Contributions to the

HOMOTOPY THEORY OF MAPPING SPACES

by

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PREFACE

This thesis contains work done at the University of Warwick in the period 1.10.69 - 15. 12. 1971 under the direction of Professor James Eells.

It is a great pleasure to express my deep gratitude to Professor Eells for, his stimulating interest in my work and for many enriching and joyful conversations on mathematics in general. It is fair to say that without his encouragement this thesis would have been finished a long time ago.

It is also a pleasure to thank Dr. David Elworthy for many valuable communications.

I would like to thank my wife here, but since she doesn't read this anyway, I have already done so.

Finally, I want to thank the University of Aarhus for financial support during my stay at the University of Warwick.

ABSTRACT

This thesis offers several contributions to the homotopy theory of mapping spaces. These contributions fall naturally into two groups. Accordingly the thesis is divided into two parts:

Part I : Spaces of continuous maps;

Part II : Spaces of differentiable maps.

Part I is mainly devoted to a study of the (path-) components in a given space of continuous maps. We show that a rich variety of homotopy types is possible thereby extending earlier fragmentary results of G.W.Whitehead, Hu and Koh in the same direction. As a new type of result we produce lots of examples of components which have the same homotopy type for non trivial reasons. In certain favourable cases, e.g. in a space of maps of a sphere into itself, we solve completely the homotopy problem for the set of components. The study of the components in a space of continuous maps is closely related to the study of certain evaluation fibrations. For these evaluation fibrations we obtain strong results on the fibre homotopy type. The methods to obtain the results mentioned above involve Whitehead products and in particular a fundamental theorem of G.W.Whitehead, which describes the boundary operator in the homotopy sequence of an evaluation fibration in terms of such products. In the final chapter of Part I, Chapter 4, we prove a variation of a theorem of Federer and, independently, Thom on the qualitative structure of the homotopy groups of a mapping space.

In Part II we study the homotopy properties of a space of differentiable embeddings or k-mersions from a compact smooth manifold into either an infinite dimensional smooth manifold or a closed expanding system of finite dimensional smooth manifolds of increasing dimension. Such a study is naturally motivated

by the work of Eells, Elworthy and Mukherjea according to which many infinite dimensional smooth manifolds are homotopy direct limits of closed expanding systems of finite dimensional smooth submanifolds of increasing dimension. Briefly stated, we show that a space of embeddings or k-mersions with infinite dimensional target, or the appropriate induced limit space over an expanding system of finite dimensional manifolds in the target has the same homotopy type as the naturally associated space of continuous maps. Using these results we prove that the functors we get in the target, when we fix the domain in a space of embeddings or k-mersions, "commute" with the homotopy direct limits given by the results of Eells, Elworthy and Mukherjea. As an application of these limit theorems we classify the k-mersions from a compact smooth manifold into a smooth (infinite dimensional) pseudo Fredholm manifold in the spirit of the Smale-Hirsch theory for immersions in finite dimensions. The class of pseudo Fredholm manifolds contains e.g. the class of separable Hilbert manifolds. We obtain hereby a strict analogue to a theorem in finite dimensions due to S.D.Feit and independently Gromov. As an application of our results for spaces of embeddings we construct models for the classifying space of a diffeomorphism group. Preliminary to all the limit arguments in Part II we study in Chapter 6 the limit space of a closed expanding system of manifolds or ANR's. We show that such a limit space usually is not metrizable, in particular it is not an ANR. It has however the homotopy type of an ANR. This result is crucial, since we in many places want to apply a fundamental theorem of J.H.C. Whitehead, according to which a weak homotopy equivalence between ANR's is a homotopy equivalence.

CONTENTS

Part II

Spaces of differentiable maps

INTRODUCTION

In this thesis we shall treat certain aspects of the homotopy theory of mapping spaces. We shall treat both spaces of continuous maps and spaces of differentiable maps. Since the methods and the results for the two cases are of a quite different nature, the thesis is naturally divided into two parts.

Before we enter into a description of the two parts separately let us discuss a little why there should be any interest at all in the topology, in particular the homotopy theory, of mapping spaces.

From an early stage of topology mapping spaces have received attention. The study of path spaces and loop spaces has occupied many topologists: already around 1930 path spaces entered into the study of geodesics on Riemannian manifolds in a significant way, notably through the work of M.Morse in America and Lusternik and Schnirelmann in Russia; spectral sequences (Leray, Serre) were invented for their study; Bott studied the double loop space of the stable unitary group in his original proof of the periodicity theorem; every H-space is homotopy equivalent to the loop space of its classifying space etc. Turning to more general mapping spaces we can mention that the study of the space of homotopy equivalences on a sphere is important e.g. in the study of spherical fibrations under fibre homotopy equivalence. Along the same lines knowledge about the structure of certain homeomorphism groups (diffeomorphism groups) find applications in questions concerning reduction of the structural group in certain locally trivial fibrations (differentiable fibrations). Many other examples could have been mentioned.

In the 1950's it was formally recognized that mapping spaces often have the structure of infinite dimensional manifolds and that they are the proper setting for variational problems, Morse theory on infinite dimensional manifolds. This development was mainly due to Eells, Sampson, Palais and Smale in America and Al'ber, Fet and Švarc in Russia. See e.g. the surveys of Eells [22] and

Al'ber [3]. Along with the successes in variational theory the fruitful interaction between the study of variational problems and the study of the topology of mapping spaces, which as already indicated had existed for a long time, was further stimulated. To mention an example of the use of topological results in variational theory let us take the theorem of Gromoll and Meyer [36], which states that if the space of circles (the free loop space) on a complete Riemannian manifold M has unbounded Betti numbers then M admits infinitely many geometrical distinct closed geodesics. Using the Serre spectral sequence this condition on the space of circles has been verified in many cases by P.Klein [54]. On the other hand methods from variational theory have given strong results on the topology of mapping spaces, in particular if the target is a manifold which admits a Riemannian metric with negative sectional curvature. See e.g. Eells and Sampson [27] and Al'ber[3].

The discovery that mapping spaces often have the structure of infinite dimensional manifolds led quite naturally to an abstract study of these manifolds. In this study it has turned out that for infinite dimensional manifolds the whole topological structure, homeomorphism type or in the differentiable case even diffeomorphism type, is completely determined by the homotopy type. See e.g. the contributions to the Nice congress by Anderson [4], Eells and Elworthy [26], Kuiper [59] and Palais [76].

There is therefore plenty of motivation to study the homotopy theory of mapping spaces and we can then safely turn to a description of the content of this thesis.

Part I consists of Chapters 1 - 4. Chapter 1 is of a preliminary character; we introduce the basic terminology, recall a fundamental theorem of G.W.Whitehead (Theorem 1.3.1) and state some facts about certain Whitehead products in spheres. The background for Part I can be set as follows. Let K and X be sufficiently nice connected topological spaces, e.g. ANR's or CW-complexes, with K locally compact. A homotopy of maps from K into X can then be described as a path in the space of continuous maps G(K,X) from K into X, equipped with the compact-open topology. We are therefore immediately led to study the (path-)components in a mapping space.

If K and X have base points the number of components in G(K,X) is given by the set of based homotopy classes of based maps $\pi(K,X)$. For $K = S^m$, the m-sphere, this simply is the m'th homotopy group $\pi_m(X)$ of X, and the structure of this group is well-known in many cases. If we e.g. take $X = S^n$, $\pi_n(S^n)$ is infinite cyclic so that $G(S^n,S^n)$ in this case has a countable number of components. It is now a typical problem in topology to divide such a collection of spaces into homotopy types. The main results in Part I will be concerned with problems of this nature. If $\alpha \in \pi(K,X)$ is a homotopy class, let $G_\alpha(K,X)$ denote the component in G(K,X) containing the maps in α. In particular $G_0(K,X)$ denotes the component containing the constant maps, and if $K = X = S^n$, $G_{\iota_n}(S^n,S^n)$ denotes the component containing the identity map. Here ι_n is the standard notation for the generator of $\pi_n(S^n)$ represented by the identity map on S^n. We shall then e.g. prove

<u>Theorem</u> (3.3.1) Consider the mapping spaces $G(S^n,S^n)$ for $n \geq 1$ and let $\alpha,\beta \in \pi_n(S^n)$.

<u>n even</u>. The components $G_\alpha(S^n,S^n)$ and $G_\beta(S^n,S^n)$ are homotopy equivalent if and only if $\alpha = \pm \beta$.

<u>n odd</u>. $G_\alpha(S^n,S^n)$ is homotype equivalent to $G_0(S^n,S^n)$ if deg α is even and to $G_{\iota_n}(S^n,S^n)$ if deg α is odd. Furthermore,

$G_0(S^n,S^n)$ and $G_{\iota_n}(S^n,S^n)$ are homotopy equivalent if and only if $n = 1,3,7$.

This theorem has a certain amount of history. In 1946 G.W.Whitehead [91] showed that $G_0(S^2,S^2)$ and $G_{\iota_2}(S^2,S^2)$ are not homotopy equivalent. This might have been some sort of a surprise at the time, since components in spaces of based maps do have the same homotopy type (Theorem 1.2.2). Also in 1946 S.T.Hu [47] showed that $\pi_1(G_\alpha(S^2,S^2)) \cong Z_{2\cdot|\deg\alpha|}$, the cyclic group of order $2\cdot|\deg\alpha|$, thus in particular proving the theorem for $n = 2$. After some work of a slightly different nature of Wada [89] in the 1950's, S.S.Koh [55] proved in 1960 the result in the theorem for $n = 2,4,6,8,10,12,14$. Koh also proved that $G_\alpha(S^n, S^n)$ and $G_\beta(S^n,S^n)$ have different homotopy type for n odd if $\deg\alpha - \deg\beta$ is odd and $n \neq 1,3,7$. The above theorem thus greatly extends these results and gives, in fact, the complete solution to the homotopy problem for the components in $G(S^n,S^n)$.

In the results of Whitehead, Hu and Koh the emphasis was to show that components have different homotopy type. In our theorem we see that it can happen that two components $G_\alpha(K,X)$ and $G_\beta(K,X)$ have the same homotopy type for non-trivial reasons. Trivial reasons being that K is a suspension and either $\alpha = \pm\beta$ (Theorem 1.2.1) or X is an H-space with a homotopy unit (Corollary 2.3.2). We believe therefore that this is the main contribution of our theorem.

The component $G_\alpha(K,X)$ is closely linked to the Hurewicz fibration $p_\alpha : G_\alpha(K,X) \to X$, also denoted $(G_\alpha(K,X), p_\alpha, X)$, defined by evaluation at the base point of K. Since the fibration has more structure than the component we can expect to prove stronger theorems about the former which might be helpful in the study of the latter. In Chapter 2 we study therefore these fibrations, called evaluation fibrations, in some detail. We touch upon the questions : 1) When do they have sections?, 2) When are they decomposable? 3) When are two evaluation fibrations fibre homotopy equivalent? and finally 4) When are two evaluation fibrations strongly fibre homotopy equivalent? Let us give examples of our theorems, often stated in a slightly less general form than in the text. Observe that $(G_o(K,X),\ p_o, X)$ trivially has a section.

<u>Theorem</u> (2.1.1 and 2.3.1). Suppose $m \geq n \geq 1$ and let $\alpha \in \pi_m(S^n)$. Then the following statements are equivalent:

1) $(G_\alpha(S^m,S^n),\ p_\alpha,\ S^n)$ and $(G_o(S^m,S^n),\ p_o,\ S^n)$ are fibre homotopy equivalent.

2) $(G_\alpha(S^m,S^n),\ p_\alpha,\ S^n)$ has a section.

3) The Whitehead product $[\alpha, \iota_n] = 0$.

Let $F_o(S^m,S^n)$ denote the fibre for the evaluation fibration $(G_o(S^m,S^n),\ p_o,\ S^n)$. $F_o(S^m,\ S^n)$ is then just the space of based maps homotopic to the constant map.

<u>Theorem</u> (2.2.3). Let $1 \leq k < n$ and $1 \leq m$ and suppose that the set of Whitehead products $[\pi_n(S^n),\ \pi_{k+m}(S^n)] \neq 0$.

Then $(G_o(S^m,S^n),\ p_o,\ S^n)$ is not decomposable, i.e. $G_o(S^m,S^n)$ does not have the same homotopy type as $S^n \times F_o(S^m,S^n)$. In particular $(G_o(S^m,S^n),\ p_o,\ S^n)$ is not fibre homotopically trivial.

The condition $[\pi_n(S^n),\ \pi_{k+m}(S^n)] \neq 0$ is satisfied so often that it could very well be that $(G_o(S^m,S^n),\ p_o, S^n)$ only is decomposable for $n = 1,3,7$. So far, I have however been unable to prove this.

<u>Theorem</u> (2.3.3) Let $m \geq n \geq 1$ and let $\alpha,\beta \in \pi_m(S^n)$.

If $[\alpha,\iota_n] \neq \pm[\beta,\iota_n]$ then $(G_\alpha(S^m,S^n),\ p_\alpha,\ S^n)$ and $(G_\beta(S^m,S^n),\ p_\beta, S^n)$ are fibre homotopy equivalent.

Here the author knows of no example where $[\alpha,\iota_n] \neq \pm [\beta,\iota_n]$ and $(G_\alpha(S^m,S^n), p_\alpha, S^n)$ and $(G_\beta(S^m,S^n),p_\beta,S^n)$ still are fibre homotopy equivalent.

If we introduce the notion of strong fibre homotopy equivalence (Definition 2.4.2) we have however

Theorem (2.4.3) Let $m \geqslant n \geqslant 1$ and let $\alpha \in \pi_m(S^n)$. Then we have

$(G_\alpha(S^m,S^n), p_\alpha, S^n)$ and $(G_\beta(S^m,S^n), p_\beta, S^n)$ are strongly fibre homotopy equivalent if and only if $[\alpha,\iota_n] = [\beta, \iota_n]$.

From all these theorems it is apparent that the Whitehead product $[\alpha,\iota_n]$ is a very strong characteristic class for the fibration $(G_\alpha(S^m,S^n), p_\alpha,S^n)$.

In Chapter 3 we study then the homotopy problem for the components in a space of maps between spheres. We need here the results from Chapter 2 as well as a fundamental result of G.W.Whitehead (Theorem 1.3.1), which describes the boundary operator in the homotopy sequence for the evaluation fibration $(G_\alpha(S^m,X),p_\alpha,X)$ in terms of Whitehead products. As an application we obtain the complete solution of the homotopy problem for the components in the mapping spaces $G(S^n,S^n)$ stated earlier in this introduction. Likewise we give the complete solution to the homotopy problem for the components in the mapping spaces $G(S^{n+1},S^n)$ (Theorem 3.3.2). As a particularly interesting thing it turns out that the countable number of components in $G(S^3,S^2)$ all have the same homotopy type. This will follow from the following characterization of the homotopy type of the component $G_o(S^m,S^n)$ among all the components in $G(S^m,S^n)$.

<u>Theorem</u> (3.2.1) Suppose $m \geq n \geq 1$ and let $\alpha \in \pi_m(S^n)$. Then the following statements are equivalent:

1) $G_\alpha(S^m,S^n)$ and $G_0(S^m,S^n)$ are homotopy equivalent.
2) The Whitehead product $[\alpha,\iota_n] = 0$.

In the final chapter of Part I, Chapter 4, we consider a different type of problem. We investigate here the qualitative structure of the homotopy groups of a mapping space from a finite CW-complex K into a space X. We shall prove the following variation of a theorem of Federer [32] and independently Thom [87].

<u>Theorem</u> (4.2.2) Let K be a finite CW-complex of dimension $\leq m$ and let X be an arbitrary topological space. Suppose that all the homotopy groups of X in dimension i with i in the interval $i_0 \leq i \leq i_0+m$ for $i_0 \geq 2$ belong to a given Serre class $\mathcal{C}$ of abelian groups.

Then all the homotopy groups of $G(K,X)$ in dimension i_0 belong to $\mathcal{C}$.

Federer and Thom do not need K to be a finite but only something like a finite dimensional CW-complex. On the other hand Federer needs X to be simple and both Federer and Thom need the integral cohomology groups of K to belong to $\mathcal{C}$. The formulation of our theorem is also slightly more general than theirs, and the proof we give is completely elementary. Thom proves his version of the theorem using a Postnikov decomposition of X. Federer gives his version as an application of the construction of a certain spectral sequence.

Our method of proof is closest to that of Federer. We use a filtration of K to construct a tower of Hurewicz fibrations in which we can identify the fibres with certain iterated loop spaces of the target. With this tower at our disposal the proof of the theorem is immediate.

Finally, in Chapter 4, §3 we give an example of a mapping space for which the fundamental group is infinitely generated although all the homotopy groups of the target are finitely generated.

We turn now to a description of Part II, which consists of Chapters 5 - 8.

The original motivation behind Part II was to extend the classification theorem for k-mersions from a compact smooth manifold M into a finite dimensional smooth manifold X, due to S.D. Feit and independently Gromov, to infinite dimensional X. Later many other sources of motivation transpired but it is still fairly descriptive to consider Part II from the point of view of this theorem. Let us therefore briefly recall the theorem of Feit and Gromov.

Let M be a compact smooth manifold with boundary (which might be empty), and let X be a finite dimensional metrizable smooth manifold without boundary. Let also $2 \leq r \leq \infty$ and $0 \leq k$. Denote by $C^r(M,X;k)$ the space of k-mersions of class C^r from M into X. Recall, that a k-mersion is a differentiable map of rank $\geq k$ everywhere. Denote by $Hom(TM,TX;k)$ the space of bundle maps from the tangent bundle of M into the tangent bundle of X of rank $\geq k$ on each fibre. Equip $C^r(M,X;k)$ with the C^r-topology and $Hom(TM,TX;k)$ with the compact-open topology.

<u>Theorem</u> Suppose that $k < \dim X$. Then the differential map

$$d: C^r(M,X;k) \to Hom(TM,TX;k)$$

is a homotopy equivalence.

This is, as already mentioned, a theorem of S.D.Feit [33] and independently Gromov [37]. It generalizes the Smale-Hirsch theory for immersions. See the survey by Smale [83] for the early history of the immersion theorem and the introduction to the paper of Feit for references to the later developments. An account of the work of Gromov can be found in the papers of Poénaru [78] and Haefliger [39].

Usually the statement in the theorem is just that d is a weak homotopy equivalence, but since M is compact both the mapping spaces are ANR's and hence a weak homotopy equivalence is automatically a homotopy equivalence by a fundamental theorem of J.H.C.Whitehead. The theorem with weak homotopy equivalence is also true for M an open manifold without the restriction on k.

Although the main trouble in proving the theorem in finite dimensions lies on the domain, the local compactness of the target is used in the essential step. Hence the finite dimensional proof does not immediately generalize to infinite dimensional targets. Since Mukherjea ([69] or [70]) has shown that many infinite dimensional manifolds up to homotopy type are the direct limits of closed expanding systems of finite dimensional submanifolds, it is natural to try to use a limit argument to extend the theorem of Feit and Gromov from finite to infinite dimensional targets. To carry this program through and in particular to investigate how mapping spaces like $C^r(M,X;k)$ and $Hom(TM,TX;k)$ behave w.r.t. homotopy direct limits in the target is just what Part II is all about.

Let us now describe the content of Part II in a little more detail. Chapter 5 contains the basic definitions, some preliminary material on ANR's and a discussion of the manifold structure on a space of maps. We state the fundamental theorem of J.H.C.Whitehead (Theorem 5.2.4) already mentioned above, and we give in Theorem 5.3.2 a slight extension of a theorem of Palais, according to which the homotopy type of a space of differentiable maps does not depend on the degree of differentiability.

As an introduction to all our limit arguments in later chapters we study in Chapter 6 the limit space of a CES (closed expanding system). Basically a CES is just a sequence of inclusions of topological spaces

$$X_{n_0} \subset X_{n_0+1} \subset \dots \subset X_n \subset \dots,$$

such that X_n is a closed subspace of X_{n+1} for each $n \geq n_0$. The limit space for such a system, denoted by $\underline{X}_\infty$, is defined as the union

$$\underline{X}_\infty = \bigcup_{n \geq n_0} X_n$$

with the weak topology (direct limit topology) w.r.t. the subspaces X_n.

If R^∞ denotes the topological vector space of finitely non-zero real sequences $(x_n)_{n \geq 1}$ topologized with the finite topology (the weak topology w.r.t. the system of finite dimensional subspaces), then we can state the main theorem in Chapter 6 as follows.

<u>Theorem</u> (6.2.5) Let $X_{n_0} \subset X_{n_0+1} \subset \dots \subset X_n \subset \dots$ be a CES of finite dimensional topological manifolds of increasing dimension, such that X_n is a locally flat submanifold of X_{n+1} for each $n \geq n_0$.

Then $\underline{X}_\infty$ is a topological manifold modelled on R^∞.

Notice here, that we always assume manifolds are Hausdorff spaces. The statement in Theorem 6.2.5. includes therefore that $\underline{X}_\infty$ is a Hausdorff space. Usually $\underline{X}_\infty$ will, in fact, be paracompact (Remark 6.2.7).

Since R^∞ is not metrizable, $\underline{X}_\infty$ is not metrizable, in particular it is not an ANR for the class of metrizable spaces.

Theorem 6.2.5 should therefore be compared with

Theorem (Corollary 6.3.4) Let $X_{n_o} \subset X_{n_o+1} \subset \cdots \subset X_n \subset \cdots$ be a CES of ANR's.

Then $\underline{X}_\infty$ has the homotopy type of an ANR.

It is only the results in §3 of Chapter 6, and mainly Corollary 6.3.4, which will be used in Chapters 7 and 8. Corollary 6.3.4 shows, that we will get no trouble from limit spaces of ANR's when we want to apply the theorem of J.H.C. Whitehead (Theorem 5.2.3). Trouble might have been expected after the result in Theorem 6.2.5.

We shall now describe the material in Chapters 7 and 8. Again let M be a compact smooth manifold with boundary, but now let X be a finite or infinite dimensional metrizable smooth manifold, which admits smooth partititions of unity. X shall be without boundary. For $0 \leq r \leq \infty$ let $C^r(M,X)$ denote the space of differentiable maps of class C^r. For $2 \leq r \leq \infty$ and $0 \leq k$ let $\mathcal{J}^r(M,X)$ denote either the space of k-mersions $C^r(M,X;k)$ or the space of embeddings $Emb^r(M,X)$ of class C^r. Equip all these spaces with the C^r-topology. Observe that $C^r(M,X;0) = C^r(M,X)$ and that the space of continuous maps in this context is denoted by $C^o(M,X)$. As before $Hom(TM,TX;k)$ denotes the space of bundle maps of rank $\geq k$ on each fibre equipped with the compact-open topology.

Suppose from now on that X is infinite dimensional and that it admits a filtration of finite dimensional smooth submanifolds of increasing dimension

$$X_{n_o} \subset X_{n_o+1} \subset \cdots \subset X_n \subset \cdots,$$

each a closed submanifold in the next, such that the natural map $\underline{X}_\infty \to X$ is a homotopy equivalence. Shortly, we say that X is a smooth HDL (homotopy direct limit) of the smooth CES $X_{n_o} \subset X_{n_o+1} \subset \cdots \subset X_n \subset \cdots$. We get then the following naturally induced CES's

$$C^r(M,X_{n_o}) \subset C^r(M,X_{n_o+1}) \subset \dots \subset C^r(M,X_n) \subset \dots$$

$$\mathfrak{F}^r(M,X_{n_o}) \subset \mathfrak{F}^r(M,X_{n_o+1}) \subset \dots \subset \mathfrak{F}^r(M,X_n) \subset \dots$$

and $\mathrm{Hom}(TM,TX_{n_o};k) \subset \mathrm{Hom}(TM,TX_{n_o+1};k) \subset \dots \subset \mathrm{Hom}(TM,TX_n;k) \subset \dots$

Let $C^r(M,\underline{X})_\infty$, $\mathfrak{F}^r(M,\underline{X})_\infty$ and $\mathrm{Hom}(TM,T\underline{X};k)_\infty$ denote the limit spaces for these systems.

The major part of Chapter 7 and Chapter 8, §2 can be summarized in this

<u>Theorem</u> All the maps in the following commutative diagram of naturally induced maps are homotopy equivalences. To each map we have attached the number of the appropriate theorem in the text

$$\begin{array}{lccccl}
 & \mathfrak{F}^r(M,\underline{X})_\infty & \xrightarrow{8.2.3} & \mathfrak{F}^r(M,X) & \\
7.4.1 & \downarrow & & \downarrow & 7.2.2 \\
 & C^r(M,\underline{X})_\infty & \xrightarrow{8.2.2} & C^r(M,X) & \\
7.4.2 & \downarrow & & \downarrow & 5.3.2 \text{ (Palais)} \\
 & C^0(M,\underline{X})_\infty & \xrightarrow[8.2.1]{} & C^0(M,X) &
\end{array}$$

On our way to establish the results indicated in this diagram we obtain many results of independent interest. We mention in particular Theorem 7.2.1, which generalizes the classical immersion and embedding theorems of Whitney to statements about the connectivities of the inclusion maps $C^r(M,X;k) \to C^r(M,X)$ and $\mathrm{Emb}^r(M,X) \to C^r(M,X)$. In Chapter 7, §5 we use our results on spaces of embeddings to construct models for the classifying space of the diffeomorphism group of M.

In Chapter 8, §3 we study the behaviour of the bundle map functor $\mathrm{Hom}(TM, \bullet\, ; k)$ w.r.t. direct limits. Suppose again that X is a smooth HDL of the smooth CES $X_{n_o} \subset X_{n_o+1} \subset \dots \subset X_n \subset \dots$

of finite dimensional manifolds of increasing dimension, but suppose now also, that

1) $\bigcup_{n \geq n_0} X_n$ is dense in X.

2) If $x \in X_{n(x)}$ then the union of tangent spaces $\bigcup_{n \geq n(x)} T_x X_n$ is dense in the tangent space $T_x X$ of X at $x \in X$.

A CES $X_{n_0} \subset X_{n_0+1} \subset \ldots \subset X_n \subset \ldots$ with these properties is called a finite dimensional presentation of X. If X admits a finite dimensional presentation, we say that X is a pseudo Fredholm manifold. In Example 8.1.6 we indicate that a lot of infinite dimensional smooth manifolds, e.g. all separable Hilbert manifolds, are pseudo Fredholm manifolds.

Theorem (8.3.1) Let $X_{n_0} \subset X_{n_0+1} \subset \ldots \subset X_n \subset \ldots$ be a finite dimensional presentation of the smooth pseudo Fredholm manifold X. Then the natural map

$$\mathrm{Hom}\,(TM, T\underline{X}; k)_\infty \to \mathrm{Hom}(TM, TX; k)$$

is a homotopy equivalence.

Using Theorem 8.2.3 and Theorem 8.3.1 the following theorem is an easy consequence of the theorem of Feit and Gromov.

Theorem (8.4.1) Let M^m be a compact smooth manifold and let X be a smooth pseudo Fredholm manifold. Let also $0 \leq k \leq m$ and $2 \leq r \leq \infty$.

Then the differential map

$$d\colon C^r(M, X; k) \to \mathrm{Hom}(TM, TX; k)$$

is a homotopy equivalence.

One could now ask why we should try to extend the theorem of Feit and Gromov to infinite dimensional targets as above. Well, first of all it is one of the outstanding theorems in differential topology and as such it deserves attention in its own right. Secondly, infinite dimensional manifolds arise quite naturally in the target by iteration of mapping spaces. Suppose e.g. that X is a finite dimensional smooth manifold and let L_r^2 (M,X) be the separable Hilbert manifold of Sobolev maps ($r > \frac{\dim M}{2}$ in order to get the manifold structure, see Eells [22]). Then we could naturally form $L_s^2(M,L_r^2(M,X);k)$, which is homotopy equivalent to $C^s(M,C^r(M,X);k)$. Thirdly, the bare question of the existence of the theorem with infinite dimensional target proved useful to us in stimulating us to develop a limit technique for mapping spaces, which hopefully can have other applications.

Finally, we mention that Part II to a large extent is based on our papers [40], [41] and [42].

PART I

SPACES OF CONTINUOUS MAPS

Chapter 1

Preliminaries to Part I.

In this chapter we introduce the basic terminology used in Part I and construct certain maps relating various components in a mapping space. Finally, we state for later reference a fundamental theorem of G.W.Whitehead and recall certain facts about Whitehead products in spheres.

§1 Basic Definitions.

In Part I K and X shall always denote sufficiently nice topological spaces. Unless otherwise said we will for simplicity stick to ANR's (Absolute Neighbourhood Retracts) or CW-complexes. K and X shall be connected and in addition K is assumed to be locally compact. All spaces shall be equipped with a base point, indiscriminately denoted *, although it might not come into play. Often K = SA and X = SB are assumed to be (reduced) suspensions of ANR's (CW-complexes) A and B. Points in a suspension will be denoted by their coordinates, like $[a,t] \in SA$.

$\pi(K,X)$ shall denote the set of based homotopy classes of based maps from K into X. As usual we say that X is simple w.r.t. K (m-simple, if $K = S^m$ is the m-sphere), if the fundamental group $\pi_1(X)$ acts trivially on $\pi(K,X)$. It is well-known that simply connected spaces and H-spaces are simple w.r.t. any space. Spaces with this property are just called simple. We remark that spheres are simple. If $\alpha \in \pi(K,X)$ is a homotopy class, $f_\alpha : K \to X$ shall denote a representative for α.

F(K,X) shall denote the mapping space of based maps from K into X. G(K,X) denotes the mapping space of free maps, no restriction on base points, from K into X. Both F(K,X) and G(K,X) are equipped with the compact-open topology.

The path-components in F(K,X) are exactly the based homotopy classes of based maps, and the path-components in $G(K,X)$ are the free homotopy classes of free maps. Path-components will from now on just be called components. If $\alpha \in \pi(K,X)$ is an arbitrary homotopy class $F_\alpha(K,X)$ and $G_\alpha(K,X)$ shall denote the component in respectively $F(K,X)$ and $G(K,X)$ which contains the maps in α. Notice that we also get all the components in $G(K,X)$ listed that way since any free map is freely homotopic to a based map.

It is well-known that evaluation at the base point of K defines a Hurewicz fibration $p: G(K,X) \to X$, i.e. p has the absolute covering homotopy property. See e.g. Hu ([48], Theorem 13.1, p. 83). Observe that $F(K,X)$ is the fibre of p. Now let $\alpha \in \pi(K,X)$. It is then clear that the restriction of p to $G_\alpha(K,X)$ defines a Hurewicz fibration $p_\alpha: G_\alpha(K,X) \to X$, also denoted $(G_\alpha(K,X), p_\alpha, X)$. Let $\overline{F}_\alpha(K,X)$ denote the fibre of p_α. $\overline{F}_\alpha(K,X)$ consists then of the based maps which are freely homotopic to a map in α. It is clear that $\overline{F}_\alpha(K,X)$ contains $F_\alpha(K,X)$ but normally $\overline{F}_\alpha(K,X)$ will contain several of the components $F_\beta(K,X)$ for $\beta \in \pi(K,X)$. Only if X is simple w.r.t. K, e.g. a sphere, we have $\overline{F}_\alpha(K,X) = F_\alpha(K,X)$.

The Hurewicz fibration $(G_\alpha(K,X), p_\alpha, X)$ will be called the <u>evaluation fibration</u> defined by $\alpha \in \pi(K,X)$.

The homotopy class in $\pi(K,X)$ containing the constant map will be denoted 0, and for obvious reasons, anyway if $K = SA$ is a suspension, 0 will be called the <u>neutral element</u>. Accordingly $F_0(K,X)$ and $G_0(K,X)$ will be called the <u>neutral component</u> in respectively $F(K,X)$ and $G(K,X)$, and $(G_0(K,X), p_0, X)$ will be called the <u>neutral evaluation fibration</u>.

Finally, $s_0: X \to G_0(K,X)$ shall denote the <u>canonical section</u> for p_0, i.e. $s_0(y)(x) = y$ for every $x \in K$ and every $y \in X$. As we shall see later on the existence of a section is a very special property for the fibration $(G_0(K,X), p_0, X)$.

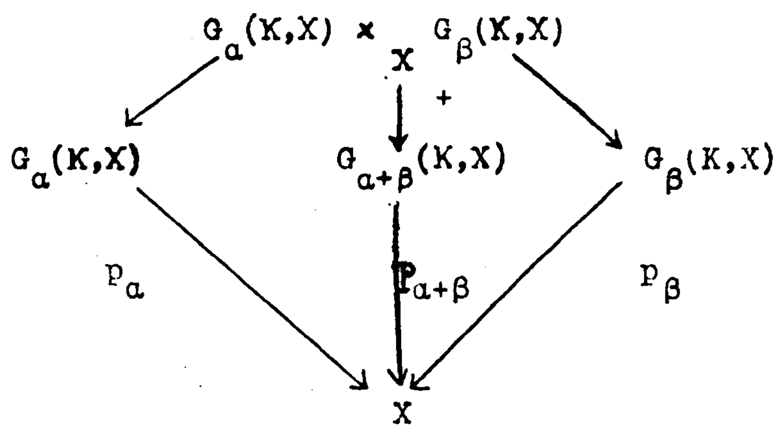
§2 Evaluation fibrations defined by a space of mappings with a suspension as domain.

In this section we assume that $K = SA$ is a suspension. Then $\pi(K,X)$ has a natural group structure defined by the standard formulas involving the suspension parameter. Using the suspension parameter in an analogeous way we shall define two maps relating the various evaluation fibrations defined by K and X.

For $\alpha, \beta \in \pi(K,X)$ let $G_\alpha(K,X) \times_X G_\beta(K,X)$ denote the fibre product of the evaluation fibrations defined by α and β. We shall then define a map

$$+ : G_\alpha(K,X) \times_X G_\beta(K,X) \to G_{\alpha+\beta}(K,X)$$

such that the following diagram is commutative



If $(f,g) \in G_\alpha(K,X) \times_X G_\beta(K,X)$ let $(f + g) \in G_{\alpha+\beta}(K,X)$ denote the image under + and define $f + g$ by the standard formula

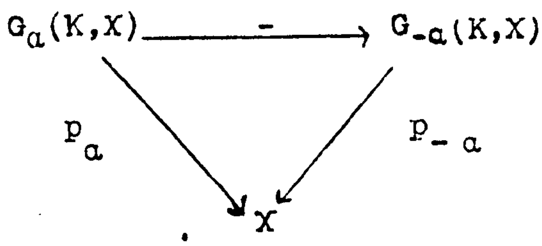
$$(f + g)([a,t]) = \begin{cases} f([a,2t]) & a \in A,\ 0 \leq t \leq \frac{1}{2} \\ g([a,2t-1]) & a \in A,\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

This formula makes sense since $f(*) = g(*)$.

For $\alpha \in \pi(K,X)$ we define a map

$-: G_\alpha(K,X) \to G_{-\alpha}(K,X)$

such that the following diagram is commutative



If $f \in G_\alpha(K,X)$ define $(-f) \in G_{-\alpha}(K,X)$ by the standard formula

$$\forall a \in A \quad \forall t \in [0,1] : \quad (-f)([a,t]) = f([a,1-t])$$

It is clear that $-: G_\alpha(K,X) \to G_{-\alpha}(K,X)$ is a bundle homeomorphism with $-: G_{-\alpha}(K,X) \to G_\alpha(K,X)$ as inverse.

We get therefore in particular

<u>Theorem 1.2.1</u> Suppose that $K = SA$ is a suspension and let $\alpha \in \pi(K,X)$. Then $G_\alpha(K,X)$ and $G_{-\alpha}(K,X)$ are homeomorphic.

Finally in this section we shall give an example of how the operations + and - will be brought into use.

The following theorem is a slight generalization of a result due to G.W.Whitehead ([91], p. 464)

<u>Theorem 1.2.2</u> Suppose that $K = SA$ is a suspension. Then $F_\alpha(K,X)$ and $F_0(K,X)$ have the same homotopy type for any $\alpha \in \pi(K,X)$.

<u>Proof</u> Let $f_\alpha : K \to X$ be a representative for $\alpha \in \pi(K,X)$ and define maps

$$\varphi: F_0(K,X) \to F_\alpha(K,X) \text{ and } \psi: F_\alpha(K,X) \to F_0(K,X)$$

by the formulas

$$\forall f \in F_0(K,X) : \varphi(f) = f_\alpha + f$$
$$\forall g \in F_\alpha(K,X) : \psi(g) = (-f_\alpha) + g$$

Using standard formulas in homotopy theory it is easy to prove that φ is a homotopy equivalence with inverse ψ. See either ([91], p. 464) or Chapter 2, §3 for similar proofs.

§3 A fundamental theorem of G.W.Whitehead and some facts about Whitehead products in spheres.

For the sake of future reference we state first a fundamental theorem of G.W.Whitehead [91] (with a correction by J.H.C.Whitehead [94]). This theorem of G.W.Whitehead will be of fundamental importance to us in Chapter 2, §4 and in most of Chapter 3.

Theorem 1.3.1 Let $m \geq 1$ and let $\alpha \in \pi_m(X)$ be a given homotopy class with f_α as representative. For any $i \geq 1$ we have then a commutative diagram

$$\begin{array}{ccccccc} \cdots \to \pi_{i+1}(X,*) & \xrightarrow{\partial_\alpha} & \pi_i(\overline{F}_\alpha(S^m,X),f_\alpha) & \to & \pi_i(G_\alpha(S^m,X),f_\alpha) & \to & \cdots \\ & \underset{\rho_\alpha}{\searrow} & \cong \downarrow H_\alpha & & & & \\ & & \pi_{m+i}(X,*), & & & & \end{array}$$

where ∂_α is the boundary operator in the exact homotopy sequence for the evaluation fibration defined by α, H_α is the Hurewicz isomorphism and $-\rho_\alpha$ is Whitehead product with α, i.e. $\rho_\alpha(\beta) = -[\alpha,\beta]$ for every $\beta \in \pi_{i+1}(X,*)$.

Normally X will be m-simple, such that $\overline{F}_\alpha(S^m,X) = F_\alpha(S^m,X)$. In this case all spaces involved in Theorem 1.3.1 will be connected and we do not need to specify base points in the homotopy groups.

In order to give concrete applications of some of our results in later chapters we need to know certain facts about Whitehead products in spheres. We collect the facts we need here.

First we recall the definition of certain standard elements in homotopy groups of spheres. $\iota_n \in \pi_n(S^n)$ for $n \geq 1$ is the element represented by the identity map on S^n. $\eta_n \in \pi_{n+1}(S^n)$ for $n \geq 2$, $\nu_n \in \pi_{n+3}(S^n)$ for $n \geq 4$ and $\sigma_n \in \pi_{n+7}(S^n)$ for $n \geq 8$

denote the elements represented by suspensions of the Hopf maps $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ respectively. We remark, that η_n generates $\pi_{n+1}(S^n)$.

What we need is facts about the Whitehead products $[\alpha_n, \iota_n]$ for $\alpha_n = \iota_n, \eta_n$ and σ_n.

Theorem 1.3.2. Consider the Whitehead products $[\iota_n, \iota_n]$ for $n > 1$.

1) n even. $[\iota_n, \iota_n]$ has infinite order in $\pi_{2n-1}(S^n)$.

2) n odd. $2 \cdot [\iota_n, \iota_n] = 0$. Furthermore, $[\iota_n, \iota_n] = 0$ if and only if $n = 1,3,7$.

The result in Theorem 1.3.2 for n even and the fact that $2 \cdot [\iota_n, \iota_n] = 0$ for n odd is due to G.W.Whitehead [92], strengthening results of Freudenthal. It is part of the so-called delicate suspension theorem. That $[\iota_n, \iota_n] = 0$ if and only if $n = 1,3,7$ is a consequence of Adam's solution of the Hopf invariant 1 problem [2].

Compiling results of Hilton [45], Hilton and J.H.C.Whitehead [46] and Mahowald [63] we get

Theorem 1.3.3 The Whitehead product $[\eta_n, \iota_n] = 0$ if and only if $n = 2,6$ or $n \equiv 3 \bmod 4$.

The following theorem is a compilation of results of Mahowald [63] and Kristensen and Madsen [57].

Theorem 1.3.4. Consider the Whitehead products $[\sigma_n, \iota_n]$ for $n > 8$.

1) $[\sigma_n, \iota_n] = 0$ for $n = 11$ and $n \equiv 15 \bmod 16$

2) $[\sigma_n, \iota_n] \neq 0$ for $n \neq 11,27$ and $n \not\equiv 15 \bmod 16$.

It is presumably still an open question whether $[\sigma_{27}, \iota_{27}] = 0$ or $\neq 0$.

Several other results on Whitehead products $[\alpha_n, \iota_n]$ for certain $\alpha_n \in \pi_m(S^n)$ and a more detailed list of known results on these products can be found in Kristensen and Madsen [57].

Chapter 2

Evaluation fibrations.

In this chapter we study evaluation fibrations. We touch upon the questions: 1) When do they have sections?, 2) When is a neutral evaluation fibration decomposable?, 3) When are two evaluation fibrations fibre homotopy equivalent?, and finally, 4) When are two evaluation fibrations strongly fibre homotopy equivalent?

For the case of spheres it will emerge that the Whitehead product $[\alpha, \iota_n]$ is a very strong characteristic class (in some sense) for the evaluation fibration defined by $\alpha \in \pi_m(S^n)$.

§1 Sections in evaluation fibrations.

We recall first the definition of generalized Whitehead products.

Let SA and SB be suspensions and let Y be an arbitrary space. Let CA and CB denote the (reduced) cones on A and B and define then the join $A*B$ of A and B by $A*B = CA \times B \cup A \times CB$. Define in the obvious way a map $W: A*B \to SA \vee SB$ from the join $A*B$ into the wedge $SA \vee SB$. W is usually called the Whitehead map. Since we are working with ANR's (CW-complexes), it is well-known that the mapping cone C_W for W is homotopy equivalent to $SA \times SB$. See e.g. [6] and [81] for this fact.

Suppose now that we are given homotopy classes $\alpha \in \pi(SA,Y)$ and $\beta \in \pi(SB,Y)$ represented by respectively $f_\alpha: SA \to Y$ and $f_\beta : SB \to Y$. Consider then the composite map

$$A*B \xrightarrow{W} SA \vee SB \xrightarrow{f_\alpha \vee f_\beta} Y \vee Y \xrightarrow{\triangledown} Y,$$

where $\triangledown$ is the folding map. The homotopy class of this map depends only on α and β and is denoted by $[\alpha, \beta]$. $[\alpha, \beta] \in \pi(A*B,Y)$ is called the generalized Whitehead product of α and β.

Denote by $\iota_X \in \pi(X,X)$ the homotopy class represented by the identity map of X. Then we have

<u>Theorem 2.1.1.</u> Suppose that $K = SA$ and $X = SB$ are suspensions and let $\alpha \in \pi(K,X)$. Then the following statements are equivalent:

1) The evaluation fibration $(G_\alpha(K,X),\ p_\alpha, X)$ has a section.

2) The generalized Whitehead product $[\alpha,\iota_X] \in \pi(A*B,X)$ is zero.

<u>Proof</u> We observe first that the generalized Whitehead product $[\alpha, \iota_X]$ is zero if and only if there exists a map $F_\alpha : K \times X \to X$ such that the following diagram is commutative

$$\begin{array}{ccccc} K \times X & & \xrightarrow{\quad F_\alpha \quad} & & \\ \uparrow & & & & \searrow \\ K \vee X & \xrightarrow{f_\alpha \vee 1_X} & X \vee X & \xrightarrow{\nabla} & X \end{array}$$

Since $F_\alpha(\cdot,x)$ is homotopic to f_α for any $x \in X$ (one gets a homotopy by moving x along a path to the wedge point) it follows easily that such a map F_α defines a section $s_\alpha : X \to G_\alpha(K,X)$ and conversely, when s_α and F_α are related by the formula

$$\forall y \in K \quad \forall x \in X \quad : s_\alpha(x)(y) = F_\alpha(y,x)$$

This proves the theorem.

Let us finish this section with just one application.

Let $n \geq 1$. Since the Whitehead product $[\iota_n,\iota_n] = 0$ if and only if $n = 1,3,7$ we get immediately

<u>Corollary 2.1.2</u> The evaluation fibration $(G_{\iota_n}(S^n,S^n), p_{\iota_n}, S^n)$ has a section if and only if $n = 1,3,7$.

§2 The structure of a neutral evaluation fibration.

Let $F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$ be a Serre fibration (p has the covering homotopy property w.r.t. polyhedra) with section s. Following James [51] we will say that this fibration is <u>decomposable</u> if E and $B \times F$ have the same homotopy type. Observe that this is the case, of course, when the fibration is fibre homotopically trivial.

In this section we shall investigate the class of neutral evaluation fibrations w.r.t. the notion of decomposability.

First we mention

<u>Theorem 2.2.1</u> Suppose that X is an H-space with a strict unit element. Then the neutral evaluation fibration $(G_0(K,X), p_0, X)$ is fibre homotopically trivial, in particular decomposable, for any K.

<u>Proof</u> Suppose that the base point $* \in X$ is a unit element for the multiplication on X. For any $x \in X$ let L_x be left multiplication with x. Define then a map θ over X,

$$\begin{array}{ccc} X \times F_0(K,X) & \xrightarrow{\theta} & G_0(K,X) \\ & \underset{\text{proj.}}{\searrow} \quad \underset{p_0}{\swarrow} & \\ & X & \end{array}$$

by the formula

$$\forall x \in X \quad \forall f \in F_0(K,X) \; : \; \theta\,(x,f) = L_x \circ f.$$

It is clear, that θ is the identity map on the fibre over $* \in X$, in particular a homotopy equivalence on that fibre. Hence θ is a fibre homotopy equivalence by a fundamental theorem of Dold ([14] , Theorem 6.3). This proves the theorem.

The main object of this section is to show that the neutral evaluation fibration

$$F_0(S^m,X) \xrightarrow{i_0} G_0(S^m,X) \underset{s_0}{\overset{p_0}{\rightleftarrows}} X$$

very seldom is decomposable.

Observe here, that if X is simple, then $G_o(S^m,X)$ and $X \times F_o(S^m,X)$ have the same homotopy groups. This follows easily, since the section s_o splits the homotopy sequence for the fibration. We need (m+1)-simplicity of X to get the statement for π_1, see Eilenberg [28].

The results in this section rely heavily on the following theorem of Federer ([32], §11, p 356).

Theorem 2.2.2 Let $p \geq 1$, $q \geq 1$ and $m \geq 1$ and suppose that the set of Whitehead products $[\pi_p(X), \pi_{q+m}(X)] \neq 0$.

Then the set of Whitehead products

$$[\pi_p(G_o(S^m,X)), \pi_q(G_o(S^m,X))] \neq 0.$$

Using this theorem we can easily prove.

Theorem 2.2.3 Let $1 < k < n$ and $1 \leq m$ and suppose that the set of Whitehead products $[\pi_n(S^n), \pi_{k+m}(S^n)] \neq 0$.

Then the neutral evaluation fibration $(G_o(S^m,S^n),p_o,S^n)$ is not decomposable.

Proof By Federer's theorem the set of Whitehead products $[\pi_n(G_o(S^m,S^n)), \pi_k(G_o(S^m,S^n))] \neq 0$.

On the other hand the set of Whitehead products $[\pi_n(S^n \times F_o(S^m,S^n)), \pi_k(S^n \times F_o(S^m,S^n))] = 0$.

This is so, since $\pi_k(S^n) = 0$ and since all Whitehead products in $F_o(S^m,S^n)$ vanish, $F_o(S^m,S^n)$ being an H-space.

$G_o(S^m,S^n)$ and $S^n \times F_o(S^m,S^n)$ must therefore have different homotopy type and the theorem is proved.

We give now some applications of Theorem 2.2.3. Before the first one observe that for $1 \leq m < n$ $G_o(S^m,S^n) = G(S^m,S^n)$.

Corollary 2.2.4 Let $1 \leq m < n$. Then the neutral evaluation fibration $(G_o(S^m,S^n), p_o,S^n)$ is decomposable if and only if $n = 3,7$.

For $n = 3,7$ the fibration is even fibre homotopically trivial.

Proof For $n = 3,7$ S^n is an H-space with a strict unit and hence $(G_o(S^m,S^n), p_o,S^n)$ is fibre homotopically trivial by Theorem 2.2.1.

For $n \neq 3,7$ $[\iota_n, \iota_n] \neq 0$ and hence the set of Whitehead products $[\pi_n(S^n), \pi_{(n-m)+m}(S^n)] \neq 0$. Therefore $(G_o(S^m,S^n), p_o, S^n)$ is not decomposable for $n \neq 3,7$ by Theorem 2.2.3.

Corollary 2.2.5 Consider the neutral evaluation fibrations $(G_o(S^n,S^n), p_o,S^n)$ for $1 \leq n$.

1) For $n = 1,3,7$ $(G_o(S^n,S^n),p_o,S^n)$ is fibre homotopically trivial.

2) For $n \geq 8$, $n \neq 11,27$ and $n \not\equiv 15 \bmod 16$ $(G_o(S^n,S^n), p_o, S^n)$ is not decomposable.

Proof 1) Follows from Theorem 2.2.1.

For $n \geq 8$ let $\sigma_n \in \pi_{n+7}(S^n)$ be the element represented by suspensions of the Hopf map $S^{15} \to S^8$. Then it is known, see Theorem 1.3.4, that $[\iota_n, \sigma_n] \neq 0$ for $n \neq 11, 27$ and $n \not\equiv 15 \bmod 16$. Hence for these values of n, we get that $[\pi_n(S^n), \pi_{7+n}(S^n)] \neq 0$. 2) follows now immediately from Theorem 2.2.3.

Let us briefly make some remarks on the fibration $(G_o(S^n,S^n), p_o,S^n)$ for $n \leq 8$. For $n = 1,3,7$ it is fibre homotopically trivial. For $n = 4,5$ it is not decomposable. This can be shown using the Whitehead product $[\iota_n, \eta_n]$, where η_n is the generator of $\pi_{n+1}(S^n)$. For $n = 2,6$ the author does not know whether it is decomposable or not.

Using Theorem 2.3.3 and all known information on Whitehead products we can prove that $(G_o(S^m, S^n), p_o, S^n)$ is not decomposable in many special cases. We shall abstain from this since it is unlikely that we can get the complete solution to the following problem using this method.

<u>Problem</u> Let $1 \leq n \leq m$. Determine when $(G_o(S^m,S^n), p_o, S^n)$ is decomposable.

It is a tempting conjecture to suggest that $(G_o(S^m,S^n),p_o,S^n)$ is decomposable if and only if $n = 1,3,7$, but I have no evidence except for the results above.

Finally in this section we touch upon the notion of brace products introduced by James in e.g. [50] and [51]. First we recall the definition. Let $F \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} B$ be a Serre fibration with section s. Choose base points in F,E and B such that they are preserved by the maps i,p and s. Now let $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$ with $p \geq 1$ and $q \geq 1$ be given. Form the Whitehead product $[s_*(\alpha), i_*(\beta)] \in \pi_{p+q-1}(E)$. Since $p_*([s_*(\alpha), i_*(\beta)]) = 0$ $(p_*s_* = 0)$ and since i_* is a monomorphism (the section s induces a splitting of the homotopy sequence of the fibration) there exists a unique element $\{\alpha, \beta\} \in \pi_{p+q-1}(F)$ such that $i_*(\{\alpha,\beta\}) = [s_*(\alpha), i_*(\beta)]$

$\{\alpha,\beta\}$ is called the <u>brace product</u> of α and β.

For a space X with base point $* \in X$ and for $m \geq 1$ consider $\bar{F}_o(S^m,X)$ and $G_o(S^m,X)$ with the constant map with value $* \in X$ as base point. Furthermore, let

$$H : \pi_i(\bar{F}_o(S^m,X)) \to \pi_{i+m}(X)$$

denote the Hurewicz isomorphism.

Examination of Federer's proof ([32], §11, p.356) shows that we can restate his theorem (Theorem 2.2.2) as follows:

<u>Theorem</u> Let $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(\bar{F}_o(S^m,X))$ and suppose that the Whitehead product $[\alpha,H(\beta)] \neq 0$. Then the brace

product $\{\alpha,\beta\} \neq 0$.

Since we know of no counter-example we pose the following

Question Is a formula like

$$H(\{\alpha,\beta\}) = [\alpha,H(\beta)]$$

true?

§3 Fibre homotopy equivalence of evaluation fibrations.

In this section we study evaluation fibrations defined by spaces K and X under the notion of fibre homotopy equivalence.

First we give a criterion for an arbitrary evaluation fibration to be fibre homotopy equivalent to the neutral evaluation fibration.

Theorem 2.3.1 Suppose that $K = SA$ is a suspension and let $\alpha \in \pi(K,X)$. Then the following statements are equivalent:

1) $(G_\alpha(K,X), p_\alpha, X)$ and $(G_o(K,X), p_o, X)$ are fibre homotopy equivalent.

2) $(G_\alpha(K,X), p_\alpha, X)$ has a section.

In case $X = SB$ is also a suspension we can add

3) The generalized Whitehead product $[\alpha, \iota_X] \in \pi(A*B,X)$ is zero.

Proof The equivalence of 1) and 2) is a slight generalization of a theorem of G.W.Whitehead ([91], p.464).

1) $\Rightarrow$ 2) is trivial since $(G_o(K,X), p_o, X)$ has a section.

Now we prove 2) $\Rightarrow$ 1). Suppose therefore that $s_\alpha : X \to G_\alpha(K,X)$ is a section for p_α and define maps

$$\varphi : G_o(K,X) \longrightarrow G_\alpha(K,X)$$

and

$$\psi : G_\alpha(K,X) \longrightarrow G_o(K,X)$$

by the formulas:

$$\forall f \in G_o(K,X) : \varphi(f) = f + s_\alpha(p_o(f))$$
$$\forall g \in G_\alpha(K,X) : \psi(g) = g + (-s_\alpha(p_\alpha(g)))$$

It is obvious that φ and ψ are maps over X. It is also clear that

$$\psi\varphi(f) = (f + s_\alpha(p_o(f))) + (-s_\alpha(p_o(f)))$$

and

$$\varphi\psi(g) = (g + (-s_\alpha(p_\alpha(g)))) + s_\alpha(p_\alpha(g)).$$

A fibre homotopy H of $\psi\varphi$ to the identity with parameter $t'(0 \leq t' \leq 1)$ can then be defined by

$$H(f,t')([a,t]) = \begin{cases} f([a,\frac{4t}{1+3t'}]) \text{ if } 0 \leq t \leq \frac{1+3t'}{4} \\ s_\alpha(p_o(f))([a,4t-(1+3t')]) \text{ if } \frac{1+3t'}{4} \leq t \leq \frac{1+t'}{2} \\ s_\alpha(p_o(f))([a,2-2t]) \text{ if } \frac{1+t'}{2} \leq t \leq 1 \end{cases}$$

for every $f \in G_o(K,X)$ and every $[a,t] \in K = SA$.

A fibre homotopy of $\varphi\psi$ to the identity can be defined by a similar formula.

φ is therefore a fibre homotopy equivalence with inverse ψ. This proves that 2) $\Rightarrow$ 1).

In case X = SB is also a suspension we know already that 2) is equivalent to 3) by Theorem 2.1.1. Hence the theorem is proved.

<u>Corollary 2.3.2</u> Suppose that K = SA is a suspension and that X is an H-space with a homotopy unit. Then $(G_\alpha(K,X), p_\alpha, X)$ is fibre homotopy equivalent to $(G_o(K,X), p_o, X)$ for any $\alpha \in \pi(K,X)$.

<u>Proof</u> Let $\mu : X \times X \to X$ be the multiplication on X and suppose that the base point $* \in X$ is a homotopy unit for μ. Let $f_\alpha : K \to X$ be a representative for α. In particular $f_\alpha(*) = *$.

Define $s'_\alpha : X \to G_\alpha(K,X)$ by the formula

$$\forall y \in K \quad \forall x \in X : s'_\alpha(x)(y) = \mu(f_\alpha(y),x).$$

Since $s'_\alpha(x)$ for fixed x is homotopic to f_α (one gets a homotopy by moving x along a path to $* \in X$), it is clear that s'_α maps into $G_\alpha(K,X)$. Observe now that $p_\alpha s'_\alpha(x) = \mu(f_\alpha(*),x) = \mu(*,x)$. Hence $p_\alpha s'_\alpha$ is homotopic to 1_X. Using now that $(G_\alpha(K,X),p_\alpha,X)$ is a Hurewicz fibration, the covering homotopy property will produce a genuine section $s_\alpha : X \to G_\alpha(K,X)$ for p_α. Hence the corollary follows from Theorem 2.3.1.

For the comparison of two arbitrary evaluation fibrations we have

<u>Theorem 2.3.3</u> Suppose that $K = SA$ and $X = SB$ are suspensions and let $\alpha, \beta \in \pi(K,X)$. Suppose also that at least one of the generalized Whitehead products $[\alpha+\beta, \iota_X]$ and $[\alpha-\beta, \iota_X]$ is zero. Then $(G_\alpha(K,X),p_\alpha,X)$ and $(G_\beta(K,X),p_\beta,X)$ are fibre homotopy equivalent.

<u>Proof</u> Suppose first that $[\alpha+\beta, \iota_X] = 0$. By Theorem 2.1.1 we can therefore find a section $s_{\alpha+\beta}: X \to G_{\alpha+\beta}(K,X)$ for $p_{\alpha+\beta}$. Define then maps over X, $\varphi : G_\alpha(K,X) \to G_\beta(K,X)$ and $\psi : G_\beta(K,X) \to G_\alpha(K,X)$, by the formulas:

$$\forall f \in G_\alpha(K,X) : \varphi(f) = (-f) + s_{\alpha+\beta}(p_\alpha(f))$$

$$\forall g \in G_\beta(K,X) : \psi(g) = s_{\alpha+\beta}(p_\beta(g)) + (-g)$$

Using formulas analogeous with those in the proof of Theorem 2.3.1 it is easy to prove that $\psi\varphi$ and $\varphi\psi$ are fibre homotopic to the respective identity maps. Thus φ is a fibre homotopy equivalence with inverse ψ.

The proof for the case $[\alpha-\beta, \iota_X] = 0$ is similar. Hence the theorem is proved.□

Theorem 2.3.3 clearly invites the question whether the vanishing of either $[\alpha+\beta, \iota_X]$ or $[\alpha-\beta, \iota_X]$ is a necessary condition for the fibrations defined by α and β to be fibre homotopy equivalent.

I know of no counter-examples to such a statement and have in fact quite a lot of evidence for the following

Conjecture Let $1 \leq n \leq m$ and let $\alpha, \beta \in \pi_m(S^n)$. Then $(G_\alpha(S^m,S^n),p_\alpha,S^n)$ and $(G_\beta(S^m,S^n),p_\beta,S^n)$ are fibre homotopy equivalent if and only if $[\alpha,\iota_n] = \pm[\beta,\iota_n]$.

§4 Strong fibre homotopy equivalence of evaluation fibrations.

Let $1 \leq n \leq m$. In this section we shall study the set of evaluation fibrations $(G_\alpha(S^m,S^n),p_\alpha,S^n)$ for $\alpha \in \pi_m(S^n)$ w.r.t. strong fibre homotopy equivalence. We will show that the strong fibre homotopy equivalence class of $(G_\alpha(S^m,S^n),p_\alpha,S^n)$ is completely determined by the Whitehead product $[\alpha,\iota_n]$.

First however, a couple of definitions.

Definition 2.4.1 A fibration (E,p,B) with basic fibre F is a Hurewicz fibration $p : E \to B$ together with a homotopy equivalence $i: F \to p^{-1}(*)$ of F into the fibre over the base point $* \in B$ fixed within a given homotopy class.

We shall consider the evaluation fibration $(G_\alpha(S^m,S^n), p_\alpha,S^n)$ as a fibration with basic fibre $F_0(S^m,S^n)$ and homotopy equivalence $i_\alpha : F_0(S^m,S^n) \to F_\alpha(S^m,S^n)$ given as follows:

Take a representative $f_\alpha : S^m \to S^n$ for α and put

$$i_\alpha(f) = f_\alpha + f$$

for $f \in F_0(S^m,S^n)$.

The homotopy class of i_α is independent of the choice of f_α, since $F(S^m,S^n)$ is connected.

Definition 2.4.2 Let (E_i,p_i,B) for $i = 1,2$ be fibrations with basic fibre F. A strong fibre homotopy equivalence is a fibre homotopy equivalence

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & B & \end{array} ,$$

such that the following condition is satisfied:

If $i_2' : p_2^{-1}(*) \to F$ denotes an arbitrary homotopy inverse to i_2, then $i_2' \circ \varphi_{\cdot} \circ i_1$ is homotopic to 1_F, the identity map on F.

If such a φ exists, we will say that (E_1, p_1, B) and (E_2, p_2, B) are <u>strongly fibre homotopy equivalent</u>.

The purpose of this section is to prove

<u>Theorem 2.4.3</u> Let $m \geqslant n \geqslant 1$ and let $\alpha, \beta \in \pi_m(S^n)$. Then $(G_\alpha(S^m,S^n), p_\alpha, S^n)$ and $(G_\beta(S^m,S^n), p_\beta, S^n)$ are strongly fibre homotopy equivalent if and only if $[\alpha, \iota_n] = [\beta, \iota_n]$.

Before we can prove this theorem we need to make some remarks on the classification of fibrations (E,p,B) with basic fibre F.

According to Dold ([15], Satz 16.8) there exists a classifying space B(F) and a universal Hurewicz fibration $E(F) \to B(F)$ with basic fibre F. This universal fibration has the property that for an arbitrary space X there is a bijective correspondence between the strong fibre homotopy equivalence classes of fibrations with basic fibre F over X and the set of based homotopy classes of based maps $\pi(X, B(F))$.

Now let $c_\alpha : S^n \to B(F_o(S^m,S^n))$ be the classifying map for the fibration $(G_\alpha(S^m,S^n), p_\alpha, S^n)$ with basic fibre $F_o(S^m,S^n)$, and let $[c_\alpha] \in \pi_n(B(F_o(S^m,S^n)))$ denote its homotopy class.

<u>Lemma 2.4.4</u> If δ denotes the boundary operator in the homotopy sequence for the universal fibration with basic fibre $F_o(S^m,S^n)$ and H is the obvious Hurewicz isomorphism, then the following

formula holds:

$$H \circ \delta([c_\alpha]) = -[\alpha, \iota_n]$$

<u>Proof</u> Consider the map between fibrations

$$\begin{array}{ccc}
F_o(S^m,S^n) & \xrightarrow{\;\approx{c}_\alpha\;} & F_o(S^m,S^n) \\
\downarrow{\scriptstyle i_\alpha} & & \downarrow{\scriptstyle i} \\
G_\alpha(S^m,S^n) & \xrightarrow{\;\tilde{c}_\alpha\;} & E(F_o(S^m,S^n)) \\
\downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p} \\
S^n & \xrightarrow{\;c_\alpha\;} & B(F_o(S^m,S^n)),
\end{array}$$

which classifies $(G_\alpha(S^m,S^n), p_\alpha, S^n)$.

Since the classification is up to strong fibre homotopy equivalence, $\approx{c}_\alpha$ is homotopic to the identity map of $F_o(S^m,S^n)$.

Passing to homotopy sequences we get therefore a commutative diagram

$$\begin{array}{ccccc}
\ldots \to \pi_n(S^n) & \xrightarrow{\;\delta_\alpha\;} & \pi_{n-1}(F_o(S^m,S^n)) \to \ldots & & \\
\downarrow{\scriptstyle (c_\alpha)_*} & & \downarrow{\scriptstyle 1} & \searrow{\scriptstyle H} & \\
 & & & & \pi_{n+m-1}(S^n) \\
 & & & \nearrow{\scriptstyle H} & \\
\ldots \to \pi_n(B(F_o(S^m,S^n))) & \xrightarrow[\delta]{} & \pi_{n-1}(F_o(S^m,S^n)) \to \ldots & &
\end{array}$$

,

where δ_α denotes the boundary operator in the evaluation fibration defined by α, ∂_α, composed with $(i_\alpha)_*^{-1}$, i.e. $\partial_\alpha = (i_\alpha)_* \circ \delta_\alpha$.

By the theorem of G.W.Whitehead (stated here as Theorem 1.3.1) we know that $H \circ \delta_\alpha(\iota_n) = -[\alpha, \iota_n]$. On the other hand it is clear that $(c_\alpha)_*(\iota_n) = [c_\alpha]$. Since the diagram is commutative we get then the formula we want.

<u>Proof of Theorem 2.4.3</u> Suppose first that $(G_\alpha(S^m,S^n), p_\alpha, S^n)$ and $(G_\beta(S^m,S^n), p_\beta, S^n)$ are strongly fibre homotopy equivalent. By the classification theorem the classifying maps c_α and c_β for these fibrations must then be homotopic, i.e. $[c_\alpha] = [c_\beta]$ in $\pi_n(B(F_o(S^m,S^n)))$. Hence $[\alpha, \iota_n] = [\beta, \iota_n]$ by Lemma 1.4.4.

Suppose next that $[\alpha, \iota_n] = [\beta, \iota_n]$. This implies that

$[\alpha - \beta, \iota_n] = 0$ and hence $(G_{\alpha-\beta}(S^m,S^n), p_{\alpha-\beta}, S^n)$ has a section by Theorem 2.1.1. Choose then such a section $s_{\alpha-\beta}: S^n \to G_{\alpha-\beta}(S^m,S^n)$ and define

$$\varphi : G_\alpha(S^m,S^n) \to G_\beta(S^m,S^n)$$

by the formula

$$\forall f \in G_\alpha(S^m,S^n) : \varphi(f) = (-s_{\alpha-\beta}(p_\alpha(f))) + f$$

Just as in the proof of Theorem 2.3.3 we can show that φ is a fibre homotopy equivalence.

We have now to make a good choice of a homotopy inverse i'_β to i_β in order to make it easy to prove that $i'_\beta \circ \varphi \circ i_\alpha$ is homotopy equivalent to the identity map on $F_0(S^m,S^n)$.

Suppose therefore that f_α and f_β are the representatives for respectively α and β which are used in the definitions of i_α and i_β. Define then $i'_\beta : F_\beta(S^m,S^n) \to F_0(S^m,S^n)$ by the formula

$$\forall g \in F_\beta(S^m,S^n) : i'_\beta(g) = ((-f_\alpha) + s_{\alpha-\beta}(p_\alpha(f_\alpha))) + g$$

Since $F_{-\beta}(S^m,S^n)$ is connected, it is clear that i'_β is homotopic to $i''_\beta : F_\beta(S^m,S^n) \to F_0(S^m,S^n)$ defined by the formula

$$\forall g \in F_\beta(S^m,S^n) : i''_\beta(g) = (-f_\beta) + g$$

Looking back at the proof for Theorem 1.2.2 we see that i''_β, and hence also i'_β, is a homotopy inverse to i_β.

Let now $h \in F_0(S^m,S^n)$ be an arbitrary element. Then we have:

$$\begin{aligned}
& i_\beta' \circ \varphi \circ i_\alpha(h) \\
&= i'_\beta \circ \varphi(f_\alpha + h) \\
&= i'_\beta((-s_{\alpha-\beta}(p_\alpha(f_\alpha))) + (f_\alpha + h)) \\
&= ((-f_\alpha) + s_{\alpha-\beta}(p_\alpha(f_\alpha))) + ((-s_{\alpha-\beta}(p_\alpha(f_\alpha))) + (f_\alpha + h))
\end{aligned}$$

Using appropriate formulas we can prove from this expression that $i'_\beta \circ \varphi \circ i_\alpha$ is homotopic to the identity map of $F_0(S^m,S^n)$.

This shows that φ is a strong fibre homotopy equivalence and hence the proof is finished.

Chapter 3

Homotopy equivalence of components in spaces of maps between spheres.

This chapter will be devoted to obtaining concrete information on the homotopy types of the components in certain spaces of maps between spheres. In §1 we determine the order of the homotopy groups $\pi_{n-1}(G_\alpha(S^n, S^n))$ for n even and $\alpha \in \pi_n(S^n)$. In §2 we characterize completely the homotopy type of the neutral component among all the components in a space of maps between spheres. Finally, in §3 we divide the set of components in the mapping spaces $G(S^n,S^n)$ and $G(S^{n+1},S^n)$ into homotopy types, thereby solving completely the fundamental problem behind Part I of the thesis in these specific cases.

The results in this chapter will extend considerably earlier results of Hu [47] and Koh [55]. Like Hu and Koh we shall use extensively the theorem of G.W.Whitehead (Theorem 1.3.1), but, in particular to get the results on homotopy equivalence of certain components, we have to rely on our constructions of certain fibre homotopy equivalences between evaluation fibrations in Chapter 2, §3.

§1 The order of $\pi_{n-1}(G_\alpha(S^n,S^n))$ for n even and $\alpha \in \pi_n(S^n)$.

The purpose of this section is to prove

Theorem 3.1.1 Suppose that $n > 2$ is even and let $\alpha \in \pi_n(S^n)$. Put $G_\alpha = G_\alpha(S^n,S^n)$ and denote by $\deg\alpha$ the degree of α.

1) $\alpha = 0$. $\pi_{n-1}(G_0)$ is an infinite group.

2) $\alpha \neq 0$. $\pi_{n-1}(G_\alpha)$ is a finite group and its order satisfies the formula

$$\text{order}\,(\pi_{n-1}(G_\alpha)) = |\deg \alpha| \cdot \text{order}\,(\pi_{n-1}(G_{\iota_n})).$$

Proof From Theorem 1.3.1 we get the exact sequence

$$\pi_n(S^n) \xrightarrow{\rho_\alpha} \pi_{2n-1}(S^n) \to \pi_{n-1}(G_\alpha) \to 0$$

If $\alpha = 0$ then ρ_α is the zero map and hence $\pi_{n-1}(G_\alpha) \cong \pi_{2n-1}(S^n)$, which is an infinite group by Serre [82]

Suppose then that $\alpha \neq 0$.

Put $A_\alpha = \rho_\alpha(\pi_n(S^n))$ and $A_{\iota_n} = \rho_{\iota_n}(\pi_n(S^n))$

Observe now that

$$\rho_\alpha(\iota_n) = -[\alpha, \iota_n] = -\deg \alpha[\iota_n, \iota_n] = \deg \alpha \cdot \rho_{\iota_n}(\iota_n)$$

and that $\rho_{\iota_n}(\iota_n) = -[\iota_n, \iota_n]$ has infinite order by the delicate suspension theorem (Theorem 1.3.2). Therefore A_α and A_{ι_n} are both subgroups of rank 1 in $\pi_{2n-1}(S^n)$ and A_α is a subgroup of A_{ι_n} with quotient group $A_{\iota_n}/A_\alpha \cong Z_{|\deg \alpha|}$. By a result of Serre [82], $\pi_{2n-1}(S^n) \cong Z \oplus H$, where H is a finite group. It is therefore clear that the quotients

$$\pi_{n-1}(G_\alpha) \cong \pi_{2n-1}(S^n)/A_\alpha \quad \text{and} \quad \pi_{n-1}(G_{\iota_n}) \cong \pi_{2n-1}(S^n)/A_{\iota_n}$$

are finite groups.

Consider then the exact sequence of finite groups

$$0 \to A_{\iota_n}/A_\alpha \to \pi_{2n-1}(S^n)/A_\alpha \to \pi_{2n-1}(S^n)/A_{\iota_n} \to 0$$

which is isomorphic to an exact sequence

$$0 \to Z_{|\deg \alpha|} \to \pi_{n-1}(G_\alpha) \to \pi_{n-1}(G_{\iota_n}) \to 0$$

From this sequence we conclude immediately that

$$\text{order}(\pi_{n-1}(G_\alpha)) = |\deg \alpha| \cdot \text{order}(\pi_{n-1}(G_{\iota_n})).$$

If more detailed information on the homotopy groups and the map ρ_α involved in the proof above is available, then one can of course compute the exact structure of $\pi_{n-1}(G_\alpha)$. This was first done by Hu [47] for n = 2 and later for n = 4,6,8,10,12, 14 by Koh [55].

Theorem 3.1.1 is the best one can hope to prove in general with the present knowledge on homotopy groups of spheres.

Example 3.1.2 (Hu [47] or Koh [55]) Let $\alpha \in \pi_2(S^2)$.

Then

$$\pi_1(G_\alpha(S^2,S^2)) \cong Z_{2\cdot|\deg\alpha|}$$

Here Z_k denotes the cyclic group of order k. For k = 0 we put $Z_0 = Z$, the integers.

Example 3.1.3 (Koh [55]). Let $\alpha \in \pi_4(S^4)$. Then

$$\pi_3(G_\alpha(S^4,S^4)) \cong Z_{24\cdot|\deg\alpha|} \oplus Z_{12}$$

Example 3.1.4 (Koh [55]). Let $\alpha \in \pi_6(S^6)$. Then

$$\pi_5(G_\alpha(S^6,S^6)) \cong Z_{|\deg\alpha|}$$

§2 Characterization of the homotopy type of the neutral component.

In this section we solve completely the problem when an arbitrary component in a space of maps between spheres is homotopy equivalent to the neutral component. The solution is expressed in terms of the vanishing of the Whitehead product $[\alpha, \iota_n]$, where $\alpha \in \pi_m(S^n)$ is the homotopy class defining the component in question. The following theorem contains this result and summarizes also the main bulk of results involving the neutral evaluation fibration we have obtained in Chapter 2.

<u>Theorem 3.2.1</u> Suppose that $m \geq n \geq 1$ and let $\alpha \in \pi_m(S^n)$. Then the following statements are equivalent:

1) $G_\alpha(S^m,S^n)$ and $G_o(S^m,S^n)$ are homotopy equivalent.

2) $(G_\alpha(S^m,S^n),p_\alpha,S^n)$ has a section.

3) $(G_\alpha(S^m,S^n),p_\alpha,S^n)$ and $(G_o(S^m,S^n),p_o,S^n)$ are fibre homotopy equivalent.

4) $(G_\alpha(S^m,S^n),\ p_\alpha,\ S^n)$ and $(G_o(S^m,S^n),p_o,S^n)$ are strongly fibre homotopy equivalent.

5) The Whitehead product $[\alpha,\iota_n] = 0$.

<u>Proof</u> The theorem is trivial for $n = 1$. We can therefore assume $m \geq n \geq 2$.

We prove the theorem in the following steps $1) \Rightarrow 5) \Rightarrow 4) \Rightarrow 3) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$.

$5) \Rightarrow 4)$ follows from Theorem 2.4.3. $4) \Rightarrow 3)$ is trivial. $3) \Rightarrow 2) \Rightarrow 3)$ is a special case of Theorem 2.3.1. $3) \Rightarrow 1)$ is trivial. We have therefore only to show that $1) \Rightarrow 5)$.

Assume for that purpose that $G_\alpha(S^m,S^n)$ and $G_o(S^m,S^n)$ are homotopy equivalent and consider the following exact sequence obtained from Theorem 1.3.1:

$$\pi_n(S^n) \xrightarrow{\rho_\alpha} \pi_{n+m-1}(S^n) \to \pi_{n-1}(G_\alpha(S^m,S^n)) \to 0.$$

If $m \geq n + 1$ or $m = n$ odd $\pi_{n+m-1}(S^n)$ is a finite group (Serre [82]), and hence it is obvious from this sequence that $\pi_{n-1}(G_\alpha(S^m,S^n))$ and $\pi_{n-1}(G_o(S^m,S^n)) \cong \pi_{n+m-1}(S^n)$ cannot be isomorphic unless $[\alpha,\iota_n] = 0$.

If $m = n$ is even it follows from Theorem 3.1.1 that $\pi_{n-1}(G_o(S^n,S^n))$ is an infinite group and that $\pi_{n-1}(G_\alpha(S^n,S^n))$ is a finite group for $\alpha \neq 0$. Therefore it is clear that $\alpha = 0$ and hence, in particular, $[\alpha,\iota_n] = 0$.

Therefore 1) $\Longrightarrow$5) and the theorem is proved.

The homotopy groups of the neutral component are easy to calculate since the neutral evaluation fibration has a section. We get then

Corollary 3.2.2 Let $m \geq n \geq 1$ and let $\alpha \in \pi_m(S^n)$. Suppose that $[\alpha, \iota_n] = 0$. Then

$$\pi_i(G_\alpha(S^m,S^n)) \cong \pi_i(S^n) \oplus \pi_{i+m}(S^n)$$

for any $i \geq 1$.

Proof The homotopy sequence for the neutral evaluation fibration splits. Hence

$$\begin{aligned}\pi_i(G_\alpha(S^m,S^n)) &\cong \pi_i(G_o(S^m,S^n)) \\ &\cong \pi_i(S^n) \oplus \pi_i(F_o(S^m,S^n)) \\ &\cong \pi_i(S^n) \oplus \pi_{i+m}(S^n).\end{aligned}$$

§3 The division into homotopy types of the components in the mapping spaces $G(S^n,S^n)$ and $G(S^{n+1},S^n)$.

We are now ready to prove one of the main results in Part I. The problem behind the following theorem was in fact the starting point for Part I. As already mentioned in the introduction the theorem extends considerably the results obtained by Koh ([55], Theorem 3.18).

Theorem 3.3.1 Consider the mapping spaces $G(S^n,S^n)$ for $n \geq 1$ and let $\alpha, \beta \in \pi_n(S^n)$.

1) n even. The components $G_\alpha(S^n,S^n)$ and $G_\beta(S^n,S^n)$ are

homotopy equivalent if and only if $\alpha = \pm \beta$.

2) n odd. $G_\alpha(S^n,S^n)$ is homotopy equivalent to $G_0(S^n,S^n)$ if $\deg\alpha$ is even and to $G_{\iota_n}(S^n,S^n)$ if $\deg\alpha$ is odd. Furthermore, $G_0(S^n,S^n)$ and $G_{\iota_n}(S^n,S^n)$ are homotopy equivalent if and only if $n = 1,3,7$.

Proof For n even the result follows immediately from Theorems 1.2.1, 3.1.1 and 3.2.1.

Suppose then that n is odd. As stated in Theorem 1.3.1 it is known that $2\cdot[\iota_n,\iota_n] = 0$ and that $[\iota_n,\iota_n] = 0$ if and only if $n = 1,3,7$. Therefore $[\alpha,\iota_n] = \deg\alpha\cdot[\iota_n,\iota_n] = 0$ if $\deg\alpha$ is even and $[\alpha+\iota_n,\iota_n] = (\deg\alpha + 1)\cdot[\iota_n,\iota_n] = 0$ if $\deg\alpha$ is odd. The result for n odd is therefore an easy consequence of Theorems 2.3.3 and 3.2.1.

As mentioned in Chapter 1, §3 quite a lot is known about vanishing or non-vanishing of Whitehead products $[\alpha_n,\iota_n]$ for $\alpha_n \in \pi_m(S^n)$. By Theorem 3.2.1 we get a statement about a component in $G(S^m,S^n)$ for each such result.

Since we are able to give the complete solution to the homotopy problem for the components in the mapping spaces $G(S^{n+1},S^n)$ we state in particular

Theorem 3.3.2 Consider the mapping spaces $G(S^{n+1},S^n)$ for $n \geq 1$.

1) n = 1. $G(S^2,S^1)$ is homotopy equivalent to S^1.

2) n = 2. $G(S^3,S^2)$ has a countable number of components all of which are homotopy equivalent.

3) n ≥ 3. $G(S^{n+1},S^n)$ has two components. These components have the same homotopy type if and only if $n = 6$ or $n \equiv 3 \bmod 4$.

Proof n = 1 Let $m > 1$. Since $\pi_m(S^1) = 0$ $G(S^m,S^1)$ and $F(S^m,S^1)$ has only one component. Now

$\pi_i(F(S^m,S^1)) \cong \pi_{i+m}(S^1) = 0$ for each $i \geq 1$. The fibre of the evaluation fibration $p: G(S^m,S^1) \to S^1$ is therefore contractible and hence p is a homotopy equivalence.

<u>$n = 2$</u> $\pi_3(S^2) \cong Z$. Thus $G(S^3,S^2)$ has indeed a countable number of components. Since an arbitrary element $\alpha \in \pi_3(S^2)$ has the form $\alpha = \ell \cdot \eta_2$ it follows from Theorem 1.3.3 that $[\alpha,\iota_2] = \ell \cdot [\eta_2,\iota_2] = 0$. Hence $G_\alpha(S^3,S^2)$ is homotopy equivalent to $G_0(S^3,S^2)$ by Theorem 3.2.1.

<u>$n \geq 3$</u> Since $\pi_{n+1}(S^n) \cong Z_2$ $G(S^{n+1},S^n)$ has two components, namely $G_0(S^{n+1},S^n)$ and $G_{\eta_n}(S^{n+1},S^n)$. Hence the result follows immediately from Theorems 1.3.3 and 3.2.1.

<u>Remark 3.3.3</u> One might believe that $G(S^3,S^2)$ and $G(S^3,S^3)$ are homotopy equivalent, so that this would explain $n = 2$ in Theorem 3.3.2. This is in fact not true, since the Hopf fibration $S^3 \to S^2$ induces a fibration $G(S^3,S^3) \to G(S^3,S^2)$ with fibre $G(S^3,S^1)$, which is homotopy equivalent to S^1. It is therefore somewhat surprising that all the components in $G(S^3,S^2)$ have the same homotopy type.

The groups $\pi_{n-1}(G_\alpha(S^n,S^n))$ have been computed by Koh [55] for small n and we have already stated some of Koh's result in §2 of this chapter. We finish with some computations of the groups $\pi_{n-1}(G_\alpha(S^{n+1},S^n))$.

<u>Example 3.3.4</u> Plugging information on homotopy groups into the exact sequence

$$\pi_n(S^n) \xrightarrow{\rho_\alpha} \pi_{2n}(S^n) \to \pi_{n-1}(G_\alpha(S^{n+1},S^n)) \to 0$$

obtained from Theorem 1.3.1 one gets easily

<u>$n = 4$</u> $\pi_3(G_0(S^5,S^4)) \cong Z_2 \oplus Z_2$

$$\pi_3(G_{\eta_4}(S^5,S^4)) \cong Z_2$$

n = 5

$$\pi_4(G_o(S^6,S^5)) \cong Z_2$$

$$\pi_4(G_{\eta_5}(S^6,S^5)) \cong 0.$$

Chapter 4

Mapping spaces with a finite CW-complex as domain.

In this chapter we take a brief look on the qualitative structure of the homotopy groups of a mapping space with a finite CW-complex as domain. The main purpose of the chapter will be to give a simple and elementary proof of a version of a theorem due independently to Federer [32] and Thom [87].

It is worthwhile mentioning that Federer gets his version of the theorem as a corollary to the construction of a spectral sequence. This spectral sequence has later been slightly generalized by M.Dyer and used by him in many concrete computations of homotopy groups of special mapping spaces, [19], [20] and [21].

§1 Towers of restriction fibrations.

This section contains the construction of a finite tower of fibrations naturally associated with a given space of maps having a finite CW-complex as domain.

The construction is made possible by

Lemma 4.1.1 Let A be a locally compact, connected ANR(CW-complex) and suppose that $K = A \cup_\varphi D^\lambda$ for $\lambda \geq 1$ is A with a λ-cell D^λ attached by $\varphi: S^{\lambda-1} \to A$. Let X be an arbitrary connected topological space.

Then the restriction map

$$p : G(K,X) \to G(A,X)$$

is a Hurewicz fibration over its image.

Furthermore, the fibre of p is homotopy equivalent to $F(S^\lambda, X)$, the space of λ-loops on X.

Proof It is well-known that K itself is a locally compact, connected ANR. See e.g. Hu ([49], Theorem 1.2, p. 178). The proof that p is a Hurewicz fibration can be found in Hu ([48], Theorem 13.1, p.83).

For the identification of the fibre suppose that $f \in G(A,X)$ is in the image of p. The fibre over f consists then of all possible extensions of f over the λ-cell D^λ. Hence it is clear that the fibre over f can be identified with the fibre over $f \circ \varphi$ in the restriction fibration

$$\bar{p} : G(D^\lambda,X) \to G(S^{\lambda-1},X).$$

Since f has an extension over K, $f \circ \varphi$ has an extension over D^λ. Therefore $f \circ \varphi$ is homotopic to a constant map, i.e. $f \circ \varphi \in G_o(S^{\lambda-1},X)$. Along the same lines it is clear that the image of $\bar{p}$ is exactly $G_o(S^{\lambda-1},X)$.

Since $\bar{p} : G(D^\lambda,X) \to G_o(S^{\lambda-1},X)$ is a Hurewicz fibration all fibres have the same homotopy type. See e.g. Spanier ([84], Corollary 13, Chapter 2, Section 8, p.101). It is however clear that the fibre of $\bar{p}$ over the constant map $S^{\lambda-1} \to * \in X$ can be identified with the fibre in the evaluation fibration $G(S^\lambda,X) \to X$.

Altogether we have then proved that the fibre of p over $f \in G(A,X)$ is homotopy equivalent to $F(S^\lambda,X)$.

This proves the lemma.

Now let K be a connected, finite CW-complex of dimension $\leq m$.

We can then choose a filtration of K, say

$$* = K_o \subset K_1 \subset \dots \subset K_r = K$$

with the following properties:

1) $K_o = *$ consists of a single 0-cell, which we choose as

base point in K.

2) For $1 \leq i \leq r$

$$K_i = K_{i-1} \cup_{\varphi_i} D^{\lambda_i}$$

with $1 \leq \lambda_i \leq m$.

3) $K_r = K$

Then let X be an arbitrary connected topological space. Using Lemma 4.1.1 we then get a tower of fibrations (over their images) as follows:

$$\begin{array}{ccc} F(S^{\lambda_r},X) & \to & G(K_r,X) \\ & & \downarrow \\ F(S^{\lambda_{r-1}},X) & \to & G(K_{r-1},X) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ F(S^{\lambda_1},X) & \to & G(K_1,X) \\ & & \downarrow \\ & & G(K_o,X) \end{array}$$

We have indicated the homotopy type of the fibres, and we remark that $G(K_r,X) = G(K,X)$ and that $G(K_o,X) = X$.

Since $\pi_i(F(S^\lambda,X),f) \cong \pi_{i+\lambda}(X,*)$ for any base point $f \in F(S^\lambda,X)$, it is clear that we quite easily can get qualitative results on the homotopy groups of $G(K,X)$ out of this tower.

§2 Homotopy modulo a Serre class of abelian groups.

We prove in this section our version of the theorem of Federer and Thom.

$\mathcal{A}$ shall denote an arbitrary Serre class of abelian groups, as e.g. the class of finitely generated abelian groups, the class of finite abelian groups or even the class containing only the trivial group.

Observe that if a space is not connected there will be more than one homotopy group of that space in each dimension, namely one for each component.

We shall prove our theorem using the tower in §1 and the following

<u>Lemma 4.2.1</u> Let $F \to E \xrightarrow{p} B$ be a fibration and let $\mathcal{Q}$ be a Serre class of abelian groups. Let also $i \geq 2$.

Suppose that all the homotopy groups of B and F in dimension i belong to $\mathcal{Q}$. Then all the homotopy groups of E in dimension i will belong to $\mathcal{Q}$.

<u>Proof</u> The lemma follows immediately from the exact homotopy sequence of the fibration and the definition of a Serre class.

We can then prove

<u>Theorem 4.2.2</u> Let K be a connected, finite CW-complex of dimension $\leq m$ and let X be an arbitrary connected, topological space. Let $\mathcal{Q}$ be a given Serre class of abelian groups and suppose that all the homotopy groups of X in dimensions i with i in the interval $i_o \leq i \leq i_o + m$ for $i_o \geq 2$ belong to $\mathcal{Q}$.

Then all the homotopy groups of G(K,X) in dimension i_o belong to $\mathcal{Q}$.

<u>Proof</u> Using the isomorphisms $\pi_i(F(S^\lambda,X), f) \cong \pi_{i+\lambda}(X,*)$ and the assumptions in the theorem it follows immediately that all the fibres in the tower constructed in §1 have their homotopy groups in dimension i_o in the Serre class $\mathcal{Q}$. By an obvious finite induction using Lemma 4.2.1 we get then the result in the theorem.

As already mentioned Theorem 4.2.2 is actually a variation of a theorem of Federer ([32], p.353) and independently Thom ([87], Theorems 1 and 4). In some sense the theorem of Federer and Thom is stronger than the one presented here, since they do not need K to be a finite but only something like a finite dimensional CW-complex. On the other hand Federer needs X to be simple and both Federer and Thom have restrictions on the homology of K, which are not satisfied in general for an arbitrary Serre class, not even when K is a finite CW-complex. The formulation of our theorem is also slightly more general than theirs.

The method we used in the proof given here is closest to that used by Federer.

Finally, in this section we give an example to show that we cannot relax the conditions on the homotopy groups of the target in Theorem 4.2.2.

Example 4.2.3 Let i_o, n and m be integers satisfying $i_o \geq 2$, $n > m \geq 2$ and $i_o + m = n$.

Then $\pi_i(S^n) = 0$ for $i_o \leq i \leq i_o + m - 1$ but $\pi_{i_o+m}(S^n) \cong Z$.

Let $f \in F(S^m, S^n)$ be an arbitrary map.

From the homotopy sequence for the evaluation fibration $F(S^m,S^n) \to G(S^m,S^n) \to S^n$

and the Hurewicz isomorphism it follows easily that

$$\pi_{i_o}(G(S^m,S^n),f)$$

$$\cong \pi_{i_o}(F(S^m,S^n),f)$$

$$\cong \pi_{i_o+m}(S^n,*)$$

$$\cong Z$$

If we take $\mathcal{A}$ to be the Serre class consisting only of the trivial group we see therefore that the conclusion in Theorem 4.2.2 fails for the mapping space $G(S^m, S^n)$.

§3 The fundamental group of a mapping space. An example.

The object of this section is to show by an example that the fundamental group of a mapping space need not to be finitely generated although all the homotopy groups of the target are finitely generated. This will explain why no statement was made about the fundamental group in §2.

First we need a purely group theoretic result. I am indebted to Gustav Lehrer for constructing a group H with the following property.

Proposition 4.3.1 There exists a group H, which is finitely generated but contains an element $h_o \in H$ for which the centralizer $C_H(H_o)$ is not finitely generated.

Recall that $C_H(h_o) = \{h \in H | hh_o = h_o h\}$

Construction of H

Let H be the group given by

Generators : A double infinite sequence of elements $\ldots, h_{-2}, h_{-1}, h_o, h_1, h_2, \ldots$ and an element $\bar{h}$.

Relations: $h_n^2 = 1$, $[h_n, h_m] = 1$ and

$\bar{h}^n h_m \bar{h}^{-n} = h_{n+m}$

$[h_n, h_m] = h_n h_m h_n^{-1} h_m^{-1}$ denotes here the commutator of h_n and h_m.

Since $h_n = \bar{h}^n h_o \bar{h}^{-n}$, H is obviously generated by the two elements h_o and $\bar{h}$, hence in particular finitely generated.

Assertion $C_H(h_o)$ is isomorphic to the double infinite direct product

$$\dots \times Z_2(h_{-1}) \times Z_2(h_o) \times Z_2(h_1) \times \dots,$$

where $Z_2(h_n)$ denotes the cyclic group of order 2 generated by h_n.

In particular $C_H(h_o)$ is not finitely generated.

Proof It is clear that all words $h_{m_1} h_{m_2} \dots h_{m_k}$ belong to $C_H(h_o)$.

By induction on the length of a word it is easy to show that $C_H(h_o)$ cannot contain words with a non-trivial power of $\bar{h}$ in them.

The statement in the assertion follows then immediately.

This finishes the construction of a group H with the properties in Proposition 4.3.1.

Take now the group H constructed above and let $X = K(H,1)$ be the corresponding Eilenberg-MacLane space. By definition X is then connected, $\pi_1(X) \cong H$ and $\pi_i(X) = 0$ for $i \geq 2$. Let $h_o : S^1 \to X$ be the map representing the element $h_o \in H$ and consider the component $G_{[h_o]}(S^1,X)$ in the space of maps from S^1 into X defined by the homotopy class $[h_o]$ of h_o.

Assertion $\pi_1(G_{[h_o]}(S^1,X)) \cong C_H(h_o)$.

Proof Consider the evaluation fibration

$$\bar{F}_{[h_o]}(S^1,X) \to G_{[h_o]}(S^1,X) \xrightarrow{p} X$$

and the following part of its homotopy sequence

$$\pi_1(\bar{F}_{[h_o]}(S^1,X),h_o) \to \pi_1(G_{[h_o]}(S^1,X),h_o) \to \pi_1(X,*)$$

Now $\pi_1(\bar{F}_{[h_o]}(S^1,X),h_o)$

$\cong \pi_1(F_{[h_o]}(S^1,X),h_o)$

$\cong \pi_2(X,*)$

$\cong 1$

Hence $p_* : \pi_1(G_{[h_o]}(S^1,X),h_o) \to \pi_1(X,*)$ is a monomorphism.

It is very easy to see that an element $\alpha \in \pi_1(X,*)$ is in the image of p_* if and only if there exists a map $S^1 \times S^1 \to X$ such that $S^1 \times \{*\} \to X$ represents α and $\{*\} \times S^1 \to X$ represents $[h_o]$. On the other hand such a map exists if and only if the Whitehead product of α and $[h_o]$ is the identity element. Since a Whitehead product of elements in a fundamental group coincides with the corresponding commutator product we conclude, that α is in the image of p_* if and only if $\alpha\,[h_o]\alpha^{-1}[h_o]^{-1} = 1$ or equivalently that $\alpha \in C_H(h_o)$.

Hence $\pi_1(G_{[h_o]}(S^1,X)) \cong C_H(h_o)$.

Altogether we have therefore shown

Proposition 4.3.2. There exists a connected topological space X with finitely generated homotopy groups such that at least one component in the mapping space $G(S^1,X)$ has infinitely generated fundamental group.

Notice however

Proposition 4.3.3 Suppose that $\pi_1(X)$ is abelian and that $\pi_1(X)$ and $\pi_2(X)$ are finitely generated.

Then all the components of $G(S^1,X)$ have finitely generated fundamental group.

Proof This follows immediately from the homotopy sequence of the evaluation fibration

$$F(S^1,X) \to G(S^1,X) \to X,$$

since a subgroup of a finitely generated abelian group is itself finitely generated.

The space X we have constructed in Proposition 4.3.2 is not particularly nice. For this reason and also since the free loop space of a manifold is interesting in the study of closed geodesics, we would like to ask the

Question Does there exist a compact manifold X for which $G(S^1,X)$ has a component with infinitely generated fundamental group?

PART II

SPACES OF DIFFERENTIABLE MAPS

Chapter 5

Preliminaries to Part II.

This chapter is purely expository. §1 contains the definitions of the mapping spaces with which we shall be concerned in Part II. In §2 we collect certain facts about ANR's, including the statement of a fundamental theorem of J.H.C.Whitehead (Theorem 5.2.4). In §3 we discuss the existence of partitions of unity on Banach spaces and the existence of manifold structures on mapping spaces. We indicate also the proof of a slight extension of a theorem of Palais, which states that the homotopy type of a space of differentiable maps is independent of the degree of differentiability (Theorem 5.3.2).

§1 Basic definitions.

In most of Part II M^m shall denote an m-dimensional compact smooth manifold with boundary (which might be empty), and X shall denote a paracompact (equivalent metrizable) smooth manifold without boundary modelled on a Banach space of finite or infinite dimension.

For $0 \leq r \leq \infty$, $C^r(M,X)$ shall denote the space of differentiable maps of class C^r from M into X. We equip $C^r(M,X)$ with the C^r-topology. Notice that for $r = 0$ $C^0(M,X)$ is just the space of continuous maps with the familiar compact-open topology.

Now let $1 \leq r \leq \infty$ and let $0 \leq k$. Then $C^r(M,X;k)$ shall denote the subspace of $C^r(M,X)$ containing the k-mersions of M into X of class C^r. Recall that $f \in C^r(M,X)$ is a k-mersion if it has rank $\geq k$ everywhere. For k big $C^r(M,X;k)$ will then, of course, be empty. Observe that for $k = 0$ $C^r(M,X;0) = C^r(M,X)$ and that for $k = \dim M$ we get the space of immersions $Imm^r(M,X)$ of M into X of class C^r.

Likewise for $1 \leq r \leq \infty$ we shall denote by $Emb^r(M,X)$ the subspace of $C^r(M,X)$ containing the embeddings of M into X of class C^r.

Since M is compact both $C^r(M,X;k)$ and $Emb^r(M,X)$ are open subspaces of $C^r(M,X)$.

Chapters 7 and 8 will be occupied with a study of the spaces $C^r(M,X)$, $C^r(M,X;k)$ and $Emb^r(M,X)$. In Chapter 8 we shall relate $C^r(M,X;k)$ to a space of bundle maps Hom(TM,TX;k), which we now proceed to define.

Let TM and TX denote the total space in the tangent bundle of respectively M and X. Tangent spaces will be denoted respectively by T_pM for $p \in M$ and T_xX for $x \in X$. Let again $0 \leq k$. Then Hom (TM,TX;k) shall denote the space of continuous maps of TM into TX, which maps each fibre of TM into a fibre of TX by a linear map of rank $\geq k$. We equip Hom (TM,TX;k) with the compact-open topology.

Let F and E denote Banach spaces with F finite dimensional and let $0 \leq k$. Then L(F,E;k) shall denote the space of linear maps of F into E of rank $\geq k$. Observe that L(F,E;k) is an open subset of the Banach space L(F,E) of linear maps.

We shall now define certain locally trivial smooth fibrations, namely

1) $\pi' : L(F,TX;k) \to X$

with fibre $L(F,T_xX;k)$ over $x \in X$.

2) $\pi'' : B(TM,TX;k) \to M \times X$

with fibre $L(T_pM,T_xX;k)$ over $(p,x) \in M \times X$.

and finally

3) $\pi : B(TM,TX;k) \to M$

with fibre $L(T_pM,TX;k)$ over $p \in M$.

The definitions are simple. If $\vee$ denotes disjoint union, then

$$L(F,TX;k) = \bigvee_{x\in X} L(F,T_xX;k)$$

and

$$B(TM,TX;k) = \bigvee_{(p,x)\in M\times X} L(T_pM,T_xX;k)$$

π' and π'' are the obvious maps. π is the composition of π'' and the projection $M \times X \to M$.

The standard technique in the category of smooth vector bundles for constructing new vector bundles out of old ones by applying a functor to each fibre can be applied to equip π' with the structure of a locally trivial smooth fibration. This construction will work, since $L(F,\cdot;k)$ is a 'functor' on isomorphisms, and that is all that is needed. See Lang [61].

It is well-known that π'' has a natural structure as a locally trivial smooth fibration. See e.g. Abraham and Robbin [1].

Finally, since a composition of locally trivial fibrations over a locally contractible space is again a locally trivial fibration (see e.g. Palais [75], Theorem 14.12), it follows that π is a locally trivial smooth fibration.

Now let $\Gamma^0(\pi)$ denote the space of continuous sections of π equipped with the compact-open topology. It is then well-known, and obvious, that Hom $(TM,TX;k)$ can be identified with $\Gamma^0(\pi)$. This description of Hom $(TM,TX;k)$ will be very useful to us in Chapter 8.

§2 Miscellaneous results on ANR's.

For us an ANR (Absolute Neighbourhood Retract) shall always be in the class of metrizable spaces. We recall that a metrizable space X is an ANR if it has the following property: For any metrizable space Y in which X is embedded as a closed subspace there exists an open neighbourhood U of X in Y and a retraction $r: U \to X$. In the class of metrizable spaces X is an ANR if and only if it is an ANE (Absolute Neighbourhood Extensor), i.e.: Any continuous map $f:A \to X$ from a closed subspace A of a metrizable space Y has a continuous extension $F:U \to X$ to an open neighbourhood U of A in Y.

The basic facts about ANR's can be found in Borsuk [8] and Hu [49]. ANR's are particularly nice spaces to work with in the homotopy theory of mapping spaces since they have the same fundamental properties as CW-complexes and since they are naturally preserved under mapping space constructions (Theorem 5.2.1 below). A lot of facts about ANR's of particular interest in the homotopy theory of infinite dimensional manifolds (mapping spaces) have been collected by Palais [74]. We mention in particular ([74], Theorem 5), which states that an arbitrary metrizable topological manifold modelled on a LCTVS (locally convex topological vector space) is an ANR. Usually the LCTVS's we shall meet will be either Banach spaces or Fréchet spaces (complete metrizable LCTVS's). Palais's paper [74] will be our main reference for unproved statements about ANR's in the following.

We shall, in fact, only need just two results which cannot be found in [74]. We list these results as Theorems 5.2.1 and 5.2.2

<u>Theorem 5.2.1</u> Let $F \to E \xrightarrow{\pi} B$ be a locally trivial fibration. Suppose that B is compact metrizable and that F is an ANR. Let $\Gamma^{0}(\pi)$ denote the space of continuous sections of π equipped with the compact-open topology.

Then E is metrizable, and $\Gamma^{0}(\pi)$ is an ANR.

If π is trivial, then $E = B \times F$ and $\Gamma^{0}(\pi) = C^{0}(B,F)$. In this case Theorem 5.2.1 is well-known. See e.g. Hu ([49], Chp. VI,

§2, especially Theorem 2.4). The proof of the slightly more general case is not difficult and is left to the reader.

For separable ANR's and countable CW-complexes the following theorem can be found in Milnor [65] and Palais [74]. I am indebted to Professor Eells for pointing out to me that separability (countability) is not needed in the theorem.

<u>Theorem 5.2.2</u> It is equivalent for a topological space X to have the homotopy type of a CW-complex and of an ANR.

<u>Proof</u> Assume first, that X is an ANR. Then by an extension of Hanner's result in the separable case Palais proves ([74], Theorem 14), that X is dominated by a simplicial complex. By a theorem of Milnor ([65], Theorem 2) X has therefore the homotopy type of a CW-complex.

Next assume that X is a CW-complex. Then by the theorem of Milnor mentioned above X has the homotopy type of a simplicial complex with the metric topology. But a simplicial complex with the metric topology is an ANR, see Hu ([49], Theorem 11.3, p. 106).

This proves the theorem.

We want now to recall a fundamental theorem of J.H.C.Whitehead. First we make however the following almost standard

<u>Definition 5.2.3</u> Let X and Y be non-empty topological spaces, and let $f:X \to Y$ be a continuous map. Let also $q \geq 0$ be an integer.

We call f a <u>0-equivalence</u> if the induced map between path-components $f_*: \pi_0(X) \to \pi_0(Y)$ is onto.

For $q \geq 1$ we call f a <u>q-equivalence</u> if $f_*: \pi_0(X) \to \pi_0(Y)$ is a bijection, and if for any base point $x \in X$ the induced map $f_*: \pi_i(X,x) \to \pi_i(Y,f(x))$ is an epimorphism for $0 < i < q$ and a monomorphism for $0 < i < q-1$.

Finally, we call f a <u>weak homotopy equivalence</u> if it is a q-equivalence for all $q \geq 0$.

In later chapters we shall frequently use the following theorem of J.H.C.Whitehead [93].

<u>Theorem 5.2.4</u> Suppose that X and Y are topological spaces with the homotopy type of ANR's.

Then a continuous map $f: X \to Y$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.

§3 <u>Structures on mapping spaces.</u>

For simplicity of exposition we shall throughout Part II assume that the Banach spaces we use as models for the smooth manifolds X are C^∞-smooth. By definition a Banach space E is said to be <u>C^∞-smooth</u> (or C^∞-paracompact) if any open covering of E admits a subordinated smooth partition of unity.

It is known that e.g. the following Banach spaces are C^∞-smooth: Finite dimensional Banach spaces, the infinite dimensional separable Hilbert space (Eells, see Lang [61]), L^p-spaces for p an even integer (Kurzweil, see e.g. Sundaresan [85]), the space c_0 of real sequences converging to 0 (Kuiper, see Bonic and Frampton [7]). On the other hand L^p-spaces for p an odd integer admits only partitions of unity of class C^{p-1}. It is still unknown whether a non-separable Hilbert space is C^∞-smooth. *) <u>Recently Wells</u> [90] has shown that they do admit partitions of

*) Latest: H.Toruńczyk has shown that it is, January 1972.

unity of class C^1. For further information on the problem concerning existence of differentiable partitions of unity on Banach spaces see Eells [22] and Bonic and Frampton [7].

If X is a metrizable smooth manifold modelled on a C^∞-smooth Banach space then it is clear that X itself admits smooth partitions of unity. This implies that X admits a smooth spray and therefore also an associated exponential map. See Lang [61]. Hence we can use the general construction principle for manifold structures on spaces of maps formulated by Eells ([22], §6) to prove the following

Theorem 5.3.1 Let M be a compact smooth manifold, and let X and Y be metrizable smooth manifolds modelled on C^∞-smooth Banach spaces

1) For $0 \leq r < \infty$ $C^r(M,X)$ can be given the structure of a smooth metrizable manifold.

2) $C^\infty(M,X)$ can be given the structure of a metrizable manifold modelled on Fréchet spaces.

3) $C^r(M,X)$ is an ANR for all $0 \leq r \leq \infty$ (consequence of 1) and 2)).

4) A smooth map $f: X \to Y$ induces a map $f_*: C^r(M,X) \to C^r(M,Y)$ by composition of maps for each $0 \leq r \leq \infty$. For $0 \leq r < \infty$ f_* is smooth and for $r = \infty$ it is continuous.

5) If $f: X \to Y$ is a smooth embedding, then $f_*: C^r(M,X) \to C^r(M,Y)$ is an embedding, smooth for $0 \leq r < \infty$ and continuous for $r = \infty$.

Since $C^r(M,X;k)$ and $Emb^r(M,X)$ for $1 \leq r \leq \infty$ are open subspaces of $C^r(M,X)$ these spaces will also be metrizable manifolds, in particular ANR's.

The only essential use we shall make of Theorem 5.3.1 is to ensure that $C^r(M,X)$ is a metrizable manifold (ANR). To achieve this it is not necessary that X admits smooth partitions of unity. Eliasson [29] showed that when X is a metrizable smooth manifold which admits a spray, then all iterated mapping spaces, like $C^s(M,C^r(M,X))$, admit a manifold structure. This is non-trivial, since $C^r(M,X)$ does not admit smooth partitions of unity even if X does, but as Eliasson shows $C^r(M,X)$ does admit a spray if X does. Next Krikorian [56] showed that for $0 < r < \infty$ $C^r(M,X)$ has the structure of a metrizable smooth manifold for any metrizable smooth manifold X. This was reproved by Penot [77] who in addition proved that $C^0(M,X)$ is a smooth manifold even if M and X are just topological manifolds. Both Krikorian and Penot used a method due to Douady [17]. It seems not to be clear that one can get the manifold structure on $C^\infty(M,X)$ without any conditions on X. Finally, we should mention that Geoghegan [35] using a completely different method has produced a Hilbert manifold structure on $C^0(M,X)$ if M and X are just polyhedra.

Since one can prove that $C^r(M,X)$ is an ANR under much more general conditions than in Theorem 5.3.1 many of the results in Chapters 7 and 8 can be slightly improved. We shall abstain from this here but mention that it is done to a certain extent in our papers [40], [41] and [42].

Finally, we shall indicate the proof of the following slightly generalized result of Palais ([75], Theorem 13.14).

<u>Theorem 5.3.2</u> Let M be a compact smooth manifold, and let X be a metrizable smooth manifold modelled on a C^∞-smooth Banach space. Let also $1 \leq r < \infty$.

Then the natural map

$$C^r(M,X) \to C^0(M,X)$$

is a homotopy equivalence.

Proof The following proof is essentially just the argument given by Palais in [75].

Without loss of generality we can identify X with a closed smooth submanifold of a Banach space E. The proof of this fact is an almost 'classical' application of smooth partitions of unity and has been carried through in all details by Penot [77]. Since any Banach space admits a spray it follows that X has a tubular neighbourhood in E (the proofs in Lang [61] work without changes). There exists therefore an open neighbourhood U of X in E and a smooth strong deformation retraction π: U $\rightarrow$ X. From the fundamental theorem of Palais ([74], Theorem 16) it follows now easily that the inclusion map $C^r(M,U) \rightarrow C^0(M,U)$ is a homotopy equivalence. Obvious use of the smooth strong deformation retraction π finishes then the proof.

Chapter 6

Expanding systems

In this chapter we study the topology of the limit space of an expanding system, in particular of a closed expanding system. §1 contains the necessary definitions and some examples of closed expanding systems are given. §2 contains the main result in this chapter, namely Theorem 6.2.5, which states, that a limit space for a closed expanding system of finite dimensional manifolds of unbounded dimension is a manifold modelled on R^∞. R^∞ is here the topological vector space of finitely non-zero real sequences topologized with the finite topology. Since R^∞ is not metrizable such limit spaces cannot be metrizable, in particular they are not ANR's. This could have caused us serious trouble in changing weak homotopy equivalences into homotopy equivalences. However trouble does not arise since in §3 we show that the limit space of a closed expanding system of ANR's has the homotopy type of an ANR (Corollary 6.3.4).

It is only the results in §3 which will be needed in later chapters.

§1 Definitions and examples.

First we recall the definition of a (closed) embedding. Let X and Y be topological spaces. A continuous map $f:X \to Y$ is called an embedding, if f is a homeomorphism of X onto f(X) considered with the subspace topology in Y. If furthermore f(X) is closed in Y then we call f a closed embedding. For X and Y smooth manifolds we have, of course, smooth counterparts to these notions.

Next we define the main objects for our investigations.

Definition 6.1.1 An expanding system of topological spaces $(\underline{X},\underline{f},n_0) = \{X_n, f_{n,n+1}\}_{n \geq n_0}$ is a system of topological spaces X_n and embeddings $f_{n,n+1}: X_n \to X_{n+1}$ indexed over the integers $n \geq n_0$.

If all the embeddings $f_{n,n+1}$ are closed embeddings, then we call $(\underline{X},\underline{f},n_0)$ a closed expanding system.

If all the topological spaces X_n are smooth manifolds, and all the embeddings $f_{n,n+1}$ are smooth embeddings, then we call $(\underline{X},\underline{f},n_0)$ a smooth expanding system.

In the obvious way we could also have defined open expanding systems. We have not included any material on such systems here, since we basically shall be concerned with closed expanding systems. The interested reader can find a few results on open expanding systems in our paper [40].

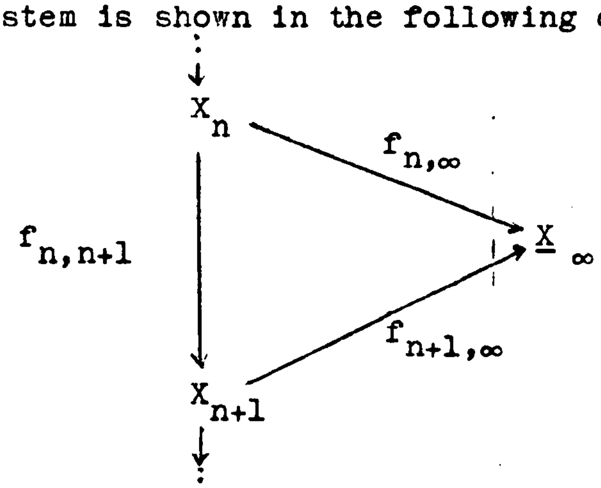
The terms in Definition 6.1.1 will occur so often in the following that we will abbreviate expanding system and closed expanding system to ES and CES respectively.

If $(\underline{X},\underline{f},n_0)$ is an arbitrary ES then we define its limit space $\underline{X}_\infty$ as the direct limit of the system $\{X_n, f_{n,n+1}\}_{n \geq n_0}$, i.e.

$$\underline{X}_\infty = \lim_{\substack{\to \\ n}} \{X_n, f_{n,n+1}\}_{n \geq n_0}.$$

As usual the direct limit $\underline{X}_\infty$ is the identification space obtained from the disjoint union $\bigvee_{n=n_0}^{\infty} X_n$ of the spaces X_n by identifying $x_n \in X_n$ with $f_{n,n+1}(x_n) \in X_{n+1}$. If $f_{n,\infty} : X_n \to \underline{X}_\infty$ denotes the composition of the inclusion of X_n into $\bigvee_{n=n_0}^{\infty} X_n$ followed by the projection of this space onto $\underline{X}_\infty$, then the topology on $\underline{X}_\infty$ can be described as the largest (finest) topology making all the maps $f_{n,\infty}$ continuous.

The whole system is shown in the following diagram



We remark that a subset of $\underline{X}_\infty$ is open or closed in the direct limit topology on $\underline{X}_\infty$ if and only if all the preimages of the set in the spaces X_n are open respectively closed.

For $n < m$ we put for convenience $f_{n,m} = f_{m-1,m} \circ \dots \circ f_{n,n+1}$. Let also $f_{n,n}$ be the identity map on X_n.

Let us record the following

<u>Lemma 6.1.2</u> Let $(\underline{X}, \underline{f}, n_0)$ be an ES.

1) The map $f_{n,\infty}$ is an embedding for all $n \geq n_0$.

2) If $(\underline{X}, \underline{f}, n_0)$ is a CES, then all the maps $f_{n,\infty}$ are closed embeddings.

Proof $f_{n,\infty}$ is clearly a continuous injective map. In order to finish the proof of 1) it will therefore be sufficient to prove that $f_{n,\infty}$ is also an open map onto its image. For that purpose let U_n be an open set in X_n. Since $f_{n,n+1}$ is an embedding there exists an open set U_{n+1} in X_{n+1}, such that $f_{n,n+1}(U_n) = f_{n,n+1}(X_n) \cap U_{n+1}$. Go on and choose sets $\{U_{n+k}\}_{k\geq 1}$ such that U_{n+k} is open in X_{n+k} and such that

$$f_{n+k,n+k+1}(U_{n+k}) = f_{n+k,n+k+1}(X_{n+k}) \cap U_{n+k+1} \qquad \text{for all } k \geq 0.$$

Now $U_\infty = \bigcup_{k=0}^{\infty} f_{n+k,\infty}(U_{n+k})$ will be open in $\underline{X}_\infty$, and clearly $f_{n,\infty}(U_n) = f_{n,\infty}(X_n) \cap U_\infty$. This shows, that $f_{n,\infty}(U_n)$ is open in $f_{n,\infty}(X_n)$. As already remarked this finishes the proof of 1).

2) is even easier and is left to the reader.

When we are just dealing with the topology of the limit space $\underline{X}_\infty$ for an ES $(\underline{X},\underline{f},n_o)$ the embeddings $f_{n,n+1}$ will not come into play. By Lemma 6.1.2 we can then identify X_n with $f_{n,\infty}(X_n)$ and think of the system as a system of inclusion maps

$$X_{n_o} \subset X_{n_o+1} \subset \cdots \subset X_n \subset \cdots \subset \underline{X}_\infty,$$

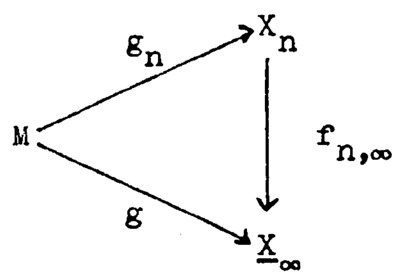
where $\underline{X}_\infty = \bigcup_{n \geq n_o} X_n$ has the weak topology w.r.t. the subspaces X_n.

The n_o in the definition of an ES $(\underline{X},\underline{f},n_o)$ is of course not important. What matters is the limit space $\underline{X}_\infty$, and it is clear that if we take the same spaces X_n and embeddings $f_{n,n+1}$ but only from $m_o > n_o$ and upwards, then $(\underline{X},\underline{f},m_o)$ gives the same limit space as $(\underline{X},\underline{f},n_o)$.

We shall often use the following straightforward but important

<u>Lemma 6.1.3</u> Let $(\underline{X},\underline{f},n_o)$ be an ES of T_1-spaces, and let $g:M \to \underline{X}_\infty$ be a continuous map such that $g(M)$ is a compact subset of $\underline{X}_\infty$.

Then there exists an $n \geq n_o$ and a continuous map $g_n: M \to X_n$ such that the following diagram commutes



<u>Proof</u> Observe that $\underline{X}_\infty = \bigcup_{n \geq n_o} f_{n,\infty}(X_n)$.

Assume, that $g(M)$ is not contained in any of the subspaces $f_{n,\infty}(X_n)$ of $\underline{X}_\infty$. We can then find an infinite subspace

$S = \{x_{n_k}\}_{k=1}^{\infty}$ of $\underline{X}_\infty$, such that $x_{n_k} \in (\underline{X}_\infty \setminus f_{n_k,\infty}(X_{n_k})) \cap g(M)$ for all $1 \leq k \leq \infty$.

Since an arbitrary subset of S by construction has at most a finite number of points in common with any $f_{n,\infty}(X_n)$, it is a closed subspace. It is here we need the spaces X_n to be T_1-spaces. S is therefore a closed discrete subspace of $g(M)$. But since $g(M)$ is compact, it cannot contain any infinite closed discrete subspace.

This shows that there exists an n such that $g(M) \subset f_{n,\infty}(X_n)$. Since $f_{n,\infty}$ is an embedding of X_n into $\underline{X}_\infty$ by Lemma 6.1.2, there exists therefore a unique continuous map $g_n : M \to X_n$ such that $g = f_{n,\infty} \circ g_n$. This proves the lemma.

Finally, in this section we give some examples of closed expanding systems.

Example 6.1.4 (CW-complexes). By definition any CW-complex X is the limit space of the CES $(\underline{X},\underline{f},0)$, where X_n is the n-skeleton of X and $f_{n,n+1} : X_n \to X_{n+1}$ is the obvious inclusion.

Example 6.1.5 (Symmetric products).

Let X be a connected topological space with base point.

For each $n \geq 2$ the symmetric group on n objects, S_n, acts in the obvious way on the n-fold product $X \times \cdots \times X$. The orbit space for this action is called the n-fold symmetric product of X and is denoted by $SP^n(X)$. Observe, that each point in $SP^n(X)$ has a representative of the form $(x_1,\ldots,x_n) \in X \times \cdots \times X$.

For each $n \geq 2$ we get a closed embedding $f_{n,n+1}:SP^n(X) \to SP^{n+1}(X)$ by viewing $SP^n(X)$ as the set of points in $SP^{n+1}(X)$ having a representative with the base point in X as the last coordinate.

The spaces $SP^n(X)$ and the embeddings $f_{n,n+1}$ form a CES $(\underline{SP}(X),\underline{f},2)$. The limit space for this CES is called the infinite symmetric product of X and is denoted by $SP^\infty(X)$, i.e.

$$SP^\infty(X) = \underline{SP}_\infty(X) = \varinjlim_n SP^n(X).$$

$SP^\infty(X)$ has been studied in great detail by Dold and Thom in [16], where the following nice theorem is proved

$$\pi_1(SP^\infty(X)) = H_1(X;Z).$$

Suppose now that X is a closed smooth manifold. Then $SP^n(X)$ will be a manifold with certain singularities, but only when X is 2-dimensional $SP^n(X)$ will actually be a smooth manifold. As an example $SP^n(S^2) = CP^n$, complex projective n-space. See the discussion in Dupont and Lusztig [18].

For each closed surface X we get therefore a smooth CES $(\underline{SP}(X), \underline{f}, 2)$.

Example 6.1.6 (Classifying spaces).

Let G be a compact topological group. We can then for each $n \geq 1$ form the G-space

$$EG_n = G * \cdots * G,$$

the n-fold join of G with the usual join topology. Let $BG_n = EG_n/G$ denote the orbit space.

There are natural closed embeddings $f_{n,n+1} : BG_n \to BG_{n+1}$, such that we get a CES $(\underline{BG}, \underline{f}, 1)$. The limit space for this CES is denoted by BG, i.e.

$$BG = \underline{BG}_\infty = \varinjlim_n BG_n$$

In Milnor's original construction of a classifying space for an arbitrary topological group G, [64], $EG_\infty = \bigcup_{n \geq 1} EG_n$, and hence also $BG = EG_\infty/G$, has a different topology. Since we have assumed that G is compact, it is still true that the BG constructed here is a classifying space for G.

For a concrete group G we can often find a smooth CES such that the limit space is a classifying space for G. If we e.g. take $G = O(k)$, the orthogonal group in Euclidian k-space, then the Grassmann manifolds of k-planes in n-space for $n > k$ form such a smooth CES for $O(k)$.

Example 6.1.7 (Homogeneous spaces).

Let E denote the separable Hilbert space and let $\{e_i\}_{i \geq 1}$

be an orthonormal basis for E.

The subspace of E spanned by the vectors $e_1,\ldots,e_n$ is then a model for Euclidian n-space R^n. We get, of course, a smooth CES

$$R^1 \subset R^2 \subset \ldots \subset R^n \subset \ldots$$

This CES induces in the obvious way a lot of other smooth CES's.

<u>Spheres</u> $(\underline{S},\underline{f},1)$:

$$S^1 \subset S^2 \subset \ldots \subset S^n \subset \ldots$$

The limit space $\underline{S}_\infty$, usually denoted by S^∞, is contractible.

<u>Projective spaces</u> $(\underline{RP},\underline{f},1)$:

$$RP^1 \subset RP^2 \subset \ldots \subset RP^n \subset \ldots$$

The limit space $\underline{RP}_\infty$, usually denoted by RP^∞, is an Eilenberg-MacLane space $K(Z_2,1)$.

<u>Stiefel manifolds</u> $(\underline{V}_k, \underline{f}, k)$:

$$V_k(k) \subset V_k(k+1) \subset \ldots \subset V_k(n) \subset \ldots,$$

where $V_k(n)$ is the space of orthonormal k-frames in R^n.

The limit space $(\underline{V}_k)_\infty$, usually denoted by $V_k(\infty)$, is contractible.

<u>Grassmann manifolds</u> $(\underline{G}_k, \underline{f}, k)$:

$$G_k(k) \subset G_k(k+1) \subset \ldots \subset G_k(n) \subset \ldots,$$

where $G_k(n)$ is the Grassmann manifold of k-planes in R^n.

The limit space $(\underline{G}_k)_\infty$, usually denoted by $G_k(\infty)$, is a classifying space for $O(k)$.

In Chapter 8, §1 we will give more examples of smooth CES's.

§2 Topology of the limit space for an expanding system.

The main purpose of this section will be to prove that the limit space for a CES of finite dimensional manifolds of unbounded dimension can be given the structure of a manifold modelled on R^∞.

First we state however some general topological facts about a limit space.

Since the topology on the limit space $\underline{X}_\infty$ for an ES $(\underline{X},\underline{f},n_o)$ is a quotient topology, we cannot expect too much of it. For a general ES we have however the following result. Recall, that a topological space is a Lindelöf space if every open covering of it contains a countable sub-covering.

Theorem 6.2.1 Let $(\underline{X},\underline{f},n_o)$ be an ES of topological spaces.

1) If all the spaces X_n are T_1-spaces, then $\underline{X}_\infty$ is a T_1-space.

2) If all the spaces X_n are locally compact Hausdorff spaces, then $\underline{X}_\infty$ is a Hausdorff space.

3) If all the spaces X_n are Lindelöf spaces, then $\underline{X}_\infty$ is a Lindelöf space.

Proof We leave the proof of 1) and 3) to the reader and concentrate on the proof of 2). As remarked in §1 we can assume that the ES under consideration has the form

$X_{n_o} \subset X_{n_o+1} \subset \dots \subset X_n \subset \dots \subset \underline{X}_\infty$.

Now let $x,y \in \underline{X}_\infty = \bigcup_{n \geq n_o} X_n$ with $x \neq y$ be given. We have to find disjoint open neighbourhoods of these points in $\underline{X}_\infty$. Pick $n_1 \geq n_o$ such that $x,y \in X_{n_1}$. Since X_{n_1} is locally compact and Hausdorff we can find open neighbourhoods U_{n_1} and V_{n_1} of respectively x and y in X_{n_1}, such that the closures

$\overline{U}_{n_1}$ and $\overline{V}_{n_1}$ in X_{n_1} are compact and disjoint. Since $\overline{U}_{n_1}$ and $\overline{V}_{n_1}$ are also compact and disjoint subsets of the Hausdorff space X_{n_1+1} they can be separated by open sets in X_{n_1+1}. Using the local compactness of X_{n_1+1} it is then easy to find open sets U_{n_1+1} and V_{n_1+1} in X_{n_1+1} which extends U_{n_1} and V_{n_1} and have disjoint compact closures $\overline{U}_{n_1+1}$ and $\overline{V}_{n_1+1}$ in X_{n_1+1}. This indicates how we can construct expanding sequences $\{U_n\}_{n \geq n_1}$ and $\{V_n\}_{n \geq n_1}$, where U_n and V_n are disjoint open neighbourhoods of x and y in X_n for each $n \geq n_1$. Then $U_\infty = \bigcup_{n \geq n_1} U_n$ and $V_\infty = \bigcup_{n \geq n_1} V_n$ will be the required disjoint open neighbourhoods of respectively x and y in $\underline{X}_\infty$.

We are particularly interested in the Lindelöf property, since it is known that a regular Lindelöf space is paracompact. For a connected, locally compact space the converse statement is also true. See Kelley [53]. We remark that paracompact includes the Hausdorff axiom in this paper.

In general the Hausdorff property is not carried on to the limit in a CES. We have however this

Theorem 6.2.2 Let $(\underline{X}, \underline{f}, n_0)$ be a CES of topological spaces.

1) If all the spaces X_n are normal, then $\underline{X}_\infty$ is normal.

2) If all the spaces X_n are regular Lindelöf spaces, then $\underline{X}_\infty$ is a regular Lindelöf space.

Proof Assume for a moment that 1) is proved. Then 2) follows in this way. Each X_n is a regular Lindelöf space, hence paracompact, in particular normal. By 1) $\underline{X}_\infty$ is therefore also normal, in particular regular. That $\underline{X}_\infty$ is Lindelöf follows from Theorem 6.2.1.

We turn then to the proof of 1). By Theorem 6.2.1 we get immediately that $\underline{X}_\infty$ is a T_1-space. It will therefore be sufficient to prove that Urysohn's lemma holds in $\underline{X}_\infty$. For that purpose, let A and B be closed, non-empty, disjoint subsets of $\underline{X}_\infty$. We have to find a continuous function $h: \underline{X}_\infty \to [0,1]$, such that $h|A \equiv 0$ and $h|B \equiv 1$. To do this we proceed as follows:

We can again assume that the CES has the form $X_{n_0} \subset X_{n_0+1} \subset \dots \subset X_n \subset \dots \subset \underline{X}_\infty$ with $\underline{X}_\infty = \bigcup_{n \geq n_0} X_n$. Choose then a sufficiently high $n_1 \geq n_0$ such that both $A \cap X_{n_1}$ and $B \cap X_{n_1}$ are non-empty. Using the normality of X_{n_1} we can now find a continuous function $h_{n_1} : X_{n_1} \to [0,1]$, such that $h_{n_1}|A \cap X_{n_1} \equiv 0$ and $h_{n_1}|B \cap X_{n_1} \equiv 1$. Consider then the closed subset

$$X_{n_1} \cup (A \cap X_{n_1+1}) \cup (B \cap X_{n_1+1})$$

in X_{n_1+1}. Using Tietze's extension theorem it follows now immediately that there exists a continuous function $h_{n_1+1} : X_{n_1+1} \to [0,1]$, such that $h_{n_1+1}|A \cap X_{n_1+1} \equiv 0$ and $h_{n_1+1}|B \cap X_{n_1+1} \equiv 1$, and such that $h_{n_1+1}|X_{n_1} \equiv h_{n_1}$.

This indicates how we can construct a family $\{h_n\}_{n \geq n_1}$ of continuous functions $h_n : X_n \to [0,1]$, such that $h_{n+1}|X_n = h_n$ and such that $h_n|A \cap X_n \equiv 0$ and $h_n|B \cap X_n \equiv 1$. But then we get an induced continuous map $h = h_\infty : \underline{X}_\infty \to [0,1]$, which by construction will satisfy the conditions $h|A \equiv 0$ and $h|B \equiv 1$.

As already remarked this finishes the proof of Theorem 6.2.2.

We shall now begin our study of the limit space of a CES of finite dimensional manifolds of unbounded dimension. We want to equip the limit space with the structure of a topological manifold modelled on a LCTVS (locally convex topological vector space). The first difficulty we run into is then that a direct limit of TVS's not always is a TVS. We have therefore to investigate this problem a little.

Recall that a Fréchet space is a complete metrizable LCTVS. Let then $E_{n_o} \subset E_{n_o+1} \subset \dots \subset E_n \subset \dots$ be an increasing sequence of Fréchet spaces E_n, such that E_n is a subspace of E_{n+1} in the sense of TVS's for all $n \geq n_o$. Put $E_\infty = \bigcup_{n \geq n_o} E_n$. Then E_∞ has a natural real vector space structure, and it is known that it can be given the structure of a Hausdorff LCTVS by taking as neighbourhoods of 0 convex sets which intersect each E_n in an open neighbourhood of $0 \in E_n$. With this LCTVS structure E_∞ is a so-called LF-space. See e.g. Treves [88] for the result just mentioned. The topology in this locally convex structure on E_∞ is usually different from the direct limit topology (the weak topology) w.r.t. the topological spaces E_n. We have however this result.

<u>Lemma 6.2.3</u> Let $E_{n_o} \subset E_{n_o+1} \subset \dots \subset E_n \subset \dots$ be an increasing sequence of finite dimensional vector spaces with their canonical Hausdorff TVS structures. The inclusions are inclusions as linear subspaces.

Then the locally convex topology and the direct limit topology on $E_\infty = \bigcup_{n \geq n_o} E_n$ w.r.t. the subspaces E_n coincides.

In particular E_∞ will therefore be a LCTVS in the direct limit topology w.r.t. the subspaces E_n.

<u>Proof</u> The locally convex topology is always smaller than the direct limit topology, so it is in the proof of the converse statement that we need the vector spaces E_n to be finite dimensional. To prove that an open set in the direct limit topology is also open in the locally convex topology, it will be sufficient to prove the following: If $U \subset E_\infty$ is an arbitrary subset of E_∞, such that $U \cap E_n$ is open in E_n for each $n > n_o$, and $x \in U$ is an arbitrary point in U, then there exists a convex neighbourhood K of x in E_∞, such that $K \cap E_n$ is open in E_n for each $n > n_o$ and $x \in K \subset U$. This statement is on the other hand easily proved using the local compactness of the finite dimensional vector spaces E_n. One merely starts in the space $E_{n(x)}$ with the lowest index $n = n(x)$ such that $x \in E_n$ and then build a K with the required properties step by step.

If E is an arbitrary vector space, the <u>finite topology</u> on E is the direct limit topology on E w.r.t. the directed set of finite dimensional subspaces of E considered with their unique Hausdorff TVS topologies. A subset $U \subset E$ is therefore open (closed) in the finite topology if and only if $U \cap F$ is open (closed) in F for every finite dimensional subspace F of E. When the subspaces E_n of E_∞ are finite dimensional as in Lemma 6.2.3, it is obvious, that the finite topology on E_∞ coincides with the direct limit topology w.r.t. the subspaces E_n and thus also with the locally convex topology. Under the assumptions in Lemma 6.2.3 E_∞ will therefore be a LCTVS in the finite topology. In general it is known that a vector space E is a TVS in its finite topology if and only if E is at most countable dimensional. See Palais [74] and the reference there to a paper by Kakutani and Klee [52]. There is however a slight mistake in Palais's argument for his Lemma 6.10 (The convex neighbourhood $N(x^0,\{\varepsilon_i\})$ is not necessarily contained in U). This rather trivial mistake is corrected by our Lemma 6.2.3.

We have a canonical countable dimensional LCTVS in its finite topology, denoted by R^∞. R^∞ is the vector space of real sequences $(x_n)_{n\geq 1}$, such that $x_n \neq 0$ for at most a finite number of indexes.

If E is an arbitrary countable dimensional TVS in its finite topology, then it is isomorphic to R^∞ as topological vector spaces, since it is obviously isomorphic to R^∞ as vector spaces and since the topologies cause no trouble in this case, because we have the finite topology on both spaces.

The following lemma is then easily proved.

<u>Lemma 6.2.4</u> Let $E_{n_0} \subset E_{n_0+1} \subset \dots \subset E_n \subset \dots$ be an increasing sequence of finite dimensional vector spaces with their canonical Hausdorff TVS structures. The inclusions are inclusions as linear subspaces. Assume also that the dimension of the vector spaces E_n is unbounded.

Then $E_\infty = \bigcup_{n\geq n_0} E_n$ is a LCTVS in the direct limit topology w.r.t. the subspaces E_n. Furthermore E_∞ is isomorphic to R^∞ as topological vector spaces.

We recall now a few definitions from the theory of topological manifolds. Let E be a TVS and let F and G be closed linear subspaces of E which splits E into $E = F \times G$. A subset X of a topological manifold Y modelled on E is then called a topological submanifold of Y modelled on F, if for each $x \in X$ there exists a coordinate chart (U,θ) on Y centred at x ($\theta(x) = 0$) such that $\theta(U) = E$ and $\theta(U \cap X) = F$.

If X and Y are topological manifolds modelled on the TVS's F and E respectively, then an embedding $f: X \to Y$ is called a locally flat embedding, if $f(X)$ is a topological submanifold of Y modelled on F. Observe that a smooth embedding is always locally flat in finite dimensions.

We can now state and prove the main theorem we have been working towards.

<u>Theorem 6.2.5</u> Let $(\underline{X}, \underline{f}, n_o)$ be a CES of finite dimensional topological manifolds X_n, where all the maps $f_{n,n+1}$ are locally flat embeddings. Assume also that the dimension of the manifolds X_n is unbounded.

Then $\underline{X}_\infty$ is a topological manifold modelled on $\mathbb{R}^\infty$.

<u>Proof</u> Let E_n be a finite dimensional model for X_n. We can assume that E_n is a linear subspace of E_{n+1} for each $n > n_o$, such that we have an increasing sequence $E_{n_o} \subset E_{n_o+1} \subset \dots \subset E_n \subset \dots$ as in Lemma 6.2.4. By this lemma we know already that $E_\infty = \bigcup_{n \geq n_o} E_n$ with the direct limit topology w.r.t. the subspaces E_n is isomorphic to $\mathbb{R}^\infty$ as topological vector spaces. Since we know from Theorem 6.2.1 2) that $\underline{X}_\infty$ is Hausdorff, it will therefore be sufficient to prove the following assertion in order to finish the proof of the theorem.

<u>Assertion</u> Each point of $\underline{X}_\infty$ has an open neighbourhood homeomorphic to an open subset of E_∞.

In order to prove this assertion we proceed as follows:

We can assume that the CES has the form

$$X_{n_o} \subset X_{n_o+1} \subset \dots \subset X_n \subset \dots \subset \underline{X}_\infty \quad \text{with } \underline{X}_\infty = \bigcup_{n \geq n_o} X_n$$

and each X_n a topological submanifold of X_{n+1}.

Now let $x \in \underline{X}_\infty$ be an arbitrary point in $\underline{X}_\infty$, and let $n(x)$ be the smallest index $n = n(x)$ such that $x \in X_n$. Choose a coordinate chart $(U_{n(x)}, \theta_{n(x)})$ on $X_{n(x)}$ centred at x such that $\theta_{n(x)}(U_{n(x)}) = E_{n(x)}$. By a theorem of Lacher ([60], Theorem 2.2) there exists a coordinate chart $(U_{n(x)+1}, \theta_{n(x)+1})$ on $X_{n(x)+1}$ centred at x, such that $\theta_{n(x)+1}(U_{n(x)+1}) = E_{n(x)+1}$, $U_{n(x)} = X_{n(x)} \cap U_{n(x)+1}$ and $\theta_{n(x)} = \theta_{n(x)+1} \restriction U_{n(x)}$. It is obvious that we can continue this extension procedure ending up with charts (U_n, θ_n) on X_n centred at x for all $n \geq n(x)$, such that $\theta_n(U_n) = E_n$ and such that (U_{n+1}, θ_{n+1}) restricts to (U_n, θ_n). Put then $U_\infty = \bigcup_{n \geq n(x)} U_n$ and define $\theta : U_\infty \to E_\infty$ by the requirements $\theta \, | U_n = \theta_n$ for all $n \geq n(x)$. By definition of the topologies on $\underline{X}_\infty$ and E_∞ it follows now immediately that $\theta : U_\infty \to E_\infty$ is a homeomorphism from the open neighbourhood U_∞ of $x \in \underline{X}_\infty$ onto E_∞.

This proves the assertion and hence the theorem.

By Theorem 6.2.5 we can create lots of examples of manifolds modelled on R^∞.

Example 6.2.6 From the smooth CES's mentioned in Example 6.1.7 we get the following manifolds modelled on R^∞:

The infinite dimensional

sphere	... ...	S^∞
projective space	... ...	RP^∞
Stiefel manifold	... ...	$V_k(\infty)$
Grassmann manifold	... ...	$G_k(\infty)$

All these examples certainly suggest than one should study the properties of infinite dimensional manifolds modelled on R^∞.

For a large class of Fréchet spaces (perhaps all) it is known, Henderson [43], that the topological type of a manifold modelled on such a Fréchet space is completely determined by the homotopy type. Hence it is natural to ask the following

Question Are two manifolds modelled on R^∞ homeomorphic if they have the same homotopy type?

Since S^∞ and $V_k(\infty)$ are contractible we could in particular ask the

Question Are S^∞ and $V_k(\infty)$ homeomorphic to R^∞?

Up to now it seems that topologists studying infinite dimensional manifolds have only dealt with metrizable spaces. In view of the following remark the problem posed above will break this tradition.

Remark 6.2.7 A topological manifold modelled on R^∞ can never be metrizable since R^∞ itself is not metrizable.

The manifolds $\underset{\sim}{X}_\infty$ modelled on R^∞ obtained from Theorem 6.2.5 are therefore never metrizable but nearly always paracompact by Theorem 6.2.2. Recall, that a regular Lindelöf space is paracompact and that these conditions actually are equivalent for locally compact, connected spaces.

Finally, in this section we make the

Remark 6.2.8 In a recent paper Henderson and West [44] have obtained a theorem like our Theorem 6.2.5. They work however with metrizable topologies in the following sense:

Let $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$ be a sequence of metric spaces such that all inclusions are isometries. In our terminology a metric expanding system. Let X_∞^{metric} denote the direct limit of this sequence in the category of metric spaces and isometries. X_∞^{metric} is then just the union $\bigcup_{n\geq 1} X_n$ with the unique metric topology, such that all inclusions $X_n \subset X_\infty^{metric}$ are isometries. Call X_∞^{metric} the metric direct limit of the sequence. Let X_∞^{weak} denote the usual direct limit of the sequence. The identity map (as sets) $X^{weak} \to X_\infty^{metric}$ is clearly continuous. When dealing with manifolds it is usually a homotopy equivalence.

If ℓ_2^f denotes the metrizable LCTVS of finitely non-zero real sequences $(x_n)_{n\geq 1}$ with its standard pre-Hilbert structure, then the result of Henderson and West can be formulated as follows:

<u>Theorem</u> If $M^1 \subset M^2 \subset \ldots \subset M^n \subset \ldots$ is a sequence of metrizable manifolds ($\dim(M^n)=n$) without boundary, each bicollared in the next, then the manifolds may be metrized, so that it is a metric expanding system, whose metric direct limit is an ℓ_2^f-manifold of the same homotopy type as the usual direct limit.

As Henderson and West remark, it is not all choices of metrics making the sequence a metric expanding system, which give the metric direct limit the structure of an ℓ_2^f-manifold.

§3 Homotopy theory of closed expanding systems.

Most of the material in this section is by now well-known. The main results will be Theorems 6.3.2 and 6.3.3. From these theorems we get easily Corollary 6.3.4, which will be used frequently in the following chapters. Proofs of Theorems 6.3.2 and 6.3.3 and of many other nice results in homotopy theory can be found in the paper of T. tom Dieck [13] and in T. Bröcker and T. tom Dieck [9]. Since these sources were not available when we wrote our paper [40] we included in that paper almost all the details in the proofs of Theorems 6.3.2 and 6.3.3. For convenience of the reader we repeat here the exposition given in [40]. This exposition owes much to T. tom Dieck anyway.

Definition 6.3.1 Let $(\underline{X},\underline{f},n_o)$ and $(\underline{Y},\underline{g},n_o)$ be expanding systems of topological spaces. A map between expanding systems $\underline{h}: (\underline{X},\underline{f},n_o) \to (\underline{Y},\underline{g},n_o)$ is a system of continuous maps $h_n : X_n \to Y_n$ making the following diagram commutative

$$\begin{array}{ccccc} \cdots \to X_n & \xrightarrow{f_{n,n+1}} & X_{n+1} \to & \cdots \\ h_n \downarrow & & \downarrow h_{n+1} & \\ \cdots \to Y_n & \xrightarrow[g_{n,n+1}]{} & Y_{n+1} \to & \cdots \end{array}$$

When composition of maps is defined in the obvious way, it is clear that we get a category consisting of ES's starting at $n=n_o$ as objects and the maps in Definition 6.3.1 as morphisms.

Call a map $\underline{h}: (\underline{X},\underline{f},n_o) \to (\underline{Y},\underline{g},n_o)$ for a homotopy equivalence between ES's, if each h_n is a homotopy equivalence in the usual sense.

A map $\underline{h}: (\underline{X},\underline{f},n_o) \to (\underline{Y},\underline{g},n_o)$ induces in the usual way a continuous <u>limit map</u>

$$\underline{h}_\infty : \underline{X}_\infty \to \underline{Y}_\infty$$

<u>Theorem 6.3.2</u> Let $(\underline{X},\underline{f},n_o)$ and $(\underline{Y},\underline{g},n_o)$ be CES's of topological spaces, such that all the embeddings $f_{n,n+1}$ and $g_{n,n+1}$ are cofibrations.

Then a homotopy equivalence between ES's $\underline{h}:(\underline{X},\underline{f},n_o) \to (\underline{Y},\underline{g},n_o)$ induces an ordinary homotopy equivalence $\underline{h}_\infty : \underline{X}_\infty \to \underline{Y}_\infty$.

<u>Theorem 6.3.3</u> Let $(\underline{X},\underline{f},n_o)$ be a CES of topological spaces, such that all the embeddings $f_{n,n+1}$ are cofibrations. Suppose also that all the spaces X_n have the homotopy type of a CW-complex.

Then $\underline{X}_\infty$ has the homotopy type of a CW-complex.

Before entering into the proofs of these theorems we mention the following important

<u>Corollary 6.3.4</u> Let $(\underline{X},\underline{f},n_o)$ be a CES of ANR's. Then $\underline{X}_\infty$ has the homotopy type of an ANR.

Notice that homotopy type is the most we can hope for by Remark 6.2.7.

Corollary 6.3.4 is an easy consequence of Theorem 5.2.2, Theorem 6.3.3 and the following lemma, which can be found in Palais ([74], Theorem 7).

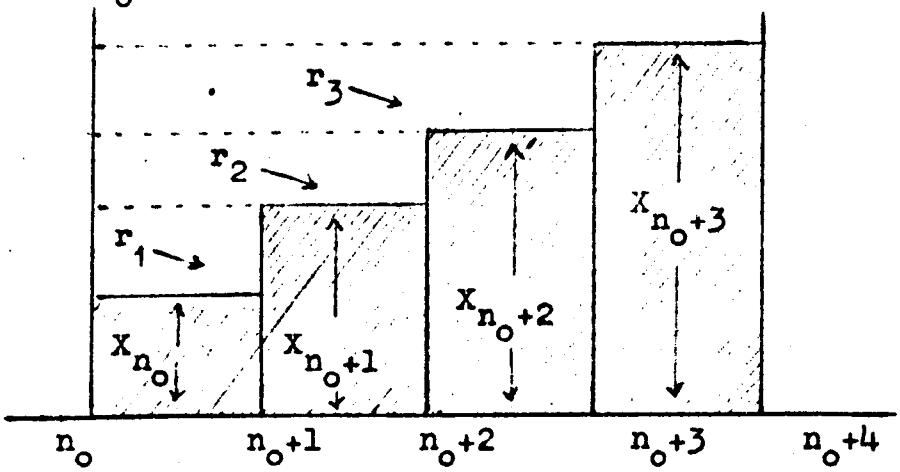
<u>Lemma 6.3.5</u> Let A and X be ANR's and let f: A → X be a closed embedding. Then f is a cofibration.

We begin now the proofs of Theorems 6.3.2 and 6.3.3. For that purpose we introduce the <u>iterated mapping cylinder</u>

(or telescope) of an ES $(\underline{X},\underline{f},n_o)$. We denote this with $Z(\underline{X},\underline{f},n_o)$ and define it as the quotient space of the disjoint union $\bigvee_{n\geq n_o} (X_n \times [n,n+1])$ modulo the relations

$(x_n,n+1) \sim (f_{n,n+1}(x_n),n+1)$ for all $x_n \in X_n$ and all $n > n_o$.

$Z(\underline{X},\underline{f},n_o)$:



The projections $X_n \times [n,n+1] \to X_n$ induce a canonical projection

$$p(\underline{X},\underline{f},n_o) : Z(\underline{X},\underline{f},n_o) \to \underline{X}_\infty$$

Taking into account the following lemma, Theorems 6.3.2 and 6.3.3 will be immediate consequences of the corresponding theorems in the appendix in Milnor [66]. We give however nearly all details here.

Lemma 6.3.6 Let $(\underline{X},\underline{f},n_o)$ be a CES of topological spaces, such that all the maps $f_{n,n+1}$ are cofibrations.

Then the canonical projection $p(\underline{X},\underline{f},n_o)$ is a homotopy equivalence.

Proof As usual we can assume that the CES has the form $X_{n_o} \subset X_{n_o+1} \subset \ldots \subset X_n \subset \ldots \subset \underline{X}_\infty \subset \ldots$ with $\underline{X}_\infty = \bigcup_{n\geq n_o} X_n$. $Z(\underline{X},\underline{f},n_o)$ can then be identified with a subspace of $\underline{X} \times [n_o,\infty[$, and the canonical projection $p(\underline{X},\underline{f},n_o)$ is just the composite map

$$Z(\underline{X},\underline{f},n_o) \hookrightarrow \underline{X}_\infty \times [n_o,\infty[\xrightarrow{\text{proj.}} \underline{X}_\infty$$

Since the projection trivially is a homotopy equivalence it will be sufficient to prove that $Z(\underline{X},\underline{f},n_o)$ is a strong deformation retract of $\underline{X}_\infty \times [n_o,\infty[$. To prove this we use a theorem of Puppe ([80], Satz 4, p.87) to construct a strong deformation retraction

$$r_n : X_{n+1} \times [n_o,n+1] \to X_n \times [n_o,n+1] \cup X_{n+1} \times \{ n+1 \}$$

for all $n \geq n_o$.

As observed by Puppe, the existence of such a strong deformation retraction follows just from the fact that the inclusion $X_n \to X_{n+1}$ is a cofibration. Using these strong deformation retractions it is easy to construct a strong deformation retraction of $\underline{X}_\infty \times [n_o,\infty[$ onto $Z(\underline{X},\underline{f},n_o)$. See the picture of $Z(\underline{X},\underline{f},n_o)$ above.

This proves the lemma.

Now let $(\underline{X},\underline{f},n_o)$ and $(\underline{Y},\underline{g},n_o)$ be ES's of topological spaces and let $\underline{h} = \{ h_n\}_{n\geq n_o}$ and $\underline{\varphi} = \{\varphi_n\}_{n\geq n_o}$ be systems of continuous maps $h_n: X_n \to Y_n$ and homotopies $\varphi_n: X_n \times [0,1] \to Y_{n+1}$, such that $(\varphi_n)_o = g_{n,n+1} \circ h_n$ and $(\varphi_n)_1 = h_{n+1} \circ f_{n,n+1}$ for all $n \geq n_o$. The following diagram is thus homotopy commutative with the homotopies φ_n as the homotopies in the squares

$$\begin{array}{ccccccc}
\cdots \to & X_n & \xrightarrow{f_{n,n+1}} & X_{n+1} & \to \cdots \\
& h_n \downarrow & \circlearrowleft \varphi_n & \downarrow h_{n+1} & \\
\cdots \to & Y_n & \xrightarrow[g_{n,n+1}]{} & Y_{n+1} & \to \cdots
\end{array}$$

To the systems of maps $\underline{h}$ and $\underline{\varphi}$ we associate a map

$$Z(\underline{h},\underline{\varphi}) : Z(\underline{X},\underline{f},n_o) \to Z(\underline{Y},\underline{g},n_o)$$

defined by

$$Z(\underline{h},\underline{\varphi})(x_n,n+t) = \begin{cases} (h_n(x_n),n+2t) & 0 \le t \le \frac{1}{2} \\ (\varphi_n(x_n,2t-1),\ n+1) & \frac{1}{2} \le t \le 1 \end{cases}$$

for all $x_n \in X_n$ and all $n \geq n_o$.

With these assumptions we have

Lemma 6.3.7 If all the maps h_n are homotopy equivalences, then the map $Z(\underline{h},\underline{\varphi})$ is a homotopy equivalence.

Proof The proof is analogous to the proof of Hilfsatz 7, p.314 in [79]. Puppe proves here the corresponding fact for the ordinary mapping cylinder.

Proof of Theorem 6.3.2 Define the homotopy φ_n as the constant homotopy $(\varphi_n)_t = g_{n,n+1} \circ h_n = h_{n+1} \circ f_{n,n+1}$ for all $t \in [0,1]$.

Then we have the commutative diagram

$$\begin{array}{ccc} Z(\underline{X},\underline{f},n_o) & \xrightarrow{p(\underline{X},\underline{f},n_o)} & \underline{X}_\infty \\ {\scriptstyle Z(\underline{h},\varphi)}\downarrow & & \downarrow{\scriptstyle \underline{h}_\infty} \\ Z(\underline{Y},\underline{g},n_o) & \xrightarrow[p(\underline{Y},\underline{g},n_o)]{} & \underline{Y}_\infty \end{array}$$

Lemma 6.3.6 an Lemma 6.3.7 finishes now the proof.

Proof of Theorem 6.3.3 It is easy to construct a commutative diagram

$$\begin{array}{ccccccc} X_{n_o} & \xrightarrow{f_{n_o,n_o+1}} & X_{n_o+1} & \xrightarrow{f_{n_o+1,n_o+2}} & X_{n_o+2} & \longrightarrow & \cdots \\ {\scriptstyle h_{n_o}}\downarrow & & \downarrow{\scriptstyle h_{n_o+1}} & & \downarrow{\scriptstyle h_{n_o+2}} & & \\ K_{n_o} & \xrightarrow[i_{n_o,n_o+1}]{} & K_{n_o+1} & \xrightarrow[i_{n_o+1,n_o+2}]{} & K_{n_o+2} & \longrightarrow & \cdots \end{array}$$

where each K_n is a CW-complex, each $i_{n,n+1}$ is an inclusion of K_n as a subcomplex of K_{n+1} and each h_n is a homotopy equivalence.

Let $\underline{\varphi} = \{\varphi_n\}_{n \geq n_o}$ be the system of homotopies in the squares.

Consider then the diagram

$$\begin{array}{ccc} Z(\underline{X},\underline{f},n_o) & \xrightarrow{p(\underline{X},\underline{f},n_o)} & \underline{X}_\infty \\ {\scriptstyle Z(\underline{h},\underline{\varphi})}\downarrow & & \\ Z(\underline{K},\underline{i},n_o) & \xrightarrow[p(\underline{K},\underline{i},n_o)]{} & \underline{K}_\infty \end{array}$$

All the maps in this diagram are homotopy equivalences by Lemma 6.3.6 and Lemma 6.3.7. Therefore $\underline{X}_\infty$ is homotopy equivalent to $\underline{K}_\infty$, which by construction is a CW-complex. Hence Theorem 6.3.3 is proved.

Chapter 7

Limit spaces of k-mersions and embeddings.

In this chapter we shall study the spaces of k-mersions $C^r(M,X;k)$ and the space of embeddings $\mathrm{Emb}^r(M,X)$ from a compact smooth manifold M into a metrizable smooth manifold X modelled on a C^∞-smooth Banach space. Here $2 \leq r \leq \infty$ and $0 \leq k$.

First we consider the inclusion maps

$$\begin{array}{ccc} C^r(M,X;k) & \searrow & \\ & & C^r(M,X) \\ \mathrm{Emb}^r(M,X) & \nearrow & \end{array}$$

In §2 we study the connectivity properties of these maps. For X finite dimensional we obtain results, which generalize the classical theorems of Whitney on existence and isotopy of immersions and embeddings (Theorem 7.2.1). For X infinite dimensional we show that these maps are homotopy equivalences (Theorem 7.2.2). To prove such theorems we need a parametrized version of Thom's transversality theorem, which we state in §1.

Next let $(\underline{X},\underline{f},n_0)$ be a smooth CES of finite dimensional manifolds of increasing dimension and consider the following diagram of naturally induced maps

$$\begin{array}{ccc} C^r(M,\underline{X};k)_\infty & \searrow & \\ & & C^r(M,\underline{X})_\infty \to C^0(M,\underline{X})_\infty \xrightarrow{\theta} C^0(M,\underline{X}_\infty) \\ \mathrm{Emb}^r(M,\underline{X})_\infty & \nearrow & \end{array}$$

The main purpose of this chapter is to show that all the maps in this diagram are homotopy equivalences. For θ this is proved in a slightly more general setting in §3. The other maps are treated in §4.

Finally, in §5, we apply the results we have obtained on spaces of embeddings to construct models of classifying spaces for diffeomorphism groups.

The results on spaces of k-mersions will be needed in Chapter 8.

§1 A transversality theorem.

In §2 we shall need a corollary to a parametrized version of Thom's transversality theorem. Before stating the theorem we will have to explain some terminology.

Let Q, M and X be finite dimensional smooth manifolds. Denote by $J^s(M,X)$ the space of s-jets of maps from M into X. For $f \in C^r(Q \times M,X)$ and $0 \leq s \leq r$ we define the partial s-jet of f after M as the map

$$j^s_M(f) : Q \times M \to J^s(M,X),$$

which maps $(q,x) \in Q \times M$ into the usual s-jet of $f_q : M \to X$ at $x \in M$, i.e. $j^s_M(f)(q,x) = j^s(f_q)(x)$.

In the following when we talk about approximations of maps in $C^r(Q \times M,X)$ we will always mean approximations w.r.t. a metric defining the C^r-topology on $C^r(Q \times M,X)$.

The transversality theorem we need reads now as follows

Theorem 7.1.1 Let Q, M and X be finite dimensional smooth manifolds, and let $A \subset Q$ and $K \subset M$ be closed subsets. Let also $W \subset J^s(M,X)$ be a smooth submanifold with closed image and suppose that $r > \max\{\dim(Q \times M) - \operatorname{codim}(W), s\}$.

Then any map $f \in C^r(Q \times M,X)$ such that $j^s_M(f)$ is transversal to W on $A \times K$ can be arbitrarily close approximated with a map $g \in C^r(Q \times M,X)$ such that $g|A \times K = f|A \times K$ and such that $j^s_M(g)$ is transversal to W on all of $Q \times M$.

If we put Q equal to a point in this theorem we get of course the classical Thom transversality theorem. The proof of the version with a parameter space Q stated here can be modelled on the proof of the special case given in e.g. Morlet [67]. The theorem is also a consequence of the general transversality theorem of Abraham (see e.g. Abraham and Robbin [1]). As usual the restriction on the degree of differentiability r is caused by the application of Sard's theorem.

The theorem has this useful

<u>Corollary 7.1.2</u> Let Q^i, M^m and X^n be finite dimensional smooth manifolds and let $A \subset Q$ and $K \subset M$ be closed subsets. Suppose also that $2 \leq r \leq \infty$, $0 \leq k \leq \min\{m,n\}$ and $0 \leq i \leq n - 2k + (m-k)(n-k)$.

Then any map $f \in C^r(Q \times M, X)$ such that f_q has rank $\geq k$ on K for $q \in A$ can be arbitrarily close approximated with a map $g \in C^r(Q \times M, X)$ such that $g|A \times K = f|A \times K$ and such that g_q has rank $\geq k$ on M for all $q \in Q$.

<u>Proof</u> Let $W(p) \subset J^1(M,X)$ be the subset of 1-jets of rank p. $W(p)$ is then a submanifold with closed image of codimension $c(p) = (m-p)(n-p)$ in $J^1(M,X)$. Observe now that a map $g \in C^r(Q \times M, X)$ will satisfy the condition rank $(g_q) \geq k$ on M for all $q \in Q$ if and only if $j^1_M(g)$ avoids $W = W(0) \cup \ldots \cup W(k-1)$. If $c(p) - (i+n) \geq 1$ for all $0 \leq p \leq k-1$ then it is clear that $j^1_M(g)$ will avoid W if and only if $j^1_M(g)$ is transversal to $W(p)$ for $0 \leq p \leq k - 1$. Since $k \leq \min\{m,n\}$ it follows from the formula $c(p) = (m-p)(n-p)$ that $c(p) \geq c(k-1)$ for all $0 \leq p \leq k-1$ and therefore that

$$c(p) - (i+m) \geq c(k-1) - (i+m) = (m-k+1)(n-k+1) - (i+m).$$

Therefore $c(p) - (i+m) \geq 1$ for all $0 \leq p \leq k - 1$ if and only if $i \leq n - 2k + (m-k)(n-k)$. Remark also that $r \geq 2$ is the degree of differentiability we need in this case in order to apply Theorem 7.1.1 since $(i+m) - c(p) \leq -1$ for

$0 \leq p \leq k - 1$.

With these observations at our disposal the corollary is an immediate consequence of Theorem 7.1.1.

§2 Homotopy properties of the inclusion maps of $C^r(M,X:k)$ and $Emb^r(M,X)$ into $C^r(M,X)$.

The purpose of this section is to prove theorems about the connectivities of the inclusion maps mentioned in the headline. Recall Lefinition 5.2.3 for the notion of q-equivalence

For any integers n,m and k we put $q(n,m,k) =$ $= n - 2k + (m-k)(n-k)$.

Theorem 7.2.1 Let M^m and X^n be finite dimensional smooth manifolds with M compact, and let k and r be integers satisfying $0 \leq k \leq \min\{m,n\}$ and $2 \leq r \leq \infty$.

1) If $0 \leq q(n,m,k)$ then $C^r(M,X;k) \neq \emptyset$ and the inclusion map

$$C^r(M,X;k) \to C^r(M,X)$$

is a $q(n,m,k)$-equivalence.

2) If $0 \leq n - 2m - 1$ then $Emb^r(M,X) \neq \emptyset$ and the inclusion map

$$Emb^r(M,X) \to C^r(M,X)$$

is an $(n-2m-1)$-equivalence.

Theorem 7.2.2 Let M^m be a compact smooth manifold and let X be a metrizable smooth manifold modelled on an infinite dimensional C^∞-smooth Banach space, E. Let also k and r be integers satisfying $0 \leq k \leq m$ and $2 \leq r \leq \infty$.

Then $C^r(M,X;k)$ and $Emb^r(M,X)$ are both non-empty and the following inclusion maps are homotopy equivalences:

1) $C^r(M,X;k) \to C^r(M,X)$

2) $Emb^r(M,X) \to C^r(M,X)$.

A path in $C^r(M,X)$ is also called a regular homotopy. If we put $k = m$ in Theorem 7.2.1 1) it follows therefore that any differentiable map is regular homotopic to an immersion when $n - 2m \geq 0$, and that any two immersions which are regular homotopic are regular homotopic through immersions when $n - 2m - 1 \geq 0$. Similar results hold by Theorem 7.2.1. 2) for embeddings when $n - 2m - 1 \geq 0$ and $n - 2m - 2 \geq 0$ respectively. These are of course the classical results of Whitney. Theorem 7.2.1 can therefore be seen as a generalization of Whitney's results.

Remark 7.2.3 For $r = \infty$ the result in Theorem 7.2.1. 2) follows also from a stronger theorem of Dax [12], which takes into account connectedness properties of M and X in the spirit of Haefliger [38].

Proof of Theorem 7.2.1 Let Q^i be a compact smooth manifold with the compact submanifold $A \subset Q^i$ and the base point $q_o \in A \subset Q^i$.

1) It is well-known (and follows in fact immediately from Corollary 7.1.2) that $C^r(M,X;k) \neq \emptyset$ when $q(n,m,k) \geq 0$.

Suppose now that $0 \leq i \leq q(n,m,k)$ and let $f_o \in C^r(M,X;k)$ be an arbitrary k-mersion. Consider then the homotopy class of a map

$$f : (Q^i, A, q_o) \to (C^r(M,X), C^r(M,X;k), f_o)$$

with the associated map

$$\hat{f} : Q^i \times M^m \to X^n.$$

We can assume w.l.o.g. that $\hat{f}$ is of class C^r.

Since $f(A) \subset C^r(M,X;k)$ $\hat{f}_q = f(q)$ has of course rank $\geq k$ on M for all $q \in A$. By Corollary 7.1.2 $\hat{f}$ can then be approximated arbitrarily close in the C^r-topology with a map $\hat{g} : Q^i \times M^m \to X^n$ such that $\hat{g}|A \times M = \hat{f}\ |A \times M$ and such that $\hat{g}_q$ has rank $\geq k$ on M for all $q \in Q^i$. Using a tubular neighbourhood for X in a Banach space E we can therefore also homotope $\hat{f}$ into a map $\hat{g}$ as above by connecting them linearly in the tubular neighbourhood and then projecting onto X. This homotopy will now induce a homotopy of f with a map $g: Q^i \to C^r(M,X;k)$ which is constant equal to $f|A$ over A. f represents therefore relatively the zero class.

Consider now the induced map

$$\pi_1(C^r(M,X;k),f_o) \to \pi_1(C^r(M,X),f_o).$$

If we put $Q^i = S^i$ and $A = q_o$ in the analysis above, we conclude that this map is epi for $0 \leq i \leq q(n,m,k)$. If we put $Q^{i+1} = D^{i+1}$ and $A = S^i$ it follows from the analogous analysis with $\dim Q = i + 1$ that this map is mono for $i + 1 \leq q(n,m,k)$. This is, however, exactly what we had to prove.

2) It is again well-known that $\mathrm{Emb}^r(M,X) \neq \emptyset$ when $n - 2m - 1 \geq 0$.

Suppose now that $0 \leq i \leq n - 2m - 1$ and let $f_o \in \mathrm{Emb}^r(M,X)$ be an arbitary embedding. Consider then the homotopy class of a map

$$f: (Q^i,A,q_o) \to (C^r(M,X),\ \mathrm{Emb}^r(M,X),f_o)$$

with the associated map

$$\hat{f} : Q^i \times M^m \to X^n,$$

which we again w.l.o.g. can assume to be of class C^r.

Since $0 \leq i \leq n - 2m - 1 \leq n - 2m$ we can as in the proof of 1) homotope f into a map $g : Q^i \to \mathrm{Imm}^r(M,X)$ fixing everything on A. We observe now that $\mathrm{Emb}^r(M,X)$ in this case is equal to the space of 1-1 immersions, since M is compact. It is easy to see that $g(q) = \hat{g}_q$ is 1-1 for all $q \in Q^i$ if and only if the map defined by the diagram

$$Q^i \times M^m \times M^m \xrightarrow{\Delta_Q \times 1_M \times 1_M} Q^i \times Q^i \times M^m \times M^m$$

$$\xrightarrow{1_Q \times \mathrm{Twist} \times 1_M} Q^i \times M^m \times Q^i \times M^m \xrightarrow{\hat{g} \times \hat{g}} X^n \times X^n$$

maps $Q^i \times (M^m \times M^m \setminus \Delta_M)$ into $X^n \times X^n \setminus \Delta_X$. This last condition is a transversality condition when $i + 2m < n - 1$ or equivalently $i \leq n - 2m - 1$. A transversality argument will therefore allow an arbitrary close approximation of $\hat{g}$ with a map $\hat{h}$ such that $\hat{h} \mid A \times M = \hat{g} \mid A \times M$ and such that $\hat{h}_q$ is 1-1 for all $q \in Q^i$ provided of course $0 \leq i \leq n - 2m - 1$. Proceeding as in the proof of 1) we can therefore homotope g into a map $h : Q^i \to \mathrm{Emb}^r(M,X)$ such that the homotopy is constant equal to $g \mid A = f \mid A$ on A. f represents therefore relatively the zero class.

The proof of 2) is now finished in analogy with the proof of 1).

Proof of Theorem 7.2.2 A chart on X provides a diffeomorphism $\theta : U \to E$ from an open set $U \subset X$ into the model Banach space E. Let $E = F \oplus R^n$ be a splitting of E into a Banach space F and a copy of euclidian n-space R^n with $n \geq 2m + 1$. Choose now an arbitrary embedding $f_2 : M^m \to R^n$ and an arbitrary differentiable map $f_1 : M^m \to F$. Then $f = \theta^{-1} \circ (f_1 \times f_2) : M \to X$ is an embedding. Therefore $\mathrm{Emb}^r(M,X) \neq \emptyset$ and hence also $C^r(M,X;k) \neq \emptyset$.

Let now again Q^i be a compact smooth manifold with the compact submanifold $A \subset Q^i$ and the base point $q_0 \in A \subset Q^i$.

1) Let $f_o \in C^r(M,X;k)$ be an arbitrary k-mersion and consider for any $i \geq 0$ the homotopy class of a map

$$f : (Q^i, A, q_o) \to (C^r(M,X), C^r(M,X;k), f_o)$$

with the associated map

$$\hat{f} : Q^i \times M^m \to X.$$

We can again assume w.l.o.g. that $\hat{f}$ is of class C^r.

We want to change $\hat{f}$ by a homotopy constant on $A \times M$ to obtain a map which is a k-mersion for each fixed $q \in Q$. We do that in a sequence of steps, in each step only making changes on a piece of the domain mapped into a chart on the target and keeping fixed what has been obtained after the previous steps. To make this precise we choose open coverings of Q by charts, say $\{V_i\}$ and $\{U_i\}$ with $i = 1, \ldots, \ell$, such that $V_i \subset \overline{V}_i \subset U_i$. Similarly, we choose for each $i = 1, \ldots, \ell$ open coverings of M by charts, say $\{V_j^i\}$ and $\{U_j^i\}$ with $j = 1, \ldots, n_i$ for fixed i, such that $V_j^i \subset \overline{V_j^i} \subset U_j^i$. Furthermore all these coverings shall be chosen, such that $\hat{f}(U_i \times U_j^i)$ is contained in a chart on X.

Consider now $U_1 \times U_1^1$. Since $\hat{f}$ maps this subset into a chart on X $\hat{f}|U_1 \times U_1^1$ corresponds by a diffeomorphism to a map

$$U_1 \times U_1^1 \to E = F \oplus R^n,$$

where we choose n sufficiently large in the splitting $E = F \oplus R^n$ of E. By the technique behind Corollary 7.1.2 we can alter the component into R^n of this map and thereby construct a homotopy (with support in $U_1 \times U_1^1$) from $\hat{f}$ to a map

$$\hat{f}_1^1 : Q \times M \to X,$$

such that the homotopy is constant on $A \times M$ and such that $(\hat{f}_1^1)_q$ has rank $\geq k$ on $\overline{V_1^1}$ for each $q \in A \cup \overline{V}_1$.

Consider then $\hat{f}_1^1$ on $U_1 \times U_2^1$. By the same method as before we can change $\hat{f}_1^1$ inside $U_1 \times U_2^1$ and thereby obtain a map

$$\hat{f}_2^1 : Q \times M \to X,$$

such that $\hat{f}_2^1$ is homotopic to $\hat{f}_1^1$ through a homotopy constant on $A \times M \cup \overline{V}_1 \times \overline{V_1^1}$. and such that $(\hat{f}_2^1)_q$ has rank $\geq k$ on $\overline{V_1^1} \cup \overline{V_2^1}$ for each $q \in A \cup \overline{V}_1$.

We construct now by induction a sequence of maps

$$\hat{f}, \hat{f}_1^1, \hat{f}_2^1, \ldots, \hat{f}_{n_1}^1, \hat{f}_1^2, \ldots, \hat{f}_{n_2}^2, \ldots, \hat{f}_{n_\ell}^\ell,$$

such that the map $\hat{f}_{n_i}^i$ for each $i = 1, \ldots, \ell$ is homotopic to $\hat{f}$ through a homotopy constant on $A \times M$ and such that $(\hat{f}_{n_i}^i)_q$ has rank $\geq k$ on $M = \bigcup_{j=1}^{n_i} \overline{V}_j^i$ for each $q \in A \cup \overline{V}_1 \cup \ldots \cup \overline{V}_i$.

Since $Q = \bigcup_{i=1}^{\ell} \overline{V}_i$ the map

$$\hat{g} = \hat{f}_{n_\ell}^\ell : Q \times M \to X$$

will induce a map $g : Q \to C^r(M,X;k)$ homotopic to f by a homotopy constant on A. This shows that f relatively represents the zero class.

Proceeding now as in the proof of Theorem 7.2.1. 1) we conclude that the induced map

$$\pi_i(C^r(M,X;k),f_o) \to \pi_i(C^r(M,X),f_o)$$

is a bijection for all $i \geq 0$. Since f_o was chosen arbitrarily the map

$$C^r(M,X;k) \to C^r(M,X)$$

is therefore a weak homotopy equivalence, and hence a homotopy equivalence, since the spaces involved are ANR's. This completes the proof of 1).

2) The proof of 2) is carried through in a manner similar to that of Theorem 7.2.1. 2) by reducing transversality questions to finite dimensional known ones as above.

§3 Induced limit spaces of continuous maps

With reasonable conditions on the spaces involved we show in this section that a continuous mapping space functor $C^0(M,\cdot)$ up to homotopy type commutes with direct limits.

Let $(\underline{X},\underline{f},n_0)$ be an ES of topological spaces, and let M be an arbitrary topological space. We get then a system starting at $n = n_0$

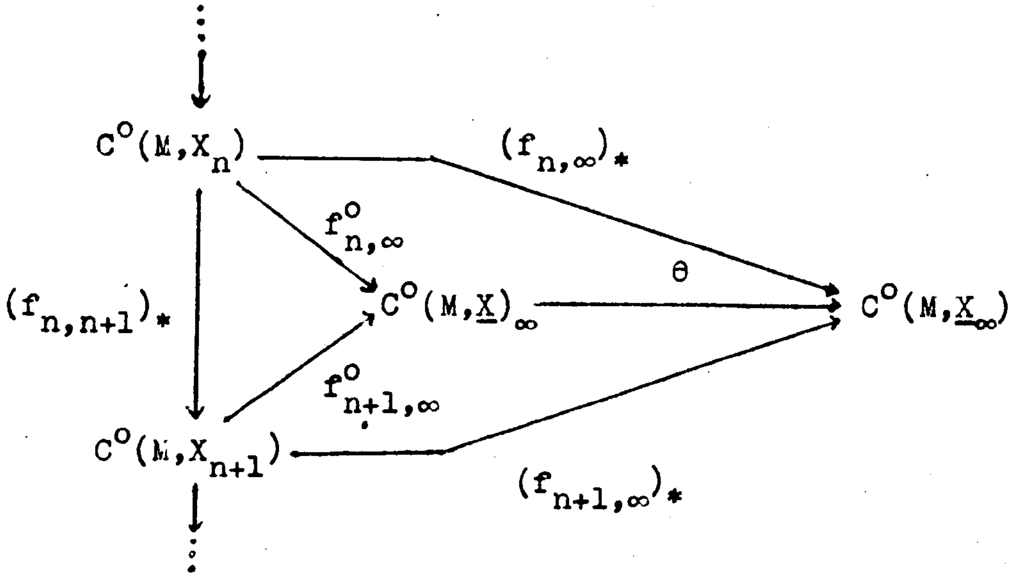
$$\ldots \to C^0(M,X_n) \xrightarrow{(f_{n,n+1})_*} C^0(M,X_{n+1}) \to \ldots,$$

where $(f_{n,n+1})_*$ is defined by composition of maps.

This system is called an induced system and is denoted by $(C^0(M,\underline{X}),\underline{f}_*,n_0)$.

The maps $(f_{n,n+1})_*$ are clearly continuous embeddings. Hence $(C^0(M,\underline{X}),\underline{f}_*, n_0)$ is an ES, and if $(\underline{X},\underline{f},n_0)$ is a CES also a CES.

Utilizing the universal property of a direct limit we get an induced continuous map $\theta : C^0(M,\underline{X})_\infty \to C^0(M,\underline{X}_\infty)$ as shown in the following diagram,



The purpose of this section is to prove

<u>Theorem 7.3.1</u> Let M be a compact metrizable space and let $(\underline{X},\underline{f},n_o)$ be a CES of ANR's.

Then θ is a homotopy equivalence.

Since all spaces involved have the homotopy type of ANR's (Theorem 5.2.1 and Corollary 6.3.4), Theorem 7.3.1 will follow immediately from the theorem of J.H.C.Whitehead (Theorem 5.2.4) and the following

<u>Lemma 7.3.2</u> Let M be a compact topological space and let $(\underline{X},\underline{f},n_o)$ be a CES of T_1-spaces.

Then θ is a weak homotopy equivalence.

<u>Proof</u> Let Q be an arbitrary compact space. It will be sufficient to prove the assertions A and B below. A will prove that the induced map θ_* in homotopy is surjective in all dimensions. B will prove that θ_* is injective in all dimensions.

<u>A</u> For each continuous map $h : Q \to C^o(M,\underline{X}_\infty)$ there exists an n and a continuous map $h_n : Q \to C^o(M,X_n)$, such that $h = (f_{n,\infty})_* \circ h_n$.

<u>B</u> If $h_n : Q \to C^o(M,X_n)$ is a continuous map, such that

$h = (f_{n,\infty})_* \circ h_n$ is homotopic to a constant map in $C^0(M,\underline{X}_\infty)$, then there exists a $k \geq 0$, such that $h_{n+k} = (f_{n,n+k})_* \circ h_n$ is homotopic to a constant map in $C^0(M,X_{n+k})$.

First we prove A. Let $Ev : C^0(M,\underline{X}_\infty) \times M \to \underline{X}_\infty$ denote the evaluation map and define $\hat{h} : Q \times M \to \underline{X}_\infty$ by $\hat{h} = Ev \circ (h \times 1_M)$. Then $\hat{h}$ is continuous if and only if h is continuous. Since $Q \times M$ is compact $\hat{h}$ can be factored continuously through X_n for some n by Lemma 6.1.3. This factorization provides us in the obvious way with the required map $h_n: Q \to C^0(M,X_n)$.

Next we prove B. From the hypothesis in B it follows by arguments similar to those under A that there exists a $k \geq 0$, a map $c \in C^0(M,X_{n+k})$ and a homotopy $H : Q \times [0,1] \to C^0(M,X_{n+k})$, such that $(f_{n+k,\infty})_* (H(q,0)) = h(q)$ and $(f_{n+k,\infty})_*(H(q,1)) = (f_{n+k,\infty})_*(c)$ for all $q \in Q$. But then it follows that $h_{n+k} = (f_{n,n+k})_* \circ h_n$ is homotopic to the constant map $Q \to C^0(M,X_{n+k})$ with value c under the homotopy H.

This proves Lemma 7.3.2.

Remark 7.3.3 It is easy to prove that θ is a bijection. It seems unlikely that θ is actually a homeomorphism. The author has however no proper counter-example.

§4 Induced limit spaces of k-mersions and embeddings

In this section we show that all naturally induced limit spaces of k-mersions and embeddings have the same homotopy type as an associated space of continuous maps.

Let M^m be a compact smooth manifold and let $(\underline{X},\underline{f},n_o)$ be a smooth CES. We get then for each $1 \leq r \leq \infty$ and each $0 \leq k \leq m$ a system starting at $n = n_o$.

$$\ldots \to C^r(M,X_n;k) \xrightarrow{(f_{n,n+1})_*} C^r(M,X_{n+1};k) \to \ldots,$$

where $(f_{n,n+1})_*$ is defined by composition of maps.

This system is called an <u>induced system</u> and is denoted by $(C^r(M,\underline{X};k),\underline{f}_*,n_o)$. From Theorem 5.3.1 it follows that

$(C^r(M,\underline{X};k),\underline{f}_*,n_o)$ is a CES.

Notice that some of the lower spaces in an induced system might be empty. If the dimension of the manifolds X_n is bounded it can even happen that all the spaces are empty.

Likewise for $1 \leq r \leq \infty$ we get an induced CES.

$$\ldots \to Emb^r(M,X_n) \xrightarrow{(f_{n,n+1})_*} Emb^r(M,X_{n+1}) \to \ldots$$

starting at $n = n_o$.

This induced system is denoted by $(Emb^r(M,\underline{X}),\underline{f}_*,n_o)$. Again it can happen that some, maybe all, the spaces in this system are empty.

The following theorem is the main result in this section

<u>Theorem 7.4.1</u> Let M^m be a compact smooth manifold and let $(\underline{X},\underline{f},n_o)$ be a smooth CES of finite dimensional manifolds of increasing dimension. Suppose also that $0 \leq k \leq m$ and $2 \leq r \leq \infty$.

Then the following limits of inclusion maps are homotopy equivalences:

1) $C^r(M,\underline{X};k)_\infty \to C^r(M,\underline{X})_\infty$

2) $\mathrm{Emb}^r(M,\underline{X})_\infty \to C^r(M,\underline{X})_\infty$.

Proof Since the homotopy functor commutes with the direct limit functor it follows immediately from Theorem 7.2.1 that both the maps are weak homotopy equivalences. From Corollary 6.3.4 we know that all the spaces involved have the homotopy type of ANR's. But then the maps are homotopy equivalences by the theorem of J.H.C.Whitehead.

This finishes the proof.

The next theorem connects the limit spaces in Theorem 7.4.1 with a space of continuous maps.

Theorem 7.4.2 Let M be a compact smooth manifold and let $(\underline{X},\underline{f},n_o)$ be a smooth CES of finite dimensional manifolds.

Then the following maps are homotopy equivalences:

1) The limit of natural maps

$$C^r(M,\underline{X})_\infty \to C^o(M,\underline{X})_\infty$$

for each $1 \leq r \leq \infty$.

2) The map θ given by the universal property of direct limits

$$\theta : C^o(M,\underline{X})_\infty \to C^o(M,\underline{X}_\infty)$$

Proof Since the expanding systems involved are closed expanding systems of ANR's 1) follows immediately from Theorem 5.3.2, Theorem 6.3.2 and Lemma 6.3.5.

2) is just a special case of Theorem 7.3.1.

Hence we have proved Theorem 7.4.2.

To sum up briefly:

For each compact smooth manifold M^m, each smooth CES $(\underline{X},\underline{f},n_0)$ of finite dimensional manifolds of increasing dimension, each $0 \leq k \leq m$ and each $2 \leq r \leq \infty$ we have shown that all the maps in the following diagram are homotopy equivalences

$$\begin{array}{l} C^r(M,\underline{X};k)_\infty \searrow \\ \qquad\qquad\qquad\qquad C^r(M,\underline{X})_\infty \to C^0(M,\underline{X})_\infty \xrightarrow{\theta} C^0(M,\underline{X}_\infty) \\ \mathrm{Emb}^r(M,\underline{X})_\infty \nearrow \end{array}$$

Corollary 7.4.3 Let M^m be a compact smooth manifold and let $(\underline{X},\underline{f},n_0)$ be a smooth CES of finite dimensional manifolds of increasing dimension. Let also $0 \leq k \leq m$ and $2 \leq r \leq \infty$.

Then all the limit spaces $C^r(M,\underline{X};k)_\infty$ and $\mathrm{Emb}^r(M,\underline{X})_\infty$ have the same homotopy type as $C^0(M,\underline{X}_\infty)$.

§5 Classifying spaces for diffeomorphism groups

In this section we shall apply Theorem 7.2.2 and Corollary 7.4.3 to construct models for the classifying space of a diffeomorphism group.

Let M be a compact smooth manifold with boundary and let X be a metrizable smooth manifold without boundary modelled on a C^∞-smooth Banach space. Let also $2 \leq r \leq \infty$. Denote by $\mathrm{Diff}^r(M)$ the space of diffeomorphisms of M equipped with the C^r-topology.

It is well-known that $\mathrm{Diff}^r(M)$ is a topological group under composition of maps. It is also well-known that composition of maps

$$\mathrm{Emb}^r(M,X) \times \mathrm{Diff}^r(M) \to \mathrm{Emb}^r(M,X)$$

defines a principal action of $\mathrm{Diff}^r(M)$ on $\mathrm{Emb}^r(M,X)$, i.e.

$$\begin{array}{ccc} \mathrm{Diff}^r(M) & \to & \mathrm{Emb}^r(M,X) \\ & & \downarrow \\ & & \mathrm{Emb}^r(M,X)/\mathrm{Diff}^r(M) \end{array}$$

is a principal $\mathrm{Diff}^r(M)$-bundle.

See e.g. Palais [72] or Lima [62] and in particular Cerf ([10] or the appendix in [11]) and Novikov [71] for the method of proof of this statement. For a general survey of mapping space fibrations see Eells [23].

If X is infinite dimensional we know by Theorem 7.2.2 in connection with Theorem 5.3.2 that $\mathrm{Emb}^r(M,X)$ is homotopy equivalent to $C^0(M,X)$. Hence $\mathrm{Emb}^r(M,X)$ is contractible if X is contractible.

By the general theory for principal bundles we get therefore the following result. Notation as above.

<u>Theorem 7.5.1</u> Suppose that X is an infinite dimensional contractible manifold. Then

$$\begin{array}{ccc} \mathrm{Diff}^r(M) & \to & \mathrm{Emb}^r(M,X) \\ & & \downarrow \\ & & \mathrm{Emb}^r(M,X)/\mathrm{Diff}^r(M) \end{array}$$

is a universal principal $\mathrm{Diff}^r(M)$-bundle.

For X the separable Hilbert space Theorem 7.5.1 was known to Eells [23]. I am indebted to Professor Eells for pointing out the application of my results on spaces of embeddings to the construction of models for classifying spaces for diffeomorphism groups.

Now let $(\underline{X},\underline{f},n_o)$ be a smooth CES of finite dimensional manifolds of increasing dimension. Since the diagram

$$\begin{array}{ccccc} \mathrm{Emb}^r(M,X_n) & \times & \mathrm{Diff}^r(M) & \rightarrow & \mathrm{Emb}^r(M,X_n) \\ (f_{n,n+1})_* \times 1 \downarrow & & & & \downarrow (f_{n,n+1})_* \\ \mathrm{Emb}^r(M,X_{n+1}) & \times & \mathrm{Diff}^r(M) & \rightarrow & \mathrm{Emb}^r(M,X_{n+1}) \end{array}$$

commutes, we get an induced action *)

$$\mathrm{Emb}^r(M,\underline{X})_\infty \times \mathrm{Diff}^r(M) \rightarrow \mathrm{Emb}^r(M,\underline{X})_\infty$$

One can again show that this action is principal, so that we get a principal $\mathrm{Diff}^r(M)$-bundle.

By Corollary 7.4.3 we know that $\mathrm{Emb}^r(M,\underline{X})_\infty$ is homotopy equivalent to $C^o(M,\underline{X}_\infty)$. Hence $\mathrm{Emb}^r(M,\underline{X})_\infty$ will be contractible, if $\underline{X}_\infty$ is contractible.

With notation as above we get therefore

<u>Theorem 7.5.2</u> Suppose that $(\underline{X},\underline{f},n_o)$ is a smooth CES of finite dimensional manifolds of increasing dimension, such that $\underline{X}_\infty$ is contractible. Then

$$\begin{array}{ccc} \mathrm{Diff}^r(M) & \longrightarrow & \mathrm{Emb}^r(M,\underline{X})_\infty \\ & & \downarrow \\ & & \mathrm{Emb}^r(M,\underline{X})_\infty / \mathrm{Diff}^r(M) \end{array}$$

is a universal principle $\mathrm{Diff}^r(M)$-bundle.

Theorem 7.5.1 and Theorem 7.5.2 give us a lot of models for the classifying space $B(\mathrm{Diff}^r(M))$ of $\mathrm{Diff}^r(M)$. Classifying spaces for diffeomorphism groups are obviously interesting.

*) *It is not obvious that the map is continuous*

<u>Example 7.5.3</u> Take $M = S^n$.

Let $E \xrightarrow{p} B$ be a smooth locally trivial fibration with fibre S^n. This fibration will then have $\mathrm{Diff}^{\infty}(S^n)$ as structural group. Hence p is up to smooth equivalence classified by a continuous map

$$f : B \to B(\mathrm{Diff}^{\infty}(S^n)).$$

Since $O(n+1) \subset \mathrm{Diff}^{\infty}(S^n)$ there exists a fibration of classifying spaces

$$\begin{array}{ccc} \mathrm{Diff}^{\infty}(S^n)/O(n+1) & \longrightarrow & BO(n+1) \\ & & \downarrow \\ & & B(\mathrm{Diff}^{\infty}(S^n)) \end{array}$$

A smooth fibration with S^n as fibre is therefore a spherical fibration ($O(n+1)$ as structural group) if and only if there exists a lifting of f

$$\begin{array}{ccc} & & BO(n+1) \\ & \nearrow & \downarrow \\ B & \xrightarrow[f]{} & B(\mathrm{Diff}^{\infty}(S^n)) \end{array}$$

This situation was studied by Novikov in [71] , where he showed that there exist smooth fibrations with fibre S^n in many dimensions $n \geq 7$ which are not equivalent to spherical fibrations.

Recently Antonelli, Burghelea and Kahn [5] has shown that there exists smooth fibrations with fibre S^n for any $n \geq 5$ which are not equivalent to spherical fibrations.

Chapter 8

Homotopy direct limits.

This final chapter contains our version of the classification theorem for k-mersions of a compact smooth manifold M into an infinite dimensional smooth manifold X (Theorem 8.4.1). The method of proof is to take the corresponding theorem in finite dimensions and then apply a limit argument. For such a limit argument to work we need to have good filtrations of X through finite dimensional submanifolds.

For many infinite dimensional smooth manifolds X it is possible to find an expanding system $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ of finite dimensional smooth submanifolds of X such that the natural map $\underline{X}_\infty \to X$ is a homotopy equivalence. When such a property holds we say that X is a smooth homotopy direct limit of the expanding system $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$. Mukherjea showed in ([69] or [70]) that every smooth separable Fredholm manifold with a C^∞-smooth model admits such filtrations. We recall this in Example 8.1.6. The filtrations of a Fredholm manifold satisfy also various other properties.

Based on the properties of the filtrations of a Fredholm manifold we define a class of infinite dimensional manifolds (Definition 8.1.7), which we call pseudo Fredholm manifolds. The relation between these manifolds and the Fredholm manifolds is not clear. Since pseudo Fredholm manifolds are just what we need, Definition 8.1.7 seems therefore justified.

After having defined (smooth) homotopy direct limits properly in §1, we show in §2 that the mapping space functors $C^o(M,\cdot)$, $C^r(M,\cdot)$, $C^r(M,\cdot;k)$ and $Emb^r(M,\cdot)$ 'commute' with such limits. In §3 we show that the bundle map functor Hom $(TM,\cdot;k)$ 'commutes' with the special smooth homotopy direct limits defining a pseudo Fredholm manifold.

With the results in §2 and §3 at our disposal it is easy to prove Theorem 8.4.1, which classifies the k-mersions from a compact smooth manifold into a smooth pseudo Fredholm manifold.

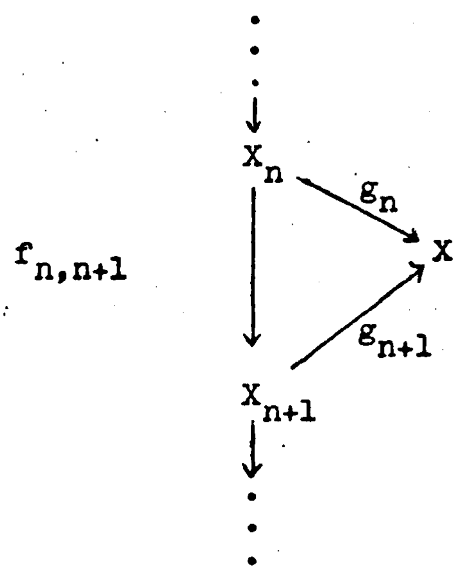
§1 Definitions and examples.

First we define the notion homotopy direct limit. The definition is split into two cases

Definition 8.1.1 (continuous case) Let $(\underline{X},\underline{f},n_o)$ be an ES of topological spaces, let X be a topological space and let $(\underline{g},n_o) = \{g_n\}_{n \geq n_o}$ be a system of continuous maps $g_n: X_n \to X$.

Then X is called a homotopy direct limit of $(\underline{X},\underline{f},n_o)$ w.r.t. $(\underline{g},n_o)$, if

1) The following diagram commutes



2) The induced continuous map

$$\underline{g}_\infty : \underline{X}_\infty \to X$$

is a homotopy equivalence.

Definition 8.1.1 (smooth case) If $(\underline{X},\underline{f},n_o)$ is a smooth ES, X is a smooth manifold and $(g,n_o) = \{g_n\}_{n \geq n_o}$ is a system of smooth maps $g_n : X_n \to X$, then we call X a smooth homotopy direct limit of $(\underline{X},\underline{f},n_o)$ w.r.t. (g,n_o), provided the conditions 1) and 2) above are still satisfied.

In the following we will abbreviate homotopy direct limit to HDL. Notice that our definition of an HDL is slightly more general than the definition in Milnor ([66], appendix).

We proceed now to give some examples of HDL's.

Example 8.1.2 Let X be a topological space with the homotopy type of a CW-complex K, and let $g:K \to X$ be a homotopy equivalence.

Let K_n denote the n-skeleton of K and let $f_{n,n+1} : K_n \to K_{n+1}$ be the obvious inclusion. Let also $g_n : K_n \to X$ denote the restriction of g to K_n.

Then X is an HDL of the CES $(\underline{K},\underline{f},0)$ w.r.t. the system of continuous maps $(g,0)$, since of course $g_\infty = g$.

A theorem of Palais creates a lot of non-trivial examples as follows:

Example 8.1.3 Let E be a LCTVS, and $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ be an increasing sequence of finite dimensional linear subspaces of E whose union $\bigcup_{n \geq 1} E_n$ is dense in E.

If X is a subset of E put $X_n = X \cap E_n$. Let also $f_{n,n+1} : X_n \to X_{n+1}$ and $g_n : X_n \to X$ be the obvious inclusions.

Assume now that X is an open subset of E. Then a theorem of Palais ([74], corollary to Theorem 17) can be restated as follows:

<u>If E is metrizable or, more generally, if X is paracompact, then X is an HDL of the CES $(\underline{X},\underline{f},1)$ w.r.t. the system of continuous maps $(\underline{g},1)$.</u>

For E a Banach space we get, of course, smooth HDL.

Next we produce a few concrete smooth HDL's.

<u>Example 8.1.4</u> This example uses the notation in Example 6.1.7.

Let E denote the separable Hilbert space and let $\{e_i\}_{i\geq 1}$ be an orthonormal basis for E.

As in Example 6.1.7 we have the CES $R^1 \subset R^2 \subset \ldots \subset R^n \subset \ldots$ of Euclidian subspaces of E.

Now let $S(E)$, $P(E)$, $V_k(E)$ and $G_k(E)$ denote respectively the unit sphere, the projective space, the Stiefel manifold of orthonormal k-frames and the Grassmann manifold of k-planes in E. All these spaces have natural topologies: $V_k(E)$ can be identified with $\mathrm{Iso}(R^k,E)$, the space of linear isometries of R^k into E equipped with the norm topology; $G_k(E)$ can be identified with a space of projection operators on E with the norm topology.

<u>Spheres</u> $S(E)$ is a smooth HDL of $(\underline{S},\underline{f},1)$ w.r.t. the system $(\underline{g},1)$ of natural embeddings $g_n : S^n \to S(E)$.

This is trivial, since both S^∞ and $S(E)$ are contractible. For the contractibility of $S(E)$ see e.g. Hu ([49], Theorem 15.2, p.61).

<u>Projective spaces</u> $P(E)$ is a smooth HDL of $(\underline{RP},\underline{f},1)$ w.r.t. the system $(\underline{g},1)$ of natural embeddings $g_n : RP^n \to P(E)$.

This can be seen as follows: Taking limits we get a fibre map

$$\begin{array}{ccc} Z_2 & \xrightarrow{1} & Z_2 \\ \downarrow & & \downarrow \\ S^\infty & \to & S(E) \\ \downarrow & & \downarrow \\ RP^\infty & \to & P(E) \end{array}$$

Using the 5-lemma on the induced map between homotopy sequences we conclude easily that $RP^\infty \to P(E)$ is a weak homotopy equivalence, and therefore a homotopy equivalence by the theorem of J.H.C.Whitehead, since both spaces have at least the homotopy type of ANR's.

Stiefel manifolds $V_k(E)$ is a smooth HDL of $(\underline{V}_k, \underline{f}, k)$ w.r.t. the system (g, k) of natural embeddings $g_n : V_k(R^n) \to V_k(E)$.

This can be proved e.g. as follows: Let Mono(R^k,R^n) and Mono(R^k,E) denote the space of monomorphisms of R^k into respectively R^n and E equipped with the norm topology. Using the sequence of finite dimensional subspaces $L(R^k,R^k) \subset \ldots \subset L(R^k,R^n) \subset \ldots$ in the Banach space $L(R^k,E)$ it follows immediately from the theorem of Palais (Example 8.1.2) that the natural map

$$\varinjlim_n \text{Mono}(R^k,R^n) \to \text{Mono}(R^k,E)$$

is a homotopy equivalence. By the standard orthogonalization procedure it is clear that Mono(R^k,R^n) (Mono(R^k,E)) has $V_k(R^n)$ ($V_k(E)$) as deformation retract. Recall, that $V_k(R^n)$ ($V_k(E)$) can be identified with the space of linear isometries Iso(R^k,R^n) (Iso(R^k,E)). Using Theorem 6.3.2 it follows then easily that $V_k(\infty) \to V_k(E)$ is a homotopy equivalence.

Grassmann manifolds $G_k(E)$ is a smooth HDL of $(\underline{G}_k, \underline{f}, k)$ w.r.t. the system (g, k) of natural embeddings $g_n : G_k(R^n) \to G_k(E)$.

This follows easily, since both $G_k(\infty)$ and $G_k(E)$ are classifying spaces for $G\ell(k)$ and since the universal bundle over

$G_k(E)$ pulls back to that over $G_k(\infty)$ via the map $G_k(\infty) \to G_k(E)$.

Spaces of operators produce also many examples of HDL's. We mention only the following fundamental

Example 8.1.5 Let E be an infinite dimensional Banach space with a flag, i.e. with a sequence of direct sum decompositions $E = E_n \oplus E^n$ $(n \geq 1)$ such that $\dim E_n = n$, $E_n \subset E_{n+1}$ and $E^{n+1} \subset E^n$.

Let $G\ell_c(E)$ denote the space of invertible operators on E of the form $I + u$, where I is the identity operator and u is a compact operator.

The spaces $G\ell(n) = G\ell(E_n)$ of invertible operators and the natural inclusions $G\ell(n) \subset G\ell(n+1)$ form a CES, denoted by $(\underline{G\ell}, \underline{f}, 1)$. The limit space is denoted by $G\ell(\infty) = \underline{G\ell}_\infty$.

Observe that we have natural inclusions $G\ell(n) \subset G\ell_c(E)$.

The following theorem was proved by Palais [73] and Švarc [86] for a wide class of Banach spaces and by Elworthy [30] and Geba [34] in general:

$G\ell_c(E)$ is an HDL of $(\underline{G\ell}, \underline{f}, 1)$ w.r.t. the natural inclusions $G\ell(n) \subset G\ell_c(E)$, i.e. the natural map $G\ell(\infty) \to G\ell_c(E)$ is a homotopy equivalence.

The following example will be the most important so far as we are concerned.

Example 8.1.6 See Eells ([24], §8) for a more detailed account of the material in this example.

Let X be a separable smooth Fredholm manifold modelled on a C^∞-smooth Banach space E with a Schauder basis.

Remark By a theorem of Elworthy ([30] or [31]) any infinite dimensional separable smooth manifold X modelled on a C^∞-smooth Banach space E, for which the structural group of the tangent bundle can be reduced from $G\ell(E)$ to $G\ell_c(E)$, admits an integrable reduction. An integrable reduction to $G\ell_c(E)$ is however by definition exactly a Fredholm structure.

Coupled with a theorem of Kuiper [58] on the contractibility of $G\ell(E)$ for E a Hilbert space, this theorem of Elworthy shows e.g. that every smooth separable Hilbert manifold admits a Fredholm structure.

With X as above, a theorem of Mukherjea ([69] or [70]) can be stated as follows. See also Eells and Elworthy [25].

Theorem There exists a smooth CES $(\underline{X},\underline{f},1)$, such that each X_n is a compact submanifold of X of dimension n, say with inclusion $g_n: X_n \to X$, and each $f_{n,n+1}$ is an inclusion of X_n as a submanifold of X_{n+1}.

Furthermore, this system can be chosen, such that the following properties are satisfied:

1) $\bigcup_{n\geq 1} X_n$ is dense in X.

2) If $x \in X_{n(x)}$ then the union of tangent spaces $\bigcup_{n\geq n(x)} T_x X_n$ is dense in the tangent space $T_x X$ of X at $x \in X$.

3) X is a smooth HDL of $(\underline{X},\underline{f},1)$ w.r.t. the system of smooth embeddings $(\underline{g},1)$, i.e. the natural map $\underline{X}_\infty \to X$ is a homotopy equivalence.

This finishes Example 8.1.6.

Based on Example 8.1.6 we make the

Definition 8.1.7 Let X be an infinite dimensional separable smooth manifold modelled on a C^∞-smooth Banach space E with a Schauder basis. We shall call X a smooth pseudo Fredholm manifold if there exists a smooth CES $X_{n_o} \subset X_{n_o+1} \subset \ldots \subset X_n \subset \ldots$ of finite dimensional submanifolds of X of increasing dimension, denoted by $(\underline{X}, n_o)$, such that

1) $\bigcup_{n \geq n_o} X_n$ is dense in X.

2) If $x \in X_{n(x)}$ then $\bigcup_{n \geq n(x)} T_x X_n$ is dense in $T_x X$.

3) The natural map $\underline{X}_\infty \to X$ is a homotopy equivalence.

A CES $(\underline{X}, n_o)$ with the properties in Definition 8.1.6 will be called a finite dimensional presentation of X.

Remark 8.1.8 Let X be a separable smooth manifold modelled on a C^∞-smooth Banach space with a Schauder basis. We say that X admits an almost Fredholm structure if the structural group of the tangent bundle of X can be reduced from $G\ell(E)$ to $G\ell_c(E)$. If a specific reduction has been chosen X is called an almost Fredholm manifold.

In the literature on G-structures the prefixes pseudo and almost are used indiscriminately. The reader is therefore warned that the definition of a pseudo Fredholm manifold given here is different from that of an almost Fredholm manifold.

By the theorem of Elworthy quoted in Example 8.1.6 any almost Fredholm structure is integrable, i.e. the underlying almost structure of a Fredholm structure. Hence there is no special theory for almost Fredholm structures and there should therefore not arise any confusion from our use of terminology.

Problems

1) Does any smooth pseudo Fredholm manifold admit a Fredholm structure?

In the corresponding theory with differentiability of class C^r for $1 \leq r < \infty$ we have also the converse problem:

2) Is any Fredholm manifold of class C^r a pseudo Fredholm manifold of class C^r?

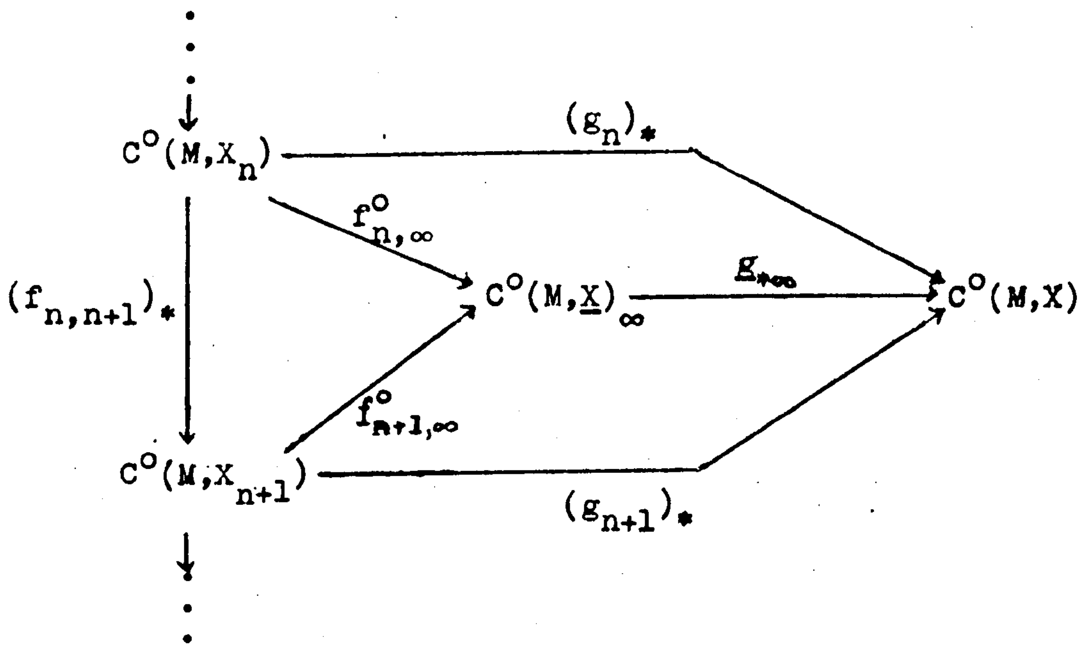
If the model for a paracompact manifold of class C^1 belongs to a certain collection of sequence spaces, e.g. ℓ_{2p} - spaces for p a natural number, then Moulis [68] has shown that any C^r-structure on the manifold has a compatible smooth structure. In such cases 2) can therefore be answered in the affirmative.

§2 Induced homotopy direct limits.

In this section we show that the mapping space functors $C^o(M,\cdot)$, $C^r(M,\cdot)$, $C^r(M,\cdot;k)$ and $Emb^r(M,\cdot)$ 'commute' with appropriate (smooth) HDL's.

First we treat the continuous case.

Let $(\underline{X},\underline{f},n_o)$ be an ES of topological spaces and suppose that the topological space X is an HDL of $(\underline{X},\underline{f},n_o)$ w.r.t. the system of continuous maps $(\underline{g},n_o)$. Then we get an induced diagram



In this diagram $C^o(M,\underline{X})_\infty$ is the limit space of the induced system $(C^o(M,\underline{X}),\underline{f}_*,n_o)$, $f^o_{n,\infty}$ is the inclusion of $C^o(M,X_n)$ into the limit space and $g_{*\infty}$ is the limit map for the induced system of continuous maps $(\underline{g}_*,n_o) = \{(g_n)_*\}_{n\geq n_o}$.

We have now

<u>Theorem 8.2.1</u> Let $(\underline{X},\underline{f},n_o)$ be a CES of ANR's, and let M be compact metrizable. Furthermore, let X be an ANR, which is an HDL of $(\underline{X},\underline{f},n_o)$ w.r.t. the system of continuous maps $(\underline{g},n_o)$.

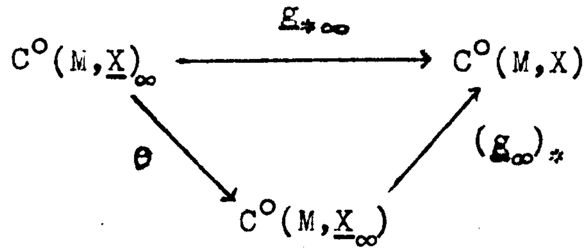
Then $C^o(M,X)$ is an HDL of the induced CES $(C^o(M,\underline{X}),\underline{f}_*,n_o)$ w.r.t. the induced system of continuous maps $(\underline{g}_*,n_o)$.

<u>Proof</u> We have to show that

$$g_{*\infty} : C^o(M,\underline{X})_\infty \to C^o(M,X)$$

is a homotopy equivalence.

For that purpose let $\theta : C^o(M,\underline{X})_\infty \to C^o(M,\underline{X}_\infty)$ be the map studied in Chapter 7, §3 and let $(\underline{g}_\infty)_* : C^o(M,\underline{X}_\infty) \longrightarrow C^o(M,X)$ be the map induced by the homotopy equivalence $\underline{g}_\infty : \underline{X}_\infty \to X$. It is clear that we get a commutative diagram



It is easy to prove that $(g_\infty)_*$ is a homotopy equivalence. Since we know that θ is a homotopy equivalence by Theorem 7.3.1, it follows immediately that $g_{*\infty}$ is a homotopy equivalence. Hence Theorem 8.2.1 is proved.

Next we turn to the differentiable case.

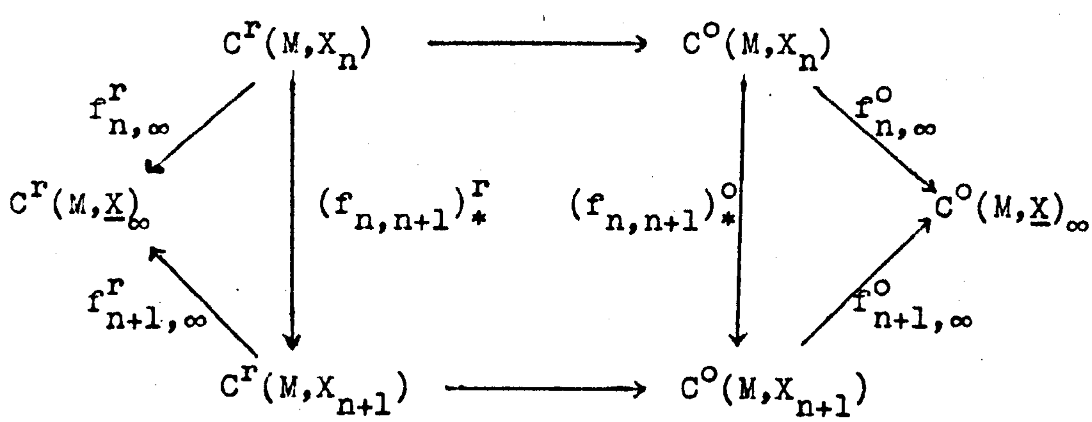
Dealing with a smooth HDL we get, of course, a diagram similar to the one preceeding Theorem 8.2.1 for each $0 \leq r \leq \infty$.

We have then

<u>Theorem 8.2.2</u> Let $(\underline{X},\underline{f},n_o)$ be a smooth CES, and let X be a smooth manifold which is a smooth HDL of $(\underline{X},\underline{f},n_o)$ w.r.t. the system (g,n_o) of smooth maps. Assume furthermore, that all the manifolds are metrizable and modelled on C^∞-smooth Banach spaces.

Then for every compact smooth manifold M and all $0 \leq r \leq \infty$, $C^r(M,X)$ is an HDL of the induced CES $(C^r(M,\underline{X}),\underline{f}_*,n_o)$ w.r.t. the induced system of continuous maps (g_*,n_o).

<u>Proof</u> Consider the following diagram



We have labelled the induced maps and the maps into the induced limit spaces with r and o respectively just to distinguish between them.

The horizontal maps are the obvious inclusions. By the theorem of Palais (Theorem 5.3.2) they are homotopy equivalences.

Both the induced systems $(C^o(M,\underline{X}),\underline{f}^o_*,n_o)$ and $(C^r(M,\underline{X}),\underline{f}^r_*,n_o)$ are CES's of ANR's, by Theorem 5.2.1 and Theorem 5.3.1 respectively. Hence it follows from Theorem 6.3.2 in connection with Lemma 6.3.5 that the limit map

$$C^r(M,\underline{X})_\infty \longrightarrow C^o(M,\underline{X})_\infty$$

is a homotopy equivalence.

Consider then finally this commutative diagram

$$\begin{array}{ccc} C^r(M,\underline{X})_\infty & \xrightarrow{g^r_{*\infty}} & C^r(M,X) \\ \downarrow & & \downarrow \\ C^o(M,\underline{X})_\infty & \xrightarrow[g^o_{*\infty}]{} & C^o(M,X) \end{array}$$

The right vertical map is again just the obvious inclusion map. It is a homotopy equivalence by the theorem of Palais. The left vertical map is the limit map above which we have just proved to be a homotopy equivalence. $g^o_{*\infty}$ is a homotopy equivalence by Theorem 8.2.1. Altogether it follows then immediately that $g^r_{*\infty}$ is a homotopy equivalence. This is however exactly what we should prove.

Finally, we state our result for the mapping space functors $C^r(M,\cdot, k)$ and $Emb^r(M,\cdot)$.

<u>Theorem 8.2.3</u> Let $(\underline{X},\underline{f},n_o)$ be a smooth CES of finite dimensional manifolds of increasing dimension and let X be a metrizable smooth manifold modelled on an infinite dimensional C^∞-smooth Banach space. Suppose also that X is a smooth HDL of $(\underline{X},\underline{f},n_o)$ w.r.t. the system of smooth embeddings $(\underline{g},n_o)$. Finally, let M^m be an arbitrary compact smooth manifold and let $0 \leqslant k < m$ and $2 \leqslant r \leqslant \infty$.

Then $C^r(M,X;k)$ and $\mathrm{Emb}^r(M,X)$ are HDL's of the corresponding induced CES's $(C^r(M,\underline{X};k), \underline{f}_*,n_o)$ and $(\mathrm{Emb}^r(M,\underline{X}),\underline{f}_*,n_o)$ w.r.t. the systems of continuous embeddings $(\underline{g}_*,n_o)$.

<u>Proof</u> Consider the following commutative diagrams of natural maps:

$$\begin{array}{ccc} C^r(M,\underline{X};k)_\infty & \xrightarrow{\;g_{*\infty}\;} & C^r(M,X;k) \\ \downarrow & & \downarrow \\ C^r(M,\underline{X})_\infty & \longrightarrow & C^r(M,X) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Emb}^r(M,\underline{X})_\infty & \xrightarrow{\;g_{*\infty}\;} & \mathrm{Emb}^r(M,X) \\ \downarrow & & \downarrow \\ C^r(M,\underline{X})_\infty & \longrightarrow & C^r(M,X) \end{array}$$

The vertical maps are homotopy equivalences by Theorem 7.2.2 and Theorem 7.4.1. The bottom horizontal map is a homotopy equivalence by Theorem 8.2.2. Then $g_{*\infty}$ must be a homotopy equivalence. This is exactly what we should prove.

§3 Bundle maps into the tangent bundle of a pseudo Fredholm manifold.

The object of this section is to prove Theorem 8.3.1 stated below. For the basic notation the reader is referred to Chapter 5, §1. Pseudo Fredholm manifolds were defined in Definition 8.1.6.

Theorem 8.3.1 Let X be a smooth pseudo Fredholm manifold and let $(\underline{X}, n_0)$ be a finite dimensional presentation of X. Let also M^m be a compact smooth manifold and let $0 \leq k \leq m$ and $2 \leq r \leq \infty$.

Then the naturally induced closed embeddings $\mathrm{Hom}(TM, TX_n; k) \to \mathrm{Hom}(TM, TX_{n+1}; k)$ form a CES with limit space $\mathrm{Hom}(TM, T\underline{X}; k)_\infty$, and the naturally induced map

$$\mathrm{Hom}(TM, T\underline{X}; k)_\infty \longrightarrow \mathrm{Hom}(TM, TX; k)$$

is a homotopy equivalence.

The proof of Theorem 8.3.1 will be given in a sequence of lemmas. The fibrations π, π' and π'' which will occur in most of these lemmas are defined in Chapter 5, §1. We shall freely use the following convention: Whenever we underbar a symbol a natural expanding system is involved and whenever we put an ∞ on a symbol a limit has been taken.

Lemma 8.3.2 Let $E_{n_0} \subset \dots \subset E_n \subset \dots$ be an increasing sequence of finite dimensional linear subspaces in a Banach space E, and suppose that $\bigcup_{n \geq n_0} E_n$ is dense in E. Let also F be a finite dimensional vector space and let $0 \leq k \leq \dim F$.

Then the natural map

$$L(F, \underline{E}; k)_\infty \longrightarrow L(F, E; k)$$

is a homotopy equivalence.

Proof Since F is finite dimensional it is clear that

1) $L(F,E_{n_o}) \subset \ldots \subset L(F,E_n) \subset \ldots$

is an expanding system of finite dimensional linear subspaces in the Banach space $L(F,E)$, such that $\bigcup_{n \geq n_o} L(F,E_n)$ is dense in $L(F,E)$.

2) $L(F,E_n;k) = L(F,E;k) \cap L(F,E_n)$

3) $L(F,E;k)$ is an open set in the Banach space $L(F,E)$.

With these observations in mind the lemma is a direct consequence of the corollary to Theorem 17 in Palais [74] (stated here as Example 8.1.3).

Lemma 8.3.3 Let X be a smooth pseudo Fredholm manifold and let $(\underline{X},n_o)$ be a finite dimensional presentation of X. Let also F be a finite dimensional vector space and let $0 \leq k \leq \dim F$.

Then the natural map

$$L(F,T\underline{X};k)_\infty \rightarrow L(F,TX;k)$$

is a homotopy equivalence.

Proof Let $x \in X_{n(x)}$ and put $E_n = T_x X_n$ for each $n \geq n(x)$ and $E = T_x X$. By assumption 2) in Definition 8.1.7 $\bigcup_{n \geq n(x)} E_n$ is then dense in E.

Taking limits we get the following commutative diagram

$$\begin{array}{ccc}
L(F,\underline{E};k)_\infty & \rightarrow & L(F,E;k) \\
\downarrow & & \downarrow \\
L(F,T\underline{X};k)_\infty & \rightarrow & L(F,TX;k) \\
\underline{\pi}'_\infty \downarrow & & \downarrow \pi \\
\underline{X}_\infty & \rightarrow & X
\end{array}$$

It is easy to prove that $\underline{\pi}'_\infty$ is a Serre fibration. The map between fibres and the base map are homotopy equivalences respectively by Lemma 8.3.2 and by assumption. Using the 5-lemma on the induced map between the homotopy sequences of the two fibrations it follows now immediately that the map

$$L(F,T\underline{X};k)_\infty \quad \to \quad L(F,TX;k)$$

is a weak homotopy equivalence and therefore a homotopy equivalence since the spaces involved have at least the homotopy type of ANR's.

This proves Lemma 8.3.3.

From now on our assumptions will be as stated in Theorem 8.3.1. We shall consider the induced limit space

$$B(TM,T\underline{X};k)_\infty \;=\; \varinjlim_n B(TM,TX_n;k)$$

and the following commutative diagram. We will get such a diagram for each $p \in M$ by taking limits

$$\begin{array}{ccc}
L(T_pM,T\underline{X};k)_\infty & \longrightarrow & L(T_pM,TX;k) \\
\downarrow & & \downarrow \\
B(TM,T\underline{X};k)_\infty & \longrightarrow & B(TM,TX;k) \\
& {\scriptstyle \underline{\pi}_\infty} \searrow \quad \swarrow {\scriptstyle \pi} & \\
& M &
\end{array}$$

<u>Lemma 8.3.4</u> In this diagram $\underline{\pi}_\infty$ is a locally trivial fibration, and the natural map

$$B(TM,T\underline{X};k)_\infty \;\to\; B(TM,TX;k)$$

is a fibre homotopy equivalence.

Proof Almost by definition $\underline{\pi}_\infty$ factors as follows

$$B(TM,T\underline{X};k) \xrightarrow{\underline{\pi}''_\infty} M \times \underline{X}_\infty \xrightarrow{\text{proj.}} M$$

It is rather easy to extend a local trivialization of the smooth fibration $\pi_n'' : B(TM,TX_n;k) \to M \times X_n$ over a coordinate chart $V \times U_n$ on $M \times X_n$ with $\overline{U}_n$ compact and contractible in X_n to a local trivialization of π''_{n+1} over a chart $V \times U_{n+1}$ on $M \times X_{n+1}$ with $\overline{U}_{n+1}$ compact and contractible in X_{n+1}. Just use that the normal bundle for X_n in X_{n+1} is trivial over $\overline{U}_n$. Taking limits of such local trivializations we get local trivializations of $\underline{\pi}''_\infty$.

Since a composition of locally trivial fibrations over a locally contractible space is again a locally trivial fibration it is clear that $\underline{\pi}_\infty$ is a locally trivial fibration.

For the second part of the lemma we remark that the map in question restricts to a homotopy equivalence on each fibre by Lemma 8.3.3 Hence the theorem of Dold ([14], Theorem 6.3) shows that it is a fibre homotopy equivalence.

This proves Lemma 8.3.4.

The next lemma will prove Theorem 8.3.1, since we know from Chapter 5, §1 that $\mathrm{Hom}(TM,TX_n;k)$ and $\mathrm{Hom}(TM,TX;k)$ can be identified with the spaces of sections $\Gamma^o(\pi_n)$ and $\Gamma^o(\pi)$ respectively.

<u>Lemma 8.3.5</u> Assumptions as in Theorem 8.3.1. Consider the induced CES

$\Gamma^o(\pi_{n_o}) \subset \ldots \subset \Gamma^o(\pi_n) \subset \ldots$ with limit space $\Gamma^o(\underline{\pi})_\infty$ and the naturally induced map

$$\Gamma^o(\underline{\pi})_\infty \to \Gamma^o(\pi).$$

This map is a homotopy equivalence.

<u>Proof</u> The map in question is one of the maps in the following commutative diagram of naturally induced maps

$$\begin{array}{ccc} \Gamma^{o}(\underline{\pi})_{\infty} & \longrightarrow & \Gamma^{o}(\pi) \\ & \searrow \quad \nearrow & \\ & \Gamma^{o}(\underline{\pi}_{\infty}) & \end{array}$$

Proceeding exactly as in the proof of Lemma 7.3.2 it is easy to show that

$$\Gamma^{o}(\underline{\pi})_{\infty} \quad \to \quad \Gamma^{o}(\underline{\pi}_{\infty})$$

is a weak homotopy equivalence. Since the map

$$B(TM,T\underline{X};k)_{\infty} \quad \to \quad B(TM,TX;k)$$

is a fibre homotopy equivalence by Lemma 8.3.4, it is easy to prove that the induced map

$$\Gamma^{o}(\underline{\pi}_{\infty}) \; \to \; \Gamma^{o}(\pi)$$

is a homotopy equivalence.

Altogether the map

$$\Gamma^{o}(\underline{\pi})_{\infty} \to \Gamma^{o}(\pi)$$

is therefore a weak homotopy equivalence. Since the spaces involved have the homotopy type of ANR's the map will therefore be a homotopy equivalence by the theorem of J.H.C.Whitehead.

This proves Lemma 8.3.5 and hence as already mentioned also Theorem 8.3.1.

§4 <u>Classification of the k-mersions from a compact manifold into a pseudo Fredholm manifold</u>.

Theorem 8.4.1 below is the strict analogue to a theorem of S.D.Feit [33] but with infinite dimensional target. On the other hand Feit's theorem is the end product of work initiated by Smale and Thom in the late 1950's and followed by work of Hirsch, Palais, Haefliger, Poenaru and Phillips (for the submersion case). See the introduction to [33] for references to this development. The theorem of Feit has also independently been discovered by Gromov [37]. Gromov gets his result as an application of a very general theorem, which has many other applications in differential topology. For an account of the work of Gromov see Haefliger [39] or Poenaru [78].

Theorem 8.4.1 Let M^m be a compact smooth manifold and let X be a smooth pseudo Fredholm manifold. Let also $0 \leq k \leq m$ and $2 \leq r \leq \infty$. Then the differential map induces a homotopy equivalence

$$d : C^r(M,X;k) \to \mathrm{Hom}(TM,TX;k)$$

Proof Let $(\underline{X},n_0)$ be a finite dimensional presentation of X and consider the induced commutative diagram

$$\begin{array}{ccc} C^r(M,\underline{X};k)_\infty & \xrightarrow{d_\infty} & \mathrm{Hom}(TM,T\underline{X};k)_\infty \\ \downarrow & & \downarrow \\ C^r(M,X;k) & \xrightarrow[d]{} & \mathrm{Hom}(TM,TX;k) \end{array}$$

In this diagram the vertical maps are homotopy equivalences by Theorem 8.2.3 and Theorem 8.3.1. d_∞ is the limit of the differential maps

$$d_n: C^r(M,X_n;k) \to \mathrm{Hom}(TM,TX_n;k)$$

which only involves finite dimensional manifolds. Since the dimension of X_n tends to infinity these maps are homotopy equivalences (= weak homotopy equivalences since the spaces are ANR's) from a certain stage by the theorem of Feit [33]. Hence

d_∞ is the limit map of a homotopy equivalence between CES's of ANR's and is therefore itself a homotopy equivalence by Theorem 6.3.2.

Since the three other maps in the diagram are homotopy equivalences, d must itself be a homotopy equivalence.

This proves Theorem 8.4.1.

Remark 8.4.2 Although it looks as though we have a theorem for each $0 \leq k \leq m$, it should be stressed that the dependence on k is rather artificial, since all spaces $C^r(M,X;k)$ are homotopy equivalent to $C^r(M,X)$ by Theorem 7.2.2.

Problem Classify the immersions $Imm^r(X,Y)$ from say a smooth separable Hilbert manifold X into an infinite dimensional, metrizable smooth manifold Y in the spirit of Theorem 8.4.1.

The first problem here is to find a 'good' topology on the space $Imm^r(X,Y)$. As soon as X is not locally compact, the C^r-topology is probably too small.

Problem Suppose that X is a smooth separable Hilbert manifold and that Y is a compact smooth manifold. Classify the submersions $Sub^r(X,Y)$ from X into Y in the spirit of Theorem 8.4.1.

This problem has interest in the study of foliations on Hilbert manifolds.

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