$.

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7.5.14:REMARK:-

Thus, we see that the semi-lattice $\text{Low}(E_\infty)$ contains that part of E_∞ that models λ-expressions. Theorem 7.5.4 shows that it would have been sensible to define only the elements of the (semi-)direct limit, $\bigcup_{i=0}^{\infty} \phi_{i,\infty}(\text{Low}(E_i))$, as finite. Also, we have in $\text{Low}(E_\infty)$ that

$(\varepsilon \prec \delta) \iff (\varepsilon \sqsubseteq \delta) \wedge (\varepsilon$ is finite) and we see that it is better behaved than E_∞ in this respect (see 4.7.12 and 4.7.17). Also, while 4.7.17 shows that the $\langle E_\infty, \mathcal{E}\rangle$-derived notions of $\prec$, $\sqsubseteq$ and finiteness on λ-expressions have our intuitive relationship, theorem 7.5.7 shows that finiteness makes sense as well.

Another good thing about $Low(E_\infty)$ is its economy; for 7.5.13 shows that the direct limit, which forms its countable basis and of which it is the directed completion, is wholly λ-definable. Incidentally, since it is a trivial induction to show that $Low(E_\infty)$ and $Low(A_\infty)$ form countable bases for E_∞ and A_∞, the (semi-)direct limits, $\bigcup_{i=0}^{\infty} \phi_{i,\infty}(Low(E_i))$ and $\bigcup_{i=0}^{\infty} \theta_{i,\infty}(Low(A_i))$, form countable bases for E_∞ and A_∞, by 0.6.25(ii). In particular, E_∞ is countably based by the "approximate normal forms" (:= the image of $\acute{\mathcal{E}}$), by 7.5.6. Further, since they consist entirely of elements that are isolated in E_∞, by 4.7.3 and 4.7.10, we see that E_∞ is an <u>algebraic</u> lattice - see Scott [74].

Maybe it would have been better to have defined initially only the $Low(E_\infty)$ system. If we want, we can always make it into a lattice by adjoining a $\top$, by 0.6.28(i). Alternatively, we could have extended the relation $\sim$ on $A_i \times E_{i+1}$ (1.1.20) to :-

$$(\alpha,\varepsilon) \sim (\alpha',\varepsilon') \text{ iff } (\alpha = \top = \alpha') \vee (\varepsilon = \top = \varepsilon')$$
$$\vee (\alpha = \bot = \alpha') \vee ((\alpha = \alpha') \wedge (\varepsilon = \varepsilon')).$$

Then, making a corresponding extension to $\sqsubseteq$ on $A_i(E_{i+1})$, carry on defining E_∞ as before.

Anyway, this is just a question of the aesthetics of the actual model used. As far as the semantics of the λ-calculus is concerned, it does not matter whether we have E_∞ or $Low(E_\infty)$ as the target of $\mathcal{E}$. In fact,

$$\langle Low(E_\infty), \mathcal{E}\rangle \equiv \langle E_\infty, \mathcal{E}\rangle.$$

7.6:Continuous Semantics:-

7.6.0:REMARK:-

In 0.7.0, we defined an ordering on arbitrary semantics of the λ-calculus by inclusion of the induced equivalence relations. Since E_∞ is so syntactic in nature (i.e. closely related to EXP) there ought to be some sense in which $<E_\infty,\mathcal{E}>$ provides a minimal semantics. It is certainly not normality, substitutivity or modelship since these all apply to the trivial term model, <EXP/cnv,[cnv]>, from which $<E_\infty,\mathcal{E}>$ is derivable. However, we have discussed many reasons why this trivial model should be rejected and for which $<E_\infty,\mathcal{E}>$ is acceptable - e.g. unsolvable expressions and fixed-point combinators.

What else is there that makes a semantic function good ? Scott's theory uses lattices whose partial orderings reflect the information content of the data being modelled : it is, then, quite natural to justify the belief that computable functions over the data are continuous (0.6.6). Similarly, we maintain that the semantic function itself should be "continuous".

$\tilde{\leqq}$ reflects precisely our intuitive notion of the information available from just looking at the syntax of an expression - i.e. it ignores unevaluated redexes and considers only those sub-parts that are fully worked out. Let us call this the syntactical information of an expression and write $\tilde{\varepsilon}$ of ε (c.f. 7.2). Clearly, any decent semantics, $<L,\mathcal{F}>$, when restricted so as only to look at the syntactical information, $<L,\tilde{\mathcal{F}}>$, should be monotone with respect to $\tilde{\leqq}$.

Further, the intuitive meaning of an expression is the union of the syntactic information held in all the expressions reducible from it. Now, any decent semantics should reflect this intuition :-

$$F[\![\varepsilon]\!] = \bigsqcup\{\tilde{F}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\},$$

and this implies a "continuity" of the semantic function from our intuition, for we can then write :-

$$F[\![\cup\{\tilde{\varepsilon}' \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}]\!] = \bigsqcup\{\tilde{F}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}.$$

This justifies the terminology of the following definition. We claim that $<E_\infty, E>$ is the minimal continuous semantics and, so , is coincident with our intuition.

7.6.1:DEF:-

Let $<L, F>$ be some semantics of the λ-calculus where L is a directedly complete semi-lattice. Then, $<L, \tilde{F}>$ is a WELL-BEHAVED APPROXIMATE semantics to $<L, F>$ if :-

(i) $\tilde{F}$ is monotone with respect to $\tilde{\sqsubseteq}$

and (ii) $F[\![\varepsilon]\!] = \bigsqcup\{\tilde{F}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$.

We say that $<L, F>$ is a CONTINUOUS semantics if it has a well-behaved approximate.

7.6.2:LEMMA:-

(i) $<E_\infty, E>$ is continuous.

(ii) $<[ENV \to D_\infty], D>$ is continuous.

(iii) $<[ENV^* \to D^*_\infty], D^*>$ is continuous.

Proof:-

(i) -By definition, $\tilde{E}$ is monotone with respect to $\tilde{\sqsubseteq}$.

-$\therefore$, since $E = V$, $\tilde{E}$ is well-behaved.

(ii) -Let $\tilde{D}[\![\varepsilon]\!] := D'[\![\tilde{\varepsilon}]\!]$, where D' and ∿ are as defined in 0.7.17.

-Then, using 4.8.4, it is a trivial structural induction to show that $\tilde{D}$ is monotonic with respect to $\tilde{\sqsubseteq}$.

-Thus, the good-behaviour of $\tilde{D}$ is just the crucial theorem Wadsworth proved (0.7.19).

(iii) -Same as part(ii), defining $\tilde{D}^*[\![\varepsilon]\!] := D^{*\prime}[\![\tilde{\varepsilon}]\!]$, using the definitions of 0.7.22 and its part (xxvi).

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<u>7.6.3:LEMMA:-</u>

Let $\langle L, F\rangle$ be a continuous semantics with a well-behaved approximate $\langle L, \tilde{F}\rangle$. Then,

(i) $\tilde{F} \sqsubseteq F$,

(ii) $\{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$ is a directed subset of L,

(iii) $F\llbracket \varepsilon \rrbracket = \bigsqcup\{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\qquad} \varepsilon'\}$

$= \bigsqcup\{\tilde{F}\llbracket i\langle \varepsilon\rangle \rrbracket \mid i \geq 1\}$,

where the sets are also directed,

(iv) $\langle L, F\rangle$ is a model,

(v) $(\varepsilon, \delta \in \text{INSOL})_{\wedge}(\gamma \in \text{EXP}) \Rightarrow (F\llbracket \varepsilon \rrbracket = F\llbracket \delta \rrbracket \sqsubseteq F\llbracket \gamma \rrbracket)$,

(vi) $\langle M, G\rangle$ is ctsly. derivable from $\langle L, F\rangle \Rightarrow \langle M, G\rangle$ is cts.

<u>Proof:-</u>

(i) $-\tilde{F}\llbracket \varepsilon \rrbracket \sqsubseteq \bigsqcup\{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\} = F\llbracket \varepsilon \rrbracket$.

(ii) -Let $\tilde{F}\llbracket \omega \rrbracket, \tilde{F}\llbracket \delta \rrbracket \in \{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$.

-Then, $\varepsilon \xrightarrow{\alpha,\beta} \omega, \delta$ and so, by the Church-Rosser theorem, we have $\omega, \delta \xrightarrow{\alpha,\beta} \sigma$. $\therefore$, $\omega, \delta \overset{\sim}{\sqsubseteq} \sigma$.

$-\therefore$, $\tilde{F}\llbracket \omega \rrbracket, \tilde{F}\llbracket \delta \rrbracket \sqsubseteq \tilde{F}\llbracket \sigma \rrbracket \in \{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$.

-Hence, the set is directed.

(iii) -The sets are directed by a similar proof to that of part (ii) using 6.2.12and 5.3.3 instead of the C-R theorem.

-Their limits equal $F\llbracket \varepsilon \rrbracket$ because of the weak completeness of inside-out and i'th reductions and part (i) of definition 7.6.1.

(iv) -Simple C-R corollary, analogous to 4.6.1.

(v) -Clearly, $\varepsilon, \delta \in \text{NOH} \Rightarrow \tilde{F}\llbracket \varepsilon \rrbracket = \tilde{F}\llbracket \delta \rrbracket$, since $\varepsilon \overset{\sim}{\equiv} \delta$, by 4.8.4.

-Hence, $\varepsilon \in \text{INSOL} \Rightarrow F\llbracket \varepsilon \rrbracket = \tilde{F}\llbracket \varepsilon \rrbracket$. Hence, the result.

-Also $(\varepsilon \in \text{INSOL})_{\wedge}(\gamma \in \text{EXP}) \Rightarrow (F\llbracket \varepsilon \rrbracket = \tilde{F}\llbracket \varepsilon \rrbracket \sqsubseteq \tilde{F}\llbracket \gamma \rrbracket \sqsubseteq F\llbracket \gamma \rrbracket)$.

(vi) -There exists an $f \in [L \to M]$ such that $G = f \circ F$.

-Let $\tilde{G} := f \circ \tilde{F}$. <u>Claim:</u> $\tilde{G}$ is well-behaved :-

$-\varepsilon \overset{\sim}{\sqsubseteq} \delta \Rightarrow \tilde{F}\llbracket \varepsilon \rrbracket \sqsubseteq \tilde{F}\llbracket \delta \rrbracket \Rightarrow \tilde{G}\llbracket \varepsilon \rrbracket \sqsubseteq \tilde{G}\llbracket \delta \rrbracket$, by the monotonicity of f.

$-G\llbracket \varepsilon \rrbracket = f \circ F\llbracket \varepsilon \rrbracket = f(\bigsqcup\{\tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\})$

$= \bigsqcup\{f \circ \tilde{F}\llbracket \varepsilon' \rrbracket \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$, since the set is directed.

$$= \bigsqcup\{\tilde{G}[\![\varepsilon']\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}.$$

-$\therefore$, $\tilde{G}$ is a well-behaved approximate to G.

-$\therefore$, $\langle M,G\rangle$ is continuous.

†

7.6.4:REMARK:-

We have said before that E_∞ is syntactic in nature - i.e. "close" to EXP. We make use of this, now, by defining a way back from part of E_∞ to EXP.

7.6.5:DEF:-

$$\mathrm{syn}_I : \mathrm{Low}(I') \longrightarrow \mathrm{EXP}$$

$$\left\{\begin{matrix} x \\ \bot \end{matrix}\right\} \longmapsto \left\{\begin{matrix} x \\ \Delta\Delta \end{matrix}\right\}.$$

$$\mathrm{syn}_{E_i} : \mathrm{Low}(E_i) \longrightarrow \mathrm{EXP}$$

$$\left\{\begin{matrix} \lambda x.\varepsilon_{i-1} \\ \alpha_{i-1} \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \lambda x.\mathrm{syn}_{E_{i-1}}(\varepsilon_{i-1}) \\ \mathrm{syn}_{A_{i-1}}(\alpha_{i-1}) \end{matrix}\right\}.$$

$$\mathrm{syn}_{A_i} : \mathrm{Low}(A_i) \longrightarrow \mathrm{EXP}$$

$$\left\{\begin{matrix} x \\ \alpha_{i-1}(\varepsilon_i) \end{matrix}\right\} \longmapsto \left\{\begin{matrix} \mathrm{syn}_I(x) \\ \mathrm{syn}_{A_{i-1}}(\alpha_{i-1})(\mathrm{syn}_{E_i}(\varepsilon_i)) \end{matrix}\right\}.$$

7.6.6:LEMMA:-

Let $P(i) \equiv$ (syn_{E_i} and syn_{A_i} are well-defined up to $\acute{\equiv}$)

& $(\mathrm{syn}_{E_i}([x/y]\varepsilon_i) \acute{\equiv} [x/y]\mathrm{syn}_{E_i}(\varepsilon_i))$

& $(\mathrm{syn}_{A_i}([x/y]\alpha_i) \acute{\equiv} [x/y]\mathrm{syn}_{A_i}(\alpha_i))$

& (y is not free in ε_i $\Rightarrow$ y is not free in $\mathrm{syn}_{E_i}(\varepsilon_i)$)

& (y is not free in α_i $\Rightarrow$ y is not free in $\mathrm{syn}_{A_i}(\alpha_i)$)

& $((\varepsilon_i \equiv \varepsilon_i') \Rightarrow (\mathrm{syn}_{E_i}(\varepsilon_i) \acute{\equiv} \mathrm{syn}_{E_i}(\varepsilon_i')))$

& $((\alpha_i \equiv \alpha_i') \Rightarrow (\mathrm{syn}_{A_i}(\alpha_i) \acute{\equiv} \mathrm{syn}_{A_i}(\alpha_i)))$

& $(\mathrm{syn}_{E_i}(\varepsilon_i) \acute{\equiv} \mathrm{syn}_{E_{i+1}}(\phi_{i,i+1}(\varepsilon_i)))$

& $(\mathrm{syn}_{A_i}(\alpha_i) \acute{\equiv} \mathrm{syn}_{A_{i+1}}(\theta_{i,i+1}(\alpha_i)))$

& $(\mathrm{syn}_{E_i}(\perp_{E_i}), \mathrm{syn}_{A_i}(\perp_{A_i}) \in \mathrm{NOH})$.

Then, $(\forall i \geq 0)P(i)$.

Proof:-

-Straightforward and trivial induction.

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7.6.7:NOTATION:-

We will just write syn_i for syn_{E_i}.

7.6.8:COR:-

$<\varepsilon_i>_{i=0}^{\infty} \in \mathrm{Low}(E_\infty) \Rightarrow \{\mathrm{syn}_i(\varepsilon_i) \mid i \geq 0\}$ is a chain under $\tilde{\leqq}$.

Proof:-

$-\mathrm{syn}_i(\varepsilon_i) \cong \mathrm{syn}_{i+1} \circ \phi_{i,i+1}(\varepsilon_i)$, by 7.6.6.

$\tilde{\leqq} \mathrm{syn}_{i+1}(\varepsilon_{i+1})$, by 7.6.6.

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7.6.9:LEMMA:-

Let $\mathcal{A}\llbracket\varepsilon\rrbracket \equiv$ (there exists $i \geq 0)(\forall j \geq i)(\mathrm{syn}_j \circ \phi_{\infty,j} \circ \tilde{\mathcal{E}}\llbracket\varepsilon\rrbracket \tilde{\leqq} \varepsilon)$.

Then, $(\forall \boldsymbol{\varepsilon} \in \mathrm{EXP})\mathcal{A}\llbracket\boldsymbol{\varepsilon}\rrbracket$.

Proof:-

-Clearly, $\mathcal{A}\llbracket x\rrbracket$, with $i = 0$.

-Claim: $\mathcal{A}\llbracket\varepsilon\rrbracket \Rightarrow \mathcal{A}\llbracket\lambda x.\boldsymbol{\varepsilon}\rrbracket$:-

-Let i be that which is in $\mathcal{A}\llbracket\varepsilon\rrbracket$.

-Then, for all $j \geq i$,

$$\begin{aligned}
\mathrm{syn}_{j+1} \circ \phi_{\infty,j+1} \circ \tilde{\mathcal{E}}\llbracket\lambda y.\boldsymbol{\varepsilon}\rrbracket &= \mathrm{syn}_{j+1} \circ \phi_{\infty,j+1}(\lambda y.\tilde{\mathcal{E}}\llbracket\boldsymbol{\varepsilon}\rrbracket) \\
&= \mathrm{syn}_{j+1}(\lambda y.\phi_{\infty,j} \circ \tilde{\mathcal{E}}\llbracket\boldsymbol{\varepsilon}\rrbracket) \\
&= \lambda y.\mathrm{syn}_j \circ \phi_{\infty,j} \circ \tilde{\mathcal{E}}\llbracket\boldsymbol{\varepsilon}\rrbracket \\
&\tilde{\leqq} \lambda y.\boldsymbol{\varepsilon}, \text{ by } \mathcal{A}\llbracket\boldsymbol{\varepsilon}\rrbracket.
\end{aligned}$$

-Claim: $\mathcal{A}\llbracket\varepsilon\rrbracket \wedge \mathcal{A}\llbracket\delta\rrbracket \Rightarrow \mathcal{A}\llbracket\boldsymbol{\varepsilon}(\delta)\rrbracket$:-

-If $\boldsymbol{\varepsilon}(\delta) \in \mathrm{NOH}$, then $\mathrm{syn}_j \circ \phi_{\infty,j} \circ \tilde{\mathcal{E}}\llbracket\boldsymbol{\varepsilon}(\delta)\rrbracket = \mathrm{syn}_j(\perp_{E_j}) \in \mathrm{NOH}$, by 7.6.6, and so is $\tilde{\leqq} \boldsymbol{\varepsilon}(\delta)$, by 4.8.4.

-If $\boldsymbol{\varepsilon}(\delta) \in \mathrm{HEAD}$, then let k be the maximum of the i's in $\mathcal{A}\llbracket\boldsymbol{\varepsilon}\rrbracket$ and $\mathcal{A}\llbracket\delta\rrbracket$ plus 1.

-Then, for all $j \geq k$,

$$\begin{aligned} syn_j \circ \phi_{\infty,j} \circ \tilde{\mathcal{E}}[\![\varepsilon(\delta)]\!] &= syn_j((\phi_{\infty,j-1} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!])(\phi_{\infty,j-1} \circ \tilde{\mathcal{E}}[\![\delta]\!])) \\ &= (syn_{j-1} \circ \phi_{\infty,j-1} \circ \tilde{\mathcal{E}}[\![\varepsilon]\!]) \\ &\qquad (syn_{j-1} \circ \phi_{\infty,j-1} \circ \tilde{\mathcal{E}}[\![\delta]\!]) \\ &\cong \varepsilon(\delta), \text{ by } \mathcal{A}[\![\varepsilon]\!] \wedge \mathcal{A}[\![\delta]\!]. \end{aligned}$$

-Hence, by structural induction, $(\forall \varepsilon \in EXP)\mathcal{A}[\![\varepsilon]\!]$.

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7.6.10:DEF:-

Let $\langle L, \mathcal{F}, \tilde{\mathcal{F}} \rangle$ be a continuous semantics. Extend L to a complete lattice by adjoining a $\top$ (see 0.6.28). Then, we define :-

$$\overline{\mathcal{F}}_i : E_i \longrightarrow L \cup \{\top\}$$

$$\varepsilon_i \longmapsto \left\{ \begin{array}{l} \top, \text{ if } \varepsilon_i \notin Low(E_i) \\ \tilde{\mathcal{F}} \circ syn_i(\varepsilon_i), \text{ if } \varepsilon_i \in Low(E_i) \end{array} \right\}$$

and

$$\overline{\mathcal{F}} : E_\infty \longrightarrow L \cup \{\top\}$$

$$\langle \varepsilon_i \rangle_{i=0}^{\infty} \longmapsto \bigsqcup_{i=0}^{\infty} \overline{\mathcal{F}}_i(\varepsilon_i).$$

7.6.11:LEMMA:-

(i) $\overline{\mathcal{F}}_i$ is well-defined, monotonic and, so, continuous.

(ii) $\{\overline{\mathcal{F}}_i(\varepsilon_i) \mid i \geq 0\}$ is a chain in L, where $\langle \varepsilon_i \rangle_{i=0}^{\infty} \in E_\infty$.

(iii) $\overline{\mathcal{F}}$ is continuous.

Proof:-

(i) -$\overline{\mathcal{F}}_i$ is well-defined since syn_i is well-defined up to $\cong$ and $\tilde{\mathcal{F}}$ does not detect such differences.

-If $\varepsilon_i \sqsubseteq \varepsilon'_i$ and $\varepsilon'_i \in Low(E_i)$, then so is ε_i, by 7.5.2(ii).

-Thus, $syn_i(\varepsilon_i) \mathrel{\tilde{\sqsubseteq}} syn_i(\varepsilon_i)$, by 7.6.6.

-∴, $\tilde{\mathcal{F}} \circ syn_i(\varepsilon_i) \sqsubseteq \tilde{\mathcal{F}} \circ syn_i(\varepsilon_i)$, since $\tilde{\mathcal{F}}$ is monotonic w.r.t. $\tilde{\sqsubseteq}$.

-If $\varepsilon'_i \notin Low(E_i)$, then $\overline{\mathcal{F}}_i(\varepsilon_i) \sqsubseteq \top = \overline{\mathcal{F}}_i(\varepsilon'_i)$.

-∴, $\overline{\mathcal{F}}_i$ is monotonic and, so, continuous, since E_i has finite depth.

(ii) -If $\langle \varepsilon_i \rangle_{i=0}^{\infty} \in Low(E_\infty)$, then $\{\overline{\mathcal{F}}_i(\varepsilon_i) \mid i \geq 0\} = \tilde{\mathcal{F}}\{ syn_i(\varepsilon_i) \mid i \geq 0\}$

which is a chain by 7.6.8 and since $\tilde{F}$ is monotone w.r.t. $\tilde{\sqsubseteq}$.

-Otherwise, there exists i ≥ 0 such that for all j ≤ i, $\varepsilon_j \in \mathrm{Low}(E_j)$

-Then, as above, $\{\overline{F}_j(\varepsilon_j) \mid j \leq i\}$ forms a chain while the rest of the set consists of just the $\top$ element.

(iii) -$\overline{F}$ is continuous since it is the limit of continuous maps, $\overline{F}_i \circ \phi_{\infty,i}$, by part (i).

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7.6.12:THEOREM:-

(i) $\overline{F} \circ \tilde{E} = \tilde{F}$.

(ii) $\overline{F} \circ E = F$.

Proof:-

(i) -$\overline{F} \circ \tilde{E}[\![\varepsilon]\!] = \bigsqcup\{\overline{F}_i \circ \phi_{\infty,i} \circ \tilde{E}[\![\varepsilon]\!] \mid i \geq 0\}$

$= \bigsqcup\{\tilde{F} \circ \mathrm{syn}_i \circ \phi_{\infty,i} \circ \tilde{E}[\![\varepsilon]\!] \mid i \geq 0\}$, by 7.5.6.

$= \tilde{F}[\![\varepsilon]\!]$, by 7.6.9 and 7.6.11(ii).

(ii) -$\overline{F} \circ E[\![\varepsilon]\!] = \overline{F}(\bigsqcup\{\tilde{E}[\![\varepsilon]\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\})$, by 6.8.1.

$= \bigsqcup\{\overline{F} \circ \tilde{E}[\![\varepsilon]\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$, by 7.6.11(iii).

$= \bigsqcup\{\tilde{F}[\![\varepsilon]\!] \mid \varepsilon \xrightarrow{\alpha,\beta} \varepsilon'\}$, by part (i).

$= F[\![\varepsilon]\!]$, since $\tilde{F}$ is well-behaved w.r.t. F.

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7.6.13:COR:-

<L,F> is a continuous semantics if and only if it is continuously derivable from <E_∞,E>.

Proof:-

-By 7.6.2(i), 7.6.3(vi), 7.6.11(iii) and 7.6.12(ii).

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7.6.14:COR:-

<E_∞,E> is the minimal continuous semantics.

Proof:-

-By 7.6.13 and 0.7.1(vii).

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7.6.15:COR:-

(i) Well-behaved approximates are unique.

(ii) Let $<L,\mathcal{F}>$ be a continuous semantics. Then, the fixed-point combinators, $\{Y_i \mid i \geq 0\}$, are all equivalenced. Further,

$$\mathcal{F}[\![Y(\varepsilon)]\!] = \bigsqcup_{n=0}^{\infty} \mathcal{F}[\![\varepsilon^{n}(\Delta\Delta)]\!].$$

(iii) Let $<L,\mathcal{F}>$ be a continuous Scott-model (see 0.7.27). Then,

$$\mathcal{F}[\![Y(\varepsilon)]\!](\rho) = \mu\mathcal{F}[\![\varepsilon]\!](\rho).$$

Proof:-

(i) -Let $\tilde{\mathcal{F}}$ and $\tilde{\tilde{\mathcal{F}}}$ both be well-behaved approximates to $\mathcal{F}$.

-Then, define, as in 7.6.10, functions $\bar{\mathcal{F}}$ and $\bar{\bar{\mathcal{F}}}$.

-∴, by 7.6.12(ii), $\bar{\mathcal{F}}\circ\mathcal{E} = \mathcal{F} = \bar{\bar{\mathcal{F}}}\circ\mathcal{E}$.

-So, $\bar{\mathcal{F}}$ and $\bar{\bar{\mathcal{F}}}$ are identical on the image of EXP under $\mathcal{E}$.

-But, the image of EXP under $\tilde{\tilde{\mathcal{E}}}$ is contained in this, by 7.5.13 and 7.5.6.

-Hence, $\bar{\mathcal{F}}\circ\tilde{\mathcal{E}} = \bar{\bar{\mathcal{F}}}\circ\tilde{\tilde{\mathcal{E}}}$.

-∴, by 7.6.12(i), $\tilde{\mathcal{F}} = \tilde{\tilde{\mathcal{F}}}$.

(ii) -$Y_i \equiv Y_j$, for all i,j ≥ 0, by 4.0.5.

-Hence, $Y_i \equiv_{<L,\mathcal{F}>} Y_j$, for all i,j ≥ 0, by 7.6.14.

-Further, $\mathcal{F}[\![Y(\varepsilon)]\!] = \bar{\mathcal{F}}\circ\mathcal{E}[\![Y(\varepsilon)]\!] = \bar{\mathcal{F}}(\mu\mathcal{E}[\![\varepsilon]\!])$, by section 7.1.

$$= \bar{\mathcal{F}}(\bigsqcup_{n=0}^{\infty}\mathcal{E}[\![\varepsilon^{n}(\Delta\Delta)]\!]), \text{ by definition of } \mu.$$

$$= \bigsqcup_{n=0}^{\infty}\bar{\mathcal{F}}\circ\mathcal{E}[\![\varepsilon^{n}(\Delta\Delta)]\!], \text{by 7.6.11(iii)}.$$

$$= \bigsqcup_{n=0}^{\infty}\mathcal{F}[\![\varepsilon^{n}(\Delta\Delta)]\!], \text{by 7.6.12(ii)}.$$

(iii) -Let L = [ENV → D], where ENV = (I → D).

-We define $\mu\mathcal{F}[\![\varepsilon]\!](\rho)$, as in section 7.1, by Currying the application function.

-One property of such Scott-models is that the value of a combinator (i.e. a closed λ-expression) is independent of the environment used.

-In particular, $\mathcal{F}[\![\Delta\Delta]\!](\rho) = \mathcal{F}[\![\Delta\Delta]\!](\rho')$, for all $\rho, \rho' \in$ ENV.

-Since the semantics is continuous, $\mathcal{F}[\![\Delta\Delta]\!] \sqsubseteq \mathcal{F}[\![x]\!]$, by 7.6.3(v).

-Choose $\rho \in$ ENV such that $\rho[\![x]\!] = \perp \in$ D.

-Then, $\mathcal{F}[\![\Delta\Delta]\!](\rho) \sqsubseteq \mathcal{F}[\![x]\!](\rho) = \perp$.

-$\therefore$, $\mathcal{F}[\![\Delta\Delta]\!](\rho) = \perp$ and, so, $\mathcal{F}[\![\Delta\Delta]\!] = \perp \in$ L, by the above remarks.

-Incidentally, this means that <u>all</u> unsolvable elements are $\perp$ in a continuous Scott-model and, so, the (Park-perturbed-Scott) model (see 0.7.24) cannot be continuous.

-Now, $\mathcal{F}[\![Y(\varepsilon)]\!](\rho) = (\bigsqcup_{n=0}^{\infty} \mathcal{F}[\![\varepsilon^{n}(\Delta\Delta)]\!])(\rho)$, by part (ii) above.

$$= \bigsqcup_{n=0}^{\infty} \mathcal{F}[\![\varepsilon^{n}(\Delta\Delta)]\!](\rho)$$

$$= \bigsqcup_{n=0}^{\infty} (\mathrm{Ap}(\mathcal{F}[\![\varepsilon]\!](\rho)))^{n}(\mathcal{F}[\![\Delta\Delta]\!](\rho))$$

$$= \bigsqcup_{n=0}^{\infty} (\mathrm{Ap}(\mathcal{F}[\![\varepsilon]\!](\rho)))^{n}(\perp)$$

$$= \mu\mathcal{F}[\![\varepsilon]\!](\rho).$$

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7.6.16:REMARK:-

The degenerate maximal semantics, <{*},const>, is clearly continuous and, so, continuity does not imply properties like normality, solvability or extensional equivalence (i.e. semantic equivalence implies $\simeq$ of 0.7.11). Neither does it imply substitutivity :-

-<u>Claim:</u> $\langle \mathrm{EXP}/\underline{cnv}_{\Lambda}^{\sim}, [\underline{cnv}_{\Lambda}^{\sim}]\rangle$ is not substitutive :-

-Well, for one thing, if it were, there would have been no need to define Morris' $\langle \mathrm{EXP}/\simeq, [\simeq]\rangle$.

-But, in particular, $Y \ \underline{cnv}_{\Lambda}^{\sim} \ \Delta\Delta$.

-And, $Y(\lambda x.y) \ \underline{cnv}_{\Lambda}^{\sim} \ y$ not$(\underline{cnv}_{\Lambda}^{\sim})$ $(\Delta\Delta)(\lambda x.y)$.

-<u>Claim:</u> $<\mathrm{EXP}/\underline{\mathrm{cnv}}_{\wedge}^{\sim},[\underline{\mathrm{cnv}}_{\wedge}^{\sim}]>$ is continuous :-

-$\mathrm{EXP}/\underline{\mathrm{cnv}}_{\wedge}^{\sim}$ becomes a directedly complete semi-lattice by defining $[\varepsilon] \sqsubseteq [\delta]$ if ε does not have a normal form or $[\varepsilon] = [\delta]$.

-Define: $¶ : \mathrm{EXP} \longrightarrow \mathrm{EXP}/\underline{\mathrm{cnv}}_{\wedge}^{\sim}$

$$\varepsilon \longmapsto \left\{\begin{matrix} [\Delta\Delta], \text{ if } \varepsilon \notin \mathrm{NF} \\ [\varepsilon]\ , \text{ if } \varepsilon \in \mathrm{NF} \end{matrix}\right\}.$$

-Then, it is trivial to check that ¶ is a well-behaved approximate to $[\underline{\mathrm{cnv}}_{\wedge}^{\sim}]$.

However, continuity is quite good at ruling out some "bad" semantics - e.g. the proof of 7.6.15(iii), where, in so doing, it provides the answer to a question of D.Park [75]. Also, continuity does imply a number of good points - e.g. 7.6.3(iii), (iv) and (v) and 7.6.15(i) and (ii). Further, by the proof of part (iii) of 7.6.15 again, we see that, in continuous Scott-models, unsolvable elements are $\perp$ and the Y combinators mimic the minimal fixed-point operator μ. Finally, continuous semantics must respect all the equivalences and orderings made by its minimal formulation, $<E_{\infty},\mathcal{E}>$; and so, since this also has all the nice properties of normality, solvability, extensional equivalence and substitutivity - not to mention continuity, what it does to the fixed-point combinators (4.0.5 and section 7.1) and the fact that it is a good (4.8.8) model - we feel that its further investigation may be worthwhile.

8: REFERENCES.

8.0: Explanation:-

References are indicated in the text of this thesis by positive numbers in square brackets, sometimes with an author's name - e.g. Wadsworth [30]. We proceed by a method of "indirect addressing". Look up the number in section 8.1 to get the particular part of the work being cited - in our case, item 3.3.8 of <P>. Finally, search alphabetically in section 8.2 for the key letter enclosed in the angle brackets.

8.1: Numeric:-

[00] §3A of <D>.

[01] <C> and <E>.

[02] Appendix II of <A>.

[03] Page VIII of <D>.

[04] Page 17 of <P>. Wadsworth

[05] §3E of <D>.

[06] Chapter 4 of <D>.

[07] §4A4 and §4B1, definition 1 of <D>.

[08] §4A4(E), §4A5(D), §4B3 and §4C2 of <D>.

[09] §4A3, theorem 4 of <D>.

[10] §4D1 of <D>.

[11] §4D2 of <D>.

[12] §4D3 of <D>.

[13] §4E1 of <D>.

[14] 1.3.16 of <A>.

[15] 3.1 of <P>.

[16] 3.2.1 of <A>.

[17] Same as [16].

[18] <N>, <M>, <K> and <L>.

[19] <J>.

[20] <P>.

[21] Page 10 of <J>.

[22] <K>.

[23] 2.2 and 2.7 of <K>; exercises 7 and 8 of <O>.

[24] 2.6 and 3.3 of <K>.

[25] 4.1 of <K>.

[26] Pages 139 - 141 of <J>.

[27] Theorems 11 and 14 and exercise 18a of <J>.

[28] §3E3 of <D>.

[29] <G>.

[30] 3.3.8 of <P>.

[31] <B>.

[32] 2.4.4 of <P>.

[33] 4.4 of <K>.

[34] 2.2 of <P>.

[35] 2.3.8 and remark following 2.4.1 of <P>.

[36] 2.5.1 and 2.5.8 of <P>.

[37] <H>.

[38] 2.6.1 of <P>.

[39] §4E2, corollary 2.1 of <D>.

[40] 4.1 of <L>.

[41] <Q>.

[42] 2.7 of <P>.

[43] 2.7.3, 2.7.4, 2.7.5, 2.7.11 and 2.7.12 of <P>.

[44] 2.7.6 of <P>.

[45] 2.7.9 of <P>.

[46] Remark after "Question 1" in section 3.1 (page 94) of <P>.

[47] <H>.

[48] 1.6 of <L>.

[49] <I>.

[50] Sections 1 and 2 of <L>.

[51] 1.2 of <L>.

[52] Definition just before 1.2 of <L>.

[53] <N>.

[54] Remark following 3.13 of <K>.

[55] Note (2) on page 133 of <J>.

[56] Chapter II, sections 1,2 and 3 of <F>.

[57] Page 97 of <P>.

[58] Question 1 (page 94) and question 2 (page 103) of <P>.

[59] <Q>.

[60] 3.2.6 of <P>.

[61] <R>.

[62] Example 4.4.1 and note on page 187 of <P>.

[63] §4A4(E) of <D>.

[64] §4C3 of <D>.

[65] §4C2 of <D>.

[66] II.5 of <F>.

[67] <Q>.

[68] Page 92 of <P>.

[69] I.1.1 of <F>.

[70] Question 2 (page 103) of <P>.

[71] II.5.6.2 of <F>.

[72] 3.3.6 of <P>.

[73] 3.8 of <K>.

[74] Remark at top of page 43 of section 5 of <L>.

[75] Page 2, line 4 : "one looks......" of <H>.

[76] Lemma 13.1 of §11E8 of <S>.

[77] §11F8 of <S>.

[78] 3.2.3 of <A>.

8.2:Alphabetic:-

<A> Barendregt,H.P. : "Some Extensional Term Models for Combinaotry Logics and λ-Calculi" :
- Ph.D. Thesis, University of Utrecht (1971).

<B> Bohm,C. : "Alcune Proprieta della Forme β-η-Normali del λ-K-Calcolo" :
- Instituto per le Applicazioni del Calcolo, no 696, Consiglio Nazionale delle Richerche, Roma (1968).

<C> Church,A. : "A Set of Postulates for the Foundation of Logic" :
- Annals of Mathematics (2) 33 : 346-366 (1932) and
" " " " 34 : 839-864 (1933).

<D> Curry,H.B. & Feys,R. : "Combinatory Logic - Volume 1" :
- North-Holland, Amsterdam (1958).

<E> Kleene,S.C. & Rosser,J.B. : "The Inconsistency of Certain Formal Logics" :
- Annals of Mathematics (2) 36 : 630-636 (1935).

<F> Lévy,J-J. : "Réductions Sures dans le Lambda-Calcul" :
- Diplome de Docteur de 3^{e} Cycle, Université Paris VII (1974).

<G> Morris,J.H. : "Lambda-Calculus Models of Programming Languages" :
- Ph.D. Thesis, MIT (1968).

<H> Park,D.M.R. : "The Y-Combinator in Scott's Lambda-Calculus Models" :
- Unpublished notes, University of Warwick (1970).

<I> Plotkin, G.D. : "A Set-Theoretic Definition of Application" :
- Memorandum MIP-R-95, School of Artificial Intelligence University of Edinburgh (1972).

<J> Reynolds,J.C. : "Notes on a Lattice-Theoretic Approach to the Theory of Computation" :
- Systems and Information Science, Syracuse University (1972).

<K> Scott,D. : "Continuous Lattices" :
- Technical Monograph PRG-7, Oxford University Computing Laboratory, Programming Research Group (1970).

<L> Scott,D. : "Data Types as Lattices" :
- Unpublished Lecture Notes, Oxford (1973).

<M> Scott,D. : "Outline of a Mathematical Theory of Computation" :
- Technical Monograph PRG-2 (1970).

<N> Scott,D. : "The Lattice of Flow Diagrams" :
- Technical Monograph PRG-3 (1970).

<O> Scott,D. : "Twenty-One Exercises on Continuous Lattices" :
- Unpublished Lecture Notes, Princeton (1970).

<P> Wadsworth,C.P. : "Semantics and Pragmatics of the Lambda-Calculus" :
- Ph.D. Thesis, Oxford University (1971).

<Q> Wadsworth,C.P. : "Typed λ-Expressions" :
- Unpublished Notes, PRG (see <K> above) (1972).

<R> Vuillemin,J. : "Proof Techniques for Recursive Programs" :
- Note de Travail, Institute de Recherche d'Informatique et d'Automatique (I.R.I.A. - Laboria), 78-Rocquencourt, France (1973).

<S> Curry,H.B., Hindley,J.R. & Seldin,J.P. : "Combinatory Logic - Volume 2" :
- North-Holland, Amsterdam (1972).