# Low-factor Market Models of Interest Rates

by


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# Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. The work is original except where indicated by specific references in the text. Chapter 3 is based on the working paper Gogala and Kennedy (2015a) and Chapter 4 is based on a working paper Gogala and Kennedy (2015b), both of which are going to be submitted for publication by the end of October 2015.

# Abstract

In this thesis we study three different, but interconnected low-factor market models: LIBOR market model, Markov-functional model, and two-currency Markov-functional model.

The LIBOR market model (LMM) is one of the most popular term structure models. However, it suffers from a major drawback, it is high-dimensional. The problem of high-dimensionality can be in part solved imposing a separability condition. We will be interested how the separability condition interacts with time-homogeneity, a desirable property of an LMM. We address this question by parametrising two- and three-factor separable and time-homogeneous LMMs and show that they are of practical interest.

Markov-functional models (MFMs) are a computationally efficient alternative to the LMMs. We consider two aspects of the MFMs, implementation and specification. First we provide two new algorithms that can be used to implement the one-dimensional MFM under the terminal and the spot measure driven by a general diffusion process. Since the existing literature has been focused exclusively on the Gaussian driving processes our algorithms open the scope for new parameterisations. We then prove that the dynamics of the one-dimensional MFM are only affected by the time dependence of the driving process, described by a copula, and not by its marginal distributions. We then shift our focus and show that the one-dimensional MFM under the terminal measure is closely related to the one-factor separable local-volatility LMM.

Finally, we move our attention to the models of a two-currency economy. We propose a new three-factor model that we calibrate to the domestic and foreign caplet prices and the foreign exchange call options. To maintain the no-arbitrage condition while calibrating to foreign exchange market we propose a predictor-corrector type step. It is our conjecture that the predictor-corrector step converges, thus the model is well defined.

# Abbreviations and operators

| Abbreviation | Meaning |
|---|---|
| FRA | Forward rate agreement |
| LIBOR | London interbank offered rate |
| LMM | LIBOR market model |
| MFM | Markov-functional models |
| ZCB | Zero-coupon bond |
| **Operator** | **Meaning** |
| $(x)_+$ | Maximum of $x$ and 0 |
| $\|x\|$ | Absolute value of $x$ |
| $\operatorname{sgn} x$ | Sign of $x$ |
| $x \wedge y$ | Minimum of $x$ and $y$ |
| $x \vee y$ | Maximum of $x$ and $y$ |
| $x * y$ | Entry-by-entry product of vectors $x$ and $y$ |
| $\langle x, y \rangle$ | Inner product of vectors $x$ and $y$ |

# Chapter 1

# Introduction

Interest rate derivatives are arguably the largest asset class trading in the global market. Bank for International Settlements (Monetary and Economic Department, 2015) estimated that at the end of December 2014 the total notional amount outstanding on the global over-the-counter derivatives market was more than 630 trillion US dollars. Of this amount approximately 505 trillion US dollars, or 80% of the total amount, was attributed to the (single-currency) interest rate contracts (including forward rate agreements and interest rate swaps). This is approximately 9 times the total notional amount outstanding attributed to the foreign exchange contracts, which are the second largest asset class. Furthermore, the over-the-counter market in interest rate contracts is estimated to have approximately 6.5 times the notional amount outstanding of its exchange traded counterpart. Not only are the interest rate contracts the biggest asset class in terms of the notional amount outstanding, but are also the largest asset class by the gross market value representing approximately 75% of the over-the-counter market.

For the reasons outlined above the models of term structure of interest rates remain an important part of mathematical finance. In this thesis we contribute to the existing literature on two well established interest rate models

- *LIBOR market model* (Chapter 3),
- *Markov-functional model* (Chapter 4)

and an emerging

- *Two-currency Markov-functional model* (Chapter 5).

The common theme throughout the thesis is the computational efficiency of the models considered, which is in part achieved by concentrating on the models driven by low-number of Markovian factors or sources of uncertainty. The second motif in the thesis is our interest in the so called *'market models'*, which model the market observable forward LIBORs or forward swap rates opposed to the models of unobservable short or instantaneous forward rates.

Chapter 2 provides the reader with the background material used throughout the thesis and introduces the notation. It gives a brief overview of the arbitrage pricing theory, describes the economy of our interest and introduces some of the interest rate related instruments encountered on the interbank markets. Finally, it gives a brief introduction to the LIBOR market model and the Markov-functional model. Readers familiar with this material may choose to skip this chapter and refer back to it when necessary.

In Chapter 3 we focus on the LIBOR market model. It is well known that the LIBOR market model is a high-dimensional model regardless of the dimension of the Brownian motion driving its dynamics. In particular, this usually forces the user to implement it using Monte Carlo techniques, which are especially cumbersome to use when valuing derivatives with early exercise features. One way to overcome the problems arising from the high-dimensionality was proposed by Pietersz et al. (2004) and involves restricting the model parameterisation by imposing a separability condition. It has been suggested that such a restriction is too strong when we also want the model to be time-homogeneous. In Chapter 3 we explore this issue further. We show that a minor generalisation of the separability condition allows for a greater variety of time-homogeneous parameterisations of the separable LIBOR market model. In particular, we characterise two- and three-factor separable and time-homogeneous LIBOR market models and show that we obtain parameterisations that are of practical interest.

In Chapter 4 we turn our attention to the Markov-functional model. While the class of Markov-functional models encompasses most existing interest rate models, we will focus on the one-dimensional Markov-functional models under the terminal and the spot measure driven by a Markov process $x$. Traditionally, $x$ has always been assumed to be a Gaussian process as this allowed for a straightforward implementation on a grid. The main contribution of Chapter 4 to the literature on the Markov-functional models are two new algorithms that can be used to implement the one-dimensional Markov-functional model under the terminal or the spot measure for any diffusion process $x$ that has continuous marginal distributions.

While the Markov-functional approach have become popular because of its computational efficiency, it's main drawback is the lack of intuition behind the model dynamics, especially when compared to the LIBOR market model. In part, this has been addressed by Bennett and Kennedy (2005) who showed that the one-factor separable (log-normal) LIBOR market model has similar dynamics as an appropriately defined one-dimensional Markov-functional model (driven by a Gaussian process). In Chapter 4 we also address this issue and generalise the approach by Bennett and Kennedy (2005) to a wider class of separable one-factor local-volatility LIBOR market models. In particular, we generalise the idea of separability to the one-factor local-volatility LIBOR market models and give a systematic approach how they can be used to define one-factor Markov-functional model, which we later demonstrate with an example. We also provide a further insight into the role of the driving process by proving that the one-factor Markov-functional models under the terminal and the spot measure are only influenced by the time-dependence of the driving process, which we characterise with a copula, and not by its marginal distributions.

In Chapter 5 we introduce a two-currency economy and show how the results from Section 2.1 can be extended to the two-currency setting. We then define the concept of a two-currency Markov-functional model under the spot measure and review the approach taken by Fries and Rott (2004). We then propose a new two-currency Markov-functional model, that we conjecture can calibrate to the domestic and foreign caplet prices and foreign exchange call option prices. The novelty of the model is a predictor-corrector type step that we use to calibrate the model to foreign exchange options and maintain no-arbitrage property. Finally, we outline how the model can be efficiently implemented on a grid using the ideas from the implementation of the one-factor Markov-functional model under the spot measure.

Chapter 6 concludes by discussing how the concepts presented in the thesis are interlinked and by pointing out some interesting open questions.

There are two appendices at the end of the thesis. In Appendix A we provide further detail about the 'basis functions' that are used in Chapter 4. Appendix B contains two longer and less instructive proofs.

# Chapter 2

# Arbitrage Pricing Theory, Interest Rate Derivatives and Market Models

In this chapter we review some basic concepts of arbitrage pricing theory, interest rate derivatives and two models of the term structure of interest rates that will be used throughout the thesis.

## 2.1 Arbitrage Pricing Theory in a Nutshell

In this section we give a brief introduction of the important concepts of the arbitrage pricing theory that will be used throughout the thesis, especially the concept of a numeraire pair. The material covered in this section can be found in many textbooks about mathematical finance such as Andersen and Piterbarg (2010), Duffie (2001) or Fries (2007). Here we follow the approach taken by Hunt and Kennedy (2004).

Throughout this section and the entire thesis we will assume we are working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ supporting Brownian motion and satisfying the 'usual conditions':

1. $\sigma$-algebra $\mathcal{F}_0$ is $\mathbb{P}$-complete: if $A \subset B \in \mathcal{F}$ such that $\mathbb{P}(B) = 0$ then $A \in \mathcal{F}_0$;

2. Filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous:

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \quad t \geq 0. \tag{2.1}$$

Let us first consider an abstract single-currency[1] economy $\mathcal{E}$ with finite time horizon $T^* < \infty$ consisting of $n+1$ assets $A^1, \ldots, A^{n+1}$. For $i \in \{1, \ldots, n+1\}$ we can write $A^i = (A^i_t)_{t\in[0,T^*]}$, where $A^i_t$ is the time $t$ price of the asset $A^i$. For notational convenience we define the vector valued process $A = (A^i)_{i=1}^{n+1}$ and we assume that each of the price processes is a semimartingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T^*]}$.

[1] We will extend the results of this section to a two-currency setting in Section 5.1.

We will model gains of trading by Itō integral. More precisely, we first define a filtration $\{\mathcal{F}^A\}_{t\in[0,T^*]}$ to be the augmented natural filtration associated with $A$, i.e.

$$\mathcal{F}_t^A := \sigma\big(\sigma(A_s; s \le t) \cup \mathcal{F}_0\big), \quad t \le T^*. \tag{2.2}$$

We say that a vector valued process $\phi = (\phi^i)_{i=1}^{n+1}$ is a *self-financing trading strategy* if $\phi$ is $\{\mathcal{F}_t^A\}_{t\in[0,T^*]}$-predictable and

$$\langle \phi_t, A_t \rangle = \langle \phi_0, A_0 \rangle + G_t(\phi), \quad t \le T^*, \tag{2.3}$$

where $G(\phi) = (G_t(\phi))_{t\in[0,T^*]}$ is the gain-process of the strategy $\phi$ defined by

$$G_t(\phi) := \int_0^t \langle \phi_s, dA_s \rangle, \quad t \le T^*. \tag{2.4}$$

Next we define the concept of a numeraire pair.

**Definition 2.1.** *A process $N = (N_t)_{t\in[0,T^*]}$ is called numeraire if it is strictly positive $\mathbb{P}$-a.s. and there exists a self-financing trading strategy $\phi$ such that*

$$N_t = N_0 + \int_0^t \langle \phi_s, dA_s \rangle, \quad t \le T. \tag{2.5}$$

Note that if a process $N$ is a numeraire, the ratio $A_t^{i,N} := \frac{A_t^i}{N_t}$ is well defined $\mathbb{P}$-*a.s.* and we will denote by $A^{i,N} := (A_t^{i,N})_{t\in[0,T^*]}$ the numeraire rebased price process of asset $A^i, i \in \{1, \dots, n+1\}$. Moreover, we can define the vector valued process $A^N := (A^{i,N})_{i=1}^{n+1}$.

**Definition 2.2.** *A probability measure $\mathbb{N}$ defined on the measurable space $(\Omega, \mathcal{F}_{T^*}^A)$ is an equivalent martingale measure (EMM) associated with the numeraire $N$ if $\mathbb{N}$ and $\mathbb{P}$ are equivalent (on $(\Omega, \mathcal{F}_{T^*}^A)$) and $A^N$ is an $\{\mathcal{F}_t^A\}_{t\in[0,T^*]}$-martingale with respect to measure $\mathbb{N}$.*

We will refer to a pair $(N, \mathbb{N})$ consisting of numeraire $N$ and an EMM $\mathbb{N}$ associated with numeraire $N$ as the *numeraire pair*.

For technical reasons we need to restrict the set of self-financing trading strategies that can be used for trading. We will call the strategies that can be used for trading *admissible* and denote the set of admissible trading strategies by $\mathcal{S}$. We will adopt the convention that when the economy does not admit a numeraire pair we take the set of admissible strategies to be an empty set.

**Definition 2.3.** *A self-financing trading strategy $\phi$ is admissible if for every numeraire pair $(N, \mathbb{N})$ the process $G^N(\phi)$ defined by*

$$G_t^N(\phi) := \int_0^t \langle \phi_s, dA_s^N \rangle, \quad t \le T^*, \tag{2.6}$$

*is an $\{\mathcal{F}_t^A\}_{t\in[0,T^*]}$-martingale with respect to measure $\mathbb{N}$.*

**Remark 2.4.** *Alternatively we could impose an appropriate integrability constraint on the trading strategies. The details of such an approach can be found in Delbaen and Schachermayer (2006) and Duffie (2001).*

Next we define the concept of arbitrage and show how it relates to the existence of numeraire pair.

**Definition 2.5** (Arbitrage)**.** *We say that a self-financing trading strategy $\phi$ is an arbitrage strategy if one of the following conditions is satisfied for some $t \leq T^*$;*

1. $\langle \phi_0, A_0 \rangle < 0$ *and* $\langle \phi_t, A_t \rangle \geq 0$, $\mathbb{P}$-*a.s.;*
2. $\langle \phi_0, A_0 \rangle \leq 0$, $\langle \phi_t, A_t \rangle \geq 0$, $\mathbb{P}$-*a.s., and* $\mathbb{P}(\langle \phi_t, A_t \rangle > 0) > 0$.

We will say that an economy $\mathcal{E}$ admits arbitrage if there exists an admissible arbitrage strategy. Otherwise we say that the economy is arbitrage-free.

**Theorem 2.6.** *The economy $\mathcal{E}$ with the set of admissible strategies as defined in Definition2.3 is arbitrage-free.*

See Theorem 7.32 in Hunt and Kennedy (2004) for the proof.

Having established when an economy is arbitrage-free, we can now define price processes of contingent claims in an arbitrage-free economy.

**Definition 2.7.** *A contingent claim with paying an amount $V_T$ at time $T \leq T^*$ is attainable if there exists an admissible trading strategy $\phi$, called the replication strategy, such that $V_T = \langle \phi_T, A_T \rangle$. We will refer to the time $T$ as the expiry of the claim.*

In other words an attainable contingent claim can be replicated by trading with some admissible trading strategy. Note that by definition any attainable claim expiring at time $T$ is $\mathcal{F}_T^A$ measurable. For an attainable contingent claim $V_T$ we can define its price process $V = (V_t)_{t \in [0,T]}$ by

$$V_t := \langle \phi_t, A_t \rangle, \quad t \leq T, \tag{2.7}$$

where $\phi$ is a replication strategy for $V_T$. Note that, the price process seems to depend on the replication strategy chosen, however in an arbitrage-free economy one can show that the price process is independent replication strategy.

**Theorem 2.8** (Law of one price)**.** *Suppose that $\phi$ and $\psi$ are replication strategies for an attainable contingent claim $V_T$ expiring at time $T$. If the economy is arbitrage-free then*

$$\langle \phi_t, A_t \rangle = \langle \psi_t, A_t \rangle \ \mathbb{P}\text{-}a.s., \quad t \leq T \tag{2.8}$$

*and the price process $V = (V_t)_{t \in [0,T]}$ is well defined up to a modification.*

See Theorem 7.33 in Hunt and Kennedy (2004) for the proof.

Theorem 2.8 is sometimes referred to as the *Law of One Price* and will be used in the next section to determine the value of simple but fundamental instruments. As a corollary to Theorem 2.8 one can prove the *fundamental pricing formula.*

**Corollary 2.9** (Fundamental pricing formula)**.** *Let* $(N, \mathbb{N})$ *be a numeraire pair and* $V_T$ *an admissible contingent claim expiring at time* $T \leq T^*$*. Then the price process* $V = (V_t)_{t\in[0,T]}$ *satisfies the fundamental pricing formula*

$$V_t = N_t \mathbb{E}_{\mathbb{N}} \left[ \frac{V_T}{N_T} \middle| \mathcal{F}_t^A \right] \mathbb{P}\text{-}a.s., \quad t \leq T. \tag{2.9}$$

See Corollary 7.34 in Hunt and Kennedy (2004) for the proof.

**Remark 2.10.** *Thought the thesis we will slightly abuse the notation and simply condition on the* $\sigma$*-algebra* $\mathcal{F}_t$ *instead of* $\mathcal{F}_t^A$ *when using the fundamental pricing formula. While the two* $\sigma$*-algebras are in general not the same, in particular* $\mathcal{F}_t^A \subset \mathcal{F}_t$*, the distinction between them is not of practical importance for most of the thesis. We will explicitly point out the distinction between them in places where interchanging the* $\sigma$*-algebras would lead to different results (see also Section 7.3.1 and Remarks 7.38 and 7.45 in Hunt and Kennedy (2004)).*

Let us complete this short introduction to the arbitrage pricing theory by briefly discussing the notion of completeness.

**Definition 2.11.** *An economy* $\mathcal{E}$ *admitting a numeraire pair* $(N, \mathbb{N})$ *is complete if for every* $\mathcal{F}_{T^*}^A$ *measurable contingent claim* $V_{T^*}$ *satisfying*

$$\mathbb{E}_{\mathbb{N}} \left[ \left| \frac{V_{T^*}}{N_{T^*}} \right| \right] < \infty \tag{2.10}$$

*there exists na admissible replicating strategy.*

It turns out (see Corollary 7.40 in Hunt and Kennedy (2004)) that Definition 2.11 is independent of the numeraire pair chosen and that for a self-financing trading strategy $\phi$ to be admissible it is enough that the gains process $G^N(\phi)$ as defined in equation (2.6) is $\{\mathcal{F}_t^A\}_{t\in[0,T^*]}$-martingale under an EMM $\mathbb{N}$ for a single numeraire pair $(N, \mathbb{N})$.

**Theorem 2.12.** *An economy* $\mathcal{E}$ *admitting a numeraire pair is complete if and only if for any two numeraire pairs* $(N, \mathbb{N}_1)$ *and* $(N, \mathbb{N}_2)$ *the measures* $\mathbb{N}_1$ *and* $\mathbb{N}_2$ *agree on* $\mathcal{F}_{T^*}^A$*.*

See Theorem 7.41 in Hunt and Kennedy (2004) for proof.

## 2.2 Interest Rate Derivatives

In the previous section we have defined the concept of a general arbitrage-free economy $\mathcal{E}$. Now we will introduce a concrete economy that will be of our interest throughout the thesis.

Suppose $0 = T_0 < \ldots < T_{n+1}$ is a set of dates. We will adopt the convention that the date $T_0 = 0$ represents the present and that the dates are expressed as fractions of the year. For example if $T_i = 1.5$ then date $T_i$ is 18 months from today. We will denote by $\alpha_i, i \in \{0, \ldots, n\}$, the *accrual factor* associated with the period $[T_i, T_{i+1}]$. Roughly speaking

$\alpha_i$ is the length of the period $[T_i, T_{i+1}]$ expressed as a fraction of the year, i.e. $\alpha_i \approx T_{i+1} - T_i$, however the exact calculation of the accrual depends on the daycount convention used and might differ between markets.

### 2.2.1 Zero-coupon Bonds and the Economy

A *zero-coupon bond* (ZCB) is a financial instrument that promises the holder a payment of one unit of a currency at a pre-agreed date called the *maturity*. We will denote the time $t \le T$ price of a $T$-maturity ZCB by $D_{t,T}$, obviously in an arbitrage-free economy $D_{T,T} = 1$. Figure 2.1 shows the cashflows associated with buying a $T$ maturity ZCB at time $t$.

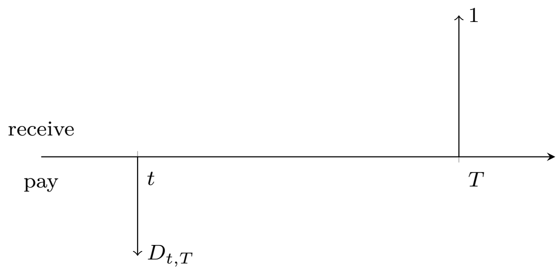



Figure 2.1: Cashflows associated with buying a $T$-maturity zero-coupon bond at time $t$.

Due to their simple structure ZCBs are fundamental assets of many interest rate models (we will explain the relationship between the ZCBs and the interest rates in the next subsection). For the purposes of this thesis we will assume, unless stated otherwise, that the economy consists of $n+1$ ZCBs maturing on dates $T_1, \dots, T_{n+1}$, which can be traded in continuous time without friction. In the context of previous section we can think of the date $T_{n+1}$ as the time horizon and the ZCBs as the assets $A^1, \dots, A^{n+1}$. For any $t < T_{n+1}$ we will refer to the map

$$T_i \mapsto D_{t,T_i}, \quad T_i > t \tag{2.11}$$

as the term structure of ZCBs and to any model of the economy consisting of ZCBs as the term structure model.

**Remark 2.13.** *To be precise, the assets price processes in the previous section were well defined up until the time horizon $T^* = T_{n+1}$. However, for $i \in \{1, \dots, n\}$ the price process of a $T_i$-maturity ZCB is defined up unitl time $T_i$, in particular only the price process of $T_{n+1}$ maturity ZCB is well defined for the entire time period $[0, T_{n+1}]$. Note that this is only a minor technical detail which can be resolved by defining an 'extended' $T_i$-maturity ZCB as*

$$D_{t,T_i} := D_{t \wedge T_i, T_i} \frac{D_{t,T_{n+1}}}{D_{t \wedge T_i, T_{n+1}}}, \quad t \le T_n. \tag{2.12}$$

*Note that slight abuse of notation is not harmful since the time $t \le T_i$ prices of the 'original' and 'extended' $T_i$-maturity ZCBs are the same, provided that $D_{t,T_{n+1}} > 0$ which is always the* *case in an arbitrage-free economy. In words, an 'extended' $T_i$-maturity ZCB is a $T_i$ maturity ZCB whose payoff is at the maturity is invested into buying $T_{n+1}$-maturity ZCBs.*

We have hinted in Remark 2.13 that in an arbitrage-free economy the price process of the $T_{n+1}$-maturity ZCB is strictly positive. This is clearly the case for the price process of any ZCB as non-positive price of a ZCB leads to trivial arbitrage strategy (buy the ZCB at a non-positive price and hold it until maturity). In particular, the $T_{n+1}$-maturity ZCB is a numeraire and we will denote the EMM assciated with it by $\mathbb{F}^{n+1}$ and refer to it as the *terminal measure*.

### 2.2.2 Deposits, Forward Rate Agreements and Swaps

**Deposit**

A *deposit* is an agreement between two counterparties where one party pays the other a fixed amount of cash in exchange for receiving it back with interest at a later date. Of our interest will be deposits starting on a date $T_i, i \in \{0, \dots, n\}$, and ending on the date $T_{i+1}$. The interest paid is proportional to the amount deposited. The interest received on a unit of a currency deposited at time $T_i$ is given by $\alpha_i L^i_{T_i}$, where $L^i_{T_i}$ is the interest rate determined at time $T_i$ for the period $[T_i, T_{i+1}]$. Figure 2.2 shows the cashflows associated with entering a unit deposit at time $T_i$.

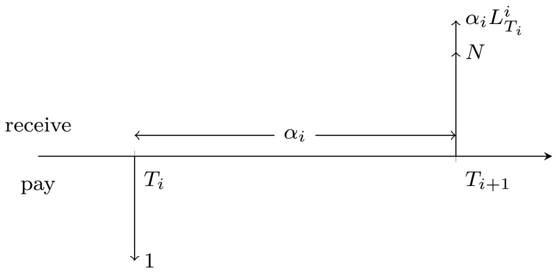



Figure 2.2: Cashflows associated with a deposit of one unit of a currency for the period $[T_i, T_{i+1}]$

On the interbank market the interest rate $L^i_{T_i}$ is called the *London Interbank Offered Rate* or simply *spot LIBOR*. It is quoted every business day by the ICE Benchmark Administration.[2] Note that the payoff of a unit deposit can be replicated by buying $1 + \alpha_i L^i_{T_i}$ ZCBs with maturity $T_{i+1}$ at time $T_i$. The cost of such a strategy at time $T_i$ is then $(1 + \alpha_i L^i_{T_i})D_{T_i,T_{i+1}}$ and in an arbitrage-free economy it has to equal to one unit of a currency. Therefore, we can express $L^i_{T_i}$ as

$$L^i_{T_i} = \frac{1 - D_{T_i,T_{i+1}}}{\alpha_i D_{T_i,T_{i+1}}}. \tag{2.13}$$

[2] More details can be found at `http://www.theice.com/iba/libor`.

### Rolling Bank Account

Closely related to the deposit is a discretely compounded *rolling bank account*. It is an instrument created by depositing one unit at time 0 for the period $[0, T_1]$. At each of the dates $T_i, \in \{1, \ldots, n\}$ we receive the deposited amount back with interest and deposit it again for the period $[T_i, T_{i+1}]$. It is easy to see that the value of the rolling bank account at time $T_i$ is given by

$$B_{T_i} = \prod_{j=0}^{i-1}(1 + \alpha_j L^j_{T_j}), \quad i = 1, \ldots, n+1. \tag{2.14}$$

Since the value of a deposit at time $T_{i+1}$ is known at time $T_i$ we can determine the value of the rolling bank account on a date $t \in (T_i, T_{i+1})$ by simply discounting the time $T_{i+1}$ value of the deposit (which is already known), i.e.

$$B_t = D_{t,T_{i+1}} B_{T_{i+1}}, \quad t \in (T_i, T_{i+1}]. \tag{2.15}$$

Note that the value of the rolling bank account can be replicated by a self-financing trading strategy involving ZCBs only. In particular, investing the amount $B_{T_i}$ in $T_{i+1}$-maturity ZCB at time $T_i, i = 0, \ldots, n$, replicates $B_{T_{n+1}}$. Furthermore, note that the process $B = (B_t)_{t\in[0,T_{n+1}]}$ is strictly positive and is therefore a numeraire. In any arbitrage-free term structure model we will denote the EMM associated with it by $\mathbb{F}^0$ and refer to it as the *spot measure*.

**Remark 2.14.** *Note that a rolling bank account provides a suitable alternative for defining 'extended' ZCBs, similarly as we have done it in Remark 2.13. For the purposes of this thesis, it is not relevant which route one chooses to take. However, if one wishes to use $T_i$-maturity ZCB $i \in \{1, \ldots, n\}$ as a numeraire and apply it to price payoffs occurring after time $T_i$ care must be taken as the two definitions will yield different EMMs.*

### Forward Rate Agreement

Another instrument closely related to the deposit is a *forward rate agreement* (FRA). An FRA is a financial instrument where two parties agree to exchange interest payments accrued over a future period $[T_i, T_{i+1}], i \in \{1, \ldots, n\}$. The payments are exchanged at time $T_{i+1}$, when one counterparty pays the other the amount $\alpha_i K$ and in exchange receives an amount $\alpha_i L^i_{T_i}$. The interest rate $K$ is agreed when the parties enter the FRA, on the other hand the spot LIBOR $L^i_{T_i}$ is determined on a later date $T_i$. The date $T_i$ is often referred to as the *reset* or *setting date*. We will say that the party paying the interest accrued at rate $K$ holds the long position in the FRA and the party paying the interest accrued at spot LIBOR holds the short position in the FRA. Figure 2.3 shows the cashflows associated with the long position in the FRA.

Suppose the parties entered an FRA with reset date $T_i$ and fixed rate $K$. We would like to calculate the time $t \leq T_i$ value of the long position in the FRA. Clearly, the time $t$ value of the cashflow $\alpha_i K$ is simply $D_{t,T_{i+1}} \alpha_i K$. On the other hand, we can replicate the

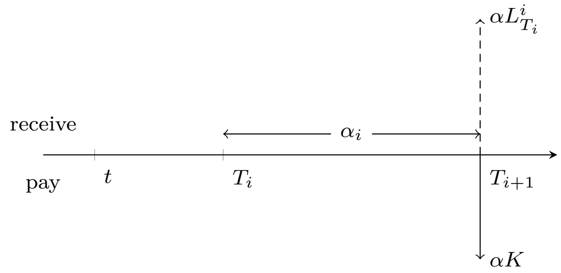



Figure 2.3: Cashflows associated with a long-position in an FRA with strike $K$ for the period $[T_i, T_{i+1}]$.

payment $\alpha_i L^i_{T_i}$ by buying a $T_i$ maturity ZCB, selling a $T_{i+1}$-maturity ZCB and depositing the payment of a ZCB at time $T_i$ for the period $[T_i, T_{i+1}]$. Then the time $t$ value of a long position in the FRA is given by

$$V_t^{\mathrm{FRA},i}(K) = D_{t,T_i} - (1 + \alpha_i K) D_{t,T_{i+1}}. \tag{2.16}$$

It is standard market practice that the FRAs are entered at zero cost. The interest rate $K$ that satisfies this condition is called the *forward LIBOR* for the period $[T_i, T_{i+1}]$ and its time $t < T_i$ value is denoted by $L^i_t$ and is given by

$$L^i_t = \frac{D_{t,T_i} - D_{t,T_{i+1}}}{\alpha_i D_{t,T_{i+1}}}. \tag{2.17}$$

For $t \leq T_n$ we will refer to the map

$$T_i \mapsto L^i_{T_i}, \quad T_i \geq t \tag{2.18}$$

as the term structure of interest.

**Remark 2.15.** *Throughout the thesis we will work in the 'single-curve' setting, that is we will assume that the term structure of ZCBs and the term structure of forward LIBORs (and later forward swap rates) are 'equivalent' in the sense that one can move between the two curves by using equation* (2.17)*. Such a setting does not take into the account the credit risk. A good overview of the 'multi-curve' approach taking into account the credit risk can be found in the books by Brigo et al. (2013) and Henrard (2014).*

Although we assumed that $t < T_i$, observe that by setting $t = T_i$ in equation (2.17) we obtain exactly the spot LIBOR $L^i_{T_i}$. This should not come as a surprise as entering into an FRA on its setting date at zero cost has to be done with fixed rate being the spot LIBOR rate (which is known on the reset date). Furthermore, note that in any arbitrage-free term structure model the process $L^i = (L^i_t)_{t \in [0,T_i]}$ has to be a martingale under the EMM $\mathbb{F}^{i+1}$ associated with $T_{i+1}$-maturity ZCB as the numeraire. The measure $\mathbb{F}^{i+1}$ is referred to as the $T_{i+1}$-*forward measure.*

### Swap

A payers interest-rate *swap* is a financial instrument where two parties agree to exchange a series of interest payments on a series of pre-agreed dates. The swap is characterised by a start date $T_i, i \in \{1, \dots, n\}$, the last payment date $T_j, j \in i+1, \dots, n+1$, and a fixed interest rate $K$ agreed when the parties enter the swap. The payments are exchanged at times $T_{i+1}, \dots, T_j$, on a date $T_{k+1}, k \in \{i, \dots, j-1\}$ one counterparty pays the other the amount $\alpha_k K$ and in exchange receives the amount $\alpha_k L^k_{T_k}$. The series of payments $(\alpha_k K)_{k=i}^{j-1}$ is referred to as the *fixed leg* of the swap and series of payments $(\alpha_k L^k_{T_k})_{k=i}^{j-1}$ is called the *floating leg*. Similarly to the FRA we adopt the convention that the party paying the fixed leg holds the long position in the swap and that the party paying the floating leg holds the short position in the swap. Figure 2.4 shows the cashflows associated with the long position in the swap.

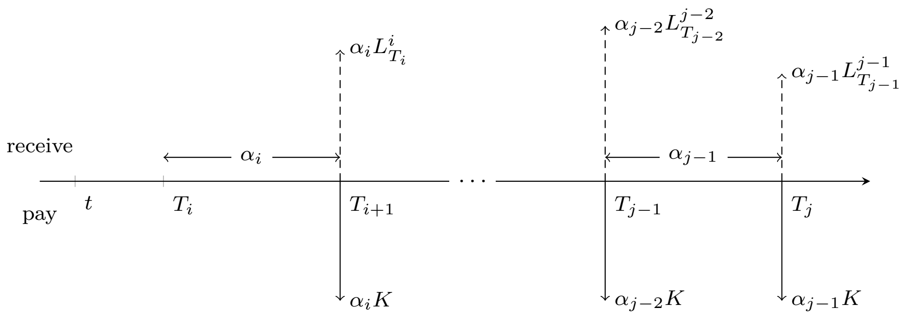



Figure 2.4: Cashflows associated with a long position in a swap starting on date $T_i$ and last payment date on $T_j$.

In order to value a swap we need to find a trading strategy replicating the swap's cashflows. One way this can be done is by observing that a swap starting on date $T_i$ and last payment date $T_j$ can be decomposed into a series of $j-i$ FRAs with reset dates $T_i, \dots, T_{j-1}$. Therefore the time $t \le T_i$ value of the swap can be expressed as a sum of values of the FRAs

$$V_t^{\text{swap},i\times j}(K) = D_{t,T_i} - D_{t,T_j} - K \sum_{k=i}^{j-1} \alpha_k D_{t,T_{k+1}}. \tag{2.19}$$

As in the case of the FRA, it is standard market practice to enter the swap at zero cost for both parties. The fixed rate $K$ that satisfies this condition is called the *swap rate*. The time $t \le T_i$ value of the swap rate for the swap starting on date $T_i$ and with last payment date $T_j$ will be denoted by $y_t^{i\times j}$ and can be expressed as

$$y_t^{i\times j} = \frac{D_{t,T_i} - D_{t,T_j}}{\sum_{k=i}^{j-1} \alpha_k D_{t,T_{k+1}}}. \tag{2.20}$$

The sum

$$P_t^{i\times j} := \sum_{k=i}^{j-1} \alpha_k D_{t,T_{k+1}} \tag{2.21}$$

is usually referred to as the *present value of a basis point.* Note that the process $(P_t^{i\times j})_{t\in[0,T_i]}$ is strictly positive and can be used as a numeraire. We will denote the EMM associated with it by $\mathbb{S}^{i\times j}$ and refer to it as the *$T_i \times T_j$-swaption measure.*

### 2.2.3 Caplets and Swaptions

In the previous subsection we introduced basic financial instruments encountered on the interbank market. They all shared a common feature that they could be replicated by a simple model-independent trading strategies involving only ZCBs. In this subsection we introduce some common instruments that in general cannot be replicated using model-independent trading strategies. The reason for this will be that the instruments' payoffs will no longer be affine functions of ZCB prices as it was the case for FRAs and swaps (see equations (2.16) and (2.19)).

#### Caplet

A (European) *caplet* with *expiry date* $T_i, i \in \{1, \dots, n\}$, payment date $T_{i+1}$ and *strike* $K > 0$ is a financial instrument that gives the holder the right to enter an FRA with reset date $T_i$, payment date $T_{i+1}$ and fixed rate $K$ on the expiry date $T_i$. Recall that the value of the underlying FRA on the date $T_i$ is given by

$$V_{T_i}^{\mathrm{FRA},i} = D_{T_i,T_{i+1}} \alpha_i (L_{T_i}^i - K). \tag{2.22}$$

The holder will only choose to exercise his right to enter the FRA if its value is positive that is when $L_{T_i}^i > K$. Therefore, the time $T_i$ value of a caplet with expiry date $T_i$ is given by

$$V_{T_i}^{\mathrm{cpl},i}(K) = \alpha_i D_{T_i,T_{i+1}} (L_{T_i}^i - K)_+, \tag{2.23}$$

where $(x)_+$ denotes the positive part of $x$, and the net payment to the holder of the caplet at time $T_{i+1}$ is given by

$$V_{T_{i+1}}^{\mathrm{cpl},i}(K) = \alpha_i (L_{T_i}^i - K)_+. \tag{2.24}$$

For time $t \leq T_i$ the value of a caplet is no longer model independent. However, if the economy admits a numeraire pair $(N, \mathbb{N})$ and the caplet payoff $V_{T_{i+1}}^{\mathrm{cpl},i}(K)$ can be replicated, the time $t$ value of a caplet is given by

$$V_t^{\mathrm{cpl},i}(K) = N_t \mathbb{E}_{\mathbb{N}} \left[ \frac{\alpha_i (L_{T_i}^i - K)_+}{N_{T_{i+1}}} \middle| \mathcal{F}_t \right] \tag{2.25}$$

or equivalently by

$$V_t^{\mathrm{cpl},i}(K) = N_t \mathbb{E}_{\mathbb{N}}\left[\frac{\alpha_i D_{T_i,T_{i+1}}(L^i_{T_i} - K)_+}{N_{T_i}}\middle|\mathcal{F}_t\right] \tag{2.26}$$

When valuing a caplet it is often convenient to choose $(D_{\cdot,T_{i+1}}, \mathbb{F}^{i+1})$ as the numeraire pair, in this case equations (2.25) and (2.26) reduce to

$$V_t^{\mathrm{cpl},i}(K) = D_{t,T_{i+1}} \mathbb{E}_{\mathbb{F}^{i+1}}\big[\alpha_i(L^i_{T_i} - K)_+\big|\mathcal{F}_t\big]. \tag{2.27}$$

One of the first models capable of valuing the caplets was proposed by Black (1976). Black's model is essentially a modification of the earlier Black and Scholes (1973) model, where instead of modelling the spot price of an asset one models its forward price, i.e. in the case of a caplet the forward LIBOR $L^i$. The original approach by Black is based on the PDE methods, however it has a straightforward probabilistic interpretation that the forward price (in our case forward LIBOR $L^i$) is a log-normal martingale under the measure $T_{i+1}$-forward measure satisfying the SDE

$$dL^i_t = L^i_t \sigma dW^{i+1}_t, \tag{2.28}$$

where $\sigma$ is a positive constant and $W^{i+1}$ is a Brownian motion under the EMM $\mathbb{F}^{i+1}$. In particular, note that the process $L^i$ is a Markov process under the measure $\mathbb{F}^{i+1}$ and the caplet price can be expressed in terms of $K, D_{t,T_{i+1}}, L^i_t$, and $\sigma$ as

$$V_t^{\mathrm{cpl},i}(K) = \alpha_i D_{t,T_{i+1}}\big(L^i_t \Phi(d_+) - K\Phi(d_-)\big), \tag{2.29}$$

where $\Phi$ is the distribution function of a standard normal random variable and

$$d_\pm := \frac{\log \frac{L^i_t}{K} \pm \frac{1}{2}\sigma^2(T_i - t)}{\sigma\sqrt{T_i - t}}. \tag{2.30}$$

We will refer to equation (2.29) as the *Black's formula for caplets.*

**Remark 2.16.** *In a term structure model for a Black's formula to be valid for time $t = 0$ only it is enough that the distribution of $L^i_{T_i}$ under the measure $\mathbb{F}^{i+1}$ is log-normal.*

While the Black's model is nowadays rarely used for the pricing of derivatives, Black's formula remains important part of the finance industry. The reason for this is that the caplet prices are quoted in terms of the value of the parameter $\sigma$ one needs to use in the Black's formula to obtain the price of the caplet. Such a value of $\sigma$ is referred to as the *implied volatility* and in practice depends on the strike and the expiry date. We will refer to the function mapping the expiry date and the strike to the implied volatility as the implied volatility surface.

### Digital Caplet

A *digital caplet* with expiry date $T_i, i \in \{1, \dots, n\}$, payment date $T_{i+1}$ and strike $K \geq 0$ is a financial instrument that pays the holder one unit of a currency at time $T_{i+1}$ if the value of $L^i_{T_i}$ is above $K$ and zero otherwise. In particular, the time $T_i$ value of a digital caplet is given by

$$V^{\mathrm{dcpl},i}_{T_i} = D_{T_i,T_{i+1}} \mathbf{1}_{\{L^i_{T_i} > K\}} \tag{2.31}$$

and the payment at $T_{i+1}$ is given by

$$V^{\mathrm{dcpl},i}_{T_{i+1}} = \mathbf{1}_{\{L^i_{T_i} > K\}}. \tag{2.32}$$

As in the case of the European caplet the price of a digital caplet is model-dependent and can be evaluated using the fundamental pricing formula provided that the option payoff can be replicated. In particular, taking the $T_{i+1}$-maturity ZCB as the numeraire yields that the time $t \leq T_i$ price of a digital caplet is given by

$$V^{\mathrm{dcpl},i}_t = D_{t,T_{i+1}} \mathbb{E}_{\mathbb{F}^{i+1}} \big[ \mathbf{1}_{\{L^i_{T_i} > K\}} \big| \mathcal{F}_t \big]. \tag{2.33}$$

In particular for $t = 0$ the price of the caplet is simply discounted probability (under the measure $\mathbb{F}^{i+1}$) of $L^i_{T_i}$ being greater than $K$, i.e.

$$V^{\mathrm{dcpl},i}_0 = D_{0,T_{i+1}} \mathbb{F}^{i+1} \big( L^i_{T_i} > K \big). \tag{2.34}$$

Furthermore, one can show that the prices of digital caplets are related to the prices of European caplets via

$$V^{\mathrm{dcpl},i}_t(K) = -\frac{1}{\alpha_i} \frac{\partial V^{\mathrm{cpl},i}_t}{\partial K} \tag{2.35}$$

since

$$\mathbf{1}_{\{L^i_{T_i} > K\}} = -\frac{\partial (L^i_{T_i} - K)_+}{\partial K} \tag{2.36}$$

in the sense of the weak derivative and we can justify the interchange of differentiation and integration using the dominated convergence theorem. Therefore, the prices of digital caplets uniquely define the prices of caplets and vice versa (this was first observed by Dupire et al. (1994)).

In particular, in the Black's model the price of a digital caplet is given by

$$V^{\mathrm{dcpl},i}_t(K) = D_{0,T_i} \Phi(d_-), \tag{2.37}$$

where $d_-$ is defined as in equation (2.30).

### Digital Caplet In-arrears

A *digital caplet in-arrears* with expiry date $T_i$ and strike $K$ is a financial instrument that pays the holder one unit of a currency at time $T_i$ if the LIBOR rate $L^i_{T_i}$ is above $K$ and zero otherwise. Therefore the time $T_i$ value/payment of the digital caplet in-arrears is given by

$$V^{\mathrm{dca},i}_{T_i} = \mathbf{1}_{\{L^i_{T_i} > K\}} \tag{2.38}$$

and the time $t < T_i$ value is given by

$$V^{\mathrm{dca},i}_t(K) = D_{t,T_{i+1}} \mathbb{E}_{\mathbb{F}^{i+1}}\left[\frac{\mathbf{1}_{\{L^i_{T_i} > K\}}}{D_{T_i,T_{i+1}}}\middle|\mathcal{F}_t\right]. \tag{2.39}$$

Note that $D^{-1}_{T_i,T_{i+1}} = 1 + \alpha_i L^i_{T_i}$ and therefore a digital caplet in-arrears can be replicated by buying $\alpha_i$ European caplets and $1 + \alpha_i K$ digital caplets, both with strike $K$ and expiry date $T_i$

$$V^{\mathrm{dca},i}_t(K) = \alpha_i V^{\mathrm{cpl},i}_t(K) + (1 + \alpha_i K) V^{\mathrm{dcpl},i}_t(K) \tag{2.40}$$

Therefore, knowing the prices of European caplets uniquely determines the prices of digital caplets in-arrears.

### Swaption

A European *swaption* with strike $K$, expiry date $T_i$ written on swap rate $y^{i\times j}_{T_i}$ is a financial instrument that gives holder the right to enter at time $T_i$ a long position in a swap with fixed rate $K$, reset date $T_i$ and last payment date $T_{i+j}$. The time $T_i$ value of such a swap can be written as

$$V^{\mathrm{swap},i\times j}_{T_i}(K) = 1 - D_{T_i,T_j} - K \sum_{k=i}^{j-1} \alpha_k D_{T_i,T_{k+1}} \tag{2.41}$$

$$= P^{i\times j}_{T_i}(y^{i\times j}_{T_i} - K) \tag{2.42}$$

Since $P^{i\times j}_{T_i}$ is strictly positive the investor will choose to exercise his right and enter the swap when the swap rate $y^{i\times j}_{T_i}$ is above the strike $K$ and the time $T_i$ value of the swaption is given by

$$V^{\mathrm{swaption},i\times j}_{T_i}(K) = P^{i\times j}_{T_i}(y^{i\times j}_{T_i} - K)_+. \tag{2.43}$$

**Remark 2.17.** *Note that swap rate $y^{i\times i+1}_{T_i}$ is exactly the LIBOR $L^i_{T_i}$ and the swaption written on $y^{i\times i+1}_{T_i}$ is the same as the caplet written on $L^i_{T_i}$.*

As in the case of caplets the time $t < T_i$ value of swaption is model dependent and can evaluated using the fundamental pricing formula

$$V^{\mathrm{swaption},i\times j}_t(K) = N_t \mathbb{E}_{\mathbb{N}}\left[\frac{P^{i\times j}_{T_i}(y^{i\times j}_{T_i} - K)_+}{N_{T_i}}\middle|\mathcal{F}_t\right], \tag{2.44}$$

where $(N, \mathbb{N})$ is a numeraire pair. In particular, it is often convenient to evaluate the swaption price under the $T_i \times T_j$-swaption measure $\mathbb{S}^{i\times j}$ associated with the numeraire $P^{i\times j}$

$$V_t^{\text{swaption},i\times j}(K) = P_t^{i\times j}\mathbb{E}_{\mathbb{S}^{i\times j}}\left[(y_{T_i}^{i\times j} - K)_+ \middle| \mathcal{F}_t\right]. \tag{2.45}$$

Note the similarity between equation (2.27) describing the caplet value under the forward measure and equation (2.45). In particular, in both cases the underlying forward rate and swap rate processes are martingales under the forward and respectively swaption measure. Therefore, it should not come as a surprise that the Black's model can be used to price the swaption.

Under the Black's model the forward swap rate process $y^{i\times j}$ is assumed to be a log-normal martingale under the measure $\mathbb{S}^{i\times j}$ given by the SDE

$$dy_t^{i\times j} = y_t^{i\times j}\sigma dW_t^{i\times j}, \tag{2.46}$$

where $\sigma$ is a positive constant and $W^{i\times j}$ is a Brownian motion under the measure $\mathbb{S}^{i\times j}$. The swaption price is then given by the *Black's formula for swaptions*

$$V_t^{\text{swaption},i\times j}(K) = P_t^{i\times j}\big(y_t^{i\times j}\Phi(d_+) - K\Phi(d_-)\big), \tag{2.47}$$

where

$$d_\pm := \frac{\log\frac{y_t^{i\times j}}{K} \pm \frac{1}{2}\sigma^2(T_i - t)}{\sigma\sqrt{T_i - t}}. \tag{2.48}$$

As in the case of caplets, the importance of Black's model today is mainly as a computational tool as the prices of swaptions are quoted in terms of *implied volatilities*. That is the parameter $\sigma$ one needs to use to determine the current value of the swaption.

### Digital swaption

A PVBP-*digital swaption* with strike $K$, expiry date $T_i$ written on swap rate $y_{T_i}^{i\times j}$ is a financial instrument paying the amount $P_{T_i}^{i\times j}$ at time $T_i$ if the swap rate $y^{i\times j}$ is above $K$ and zero otherwise. In particular, the value of digital swaption on the expiry date is given by

$$V_{T_i}^{\text{dswaption},i\times j}(K) = P_{T_i}^{i\times j}\mathbf{1}_{\{y_{T_i}^{i\times j} > K\}}. \tag{2.49}$$

**Remark 2.18.** *Note that a PVBP-digital swaption written on $y_{T_i}^{i\times i+1}$ is the same as $\alpha_i$ digital caplets written on $L_{T_i}^i$.*

We can then evaluate the time $t < T_i$ price of a digital swaption by applying the fundamental pricing formula to its payoff. Under the $T_i \times T_j$-swaption measure $\mathbb{S}^{i\times j}$ corresponding to $P^{i\times j}$ as the numeraire the digital swaption price is given by

$$V_t^{\text{dswaption},i\times j}(K) = P_t^{i\times j}\mathbb{E}_{\mathbb{S}^{i\times j}}\left[\mathbf{1}_{\{y_{T_i}^{i\times j} > K\}} \middle| \mathcal{F}_t\right]. \tag{2.50}$$

In particular for $t = 0$

$$V_0^{\text{dswaption},i\times j}(K) = P_0^{i\times j}\mathbb{S}^{i\times j}\big(y_{T_i}^{i\times j} > K\big). \tag{2.51}$$

As for the European caplets and digital caplets, we can derive a similar relationship between prices of European swaptions and PVBP-digital swaptions, namely

$$V_t^{\text{dswaption},i\times j}(K) = -\frac{\partial V_t^{\text{swaption},i\times j}}{\partial K}. \tag{2.52}$$

Finally, let us note that under the Black's model the digital swaption price can be expressed as

$$V_t^{\text{dswaption},i\times j}(K) = P_t^{i\times j}\Phi(d_-), \tag{2.53}$$

where $d_-$ is defined as in equation (2.48).

## 2.3 Market Models

Having described some of the instruments encountered on the interbank market let us outline two models that will be of our interest throughout the thesis: LIBOR market models and Markov-functional models.

### 2.3.1 LIBOR Market Model

The *LIBOR market model* (LMM) is one of the most popular models of interest rates. It was developed in the 1990s by Miltersen et al. (1997), Brace et al. (1997), Musiela and Rutkowski (1997), and Jamshidian (1997). The introduction of LMM also brought a major shift in the prevailing term structure modelling paradigm.

Before the introduction of LMM the term structure models were usually based on the unobservable short rates (e.g. Vasicek (1977), Cox et al. (1985), Hull and White (1990) etc.) and after the publication of the seminal paper Heath et al. (1992) also on unobservable instantaneous forward rates (e.g. Cheyette (1992), Ritchken and Sankarasubramanian (1995), etc.).

The LMM represented the shift from modelling the unobservable short rates or a continuum of instantaneous forward rates to a finite set of market observable forward LIBORs.[3] In particular, the underlying economy in an LMM consists of $n + 1$ ZCBs maturing on dates $T_1, \ldots, T_{n+1}$ and is exactly the economy described in Subsection 2.2.1.

The basic idea behind the LMM is to model the prices of ZCBs indirectly by specifying the dynamics of forward LIBORs $L^1, \ldots, L^n$. In particular, for each $i \in \{1, \ldots, n\}$ the process $L^i$ is assumed to be a log-normal martingale under the $T_{i+1}$-forward measure. Therefore, in the LMM the prices of caplets written on $L^i_{T_i}, i \in \{1, \ldots, n\}$ are given by the Black's formula. However, the LMM is fundamentally different from the Black's model. The Black's

[3] And with the extension by Jamshidian (1997) also the forward swap rates.

model is a model of a single forward rate whereas the LMM is a model of the entire term structure of forward rates.

The LMM can be specified under the spot measure or any $T_i$-forward measure, $i \in \{1, \ldots, n+1\}$. However, the spot and the terminal measure are the most common choices. From the theoretical standpoint the choice of measure is irrelevant, however in practical applications one might prefer one EMM over the other (more details can be found in Chapter 5 Brace (2007), Section 2.5 Gatarek et al. (2007) and Chapter 20 Joshi (2011)). In this thesis we will mainly consider LMMs under the terminal measure.

A $d$-factor LMM under the terminal measure, is given by an initial term structure $(L_0^i)_{i=0}^n$ and the system of SDEs

$$dL_t^i = L_t^i \langle \sigma^i(t), dW_t \rangle - L_t^i \sum_{j=i+1}^{n} \frac{\alpha_j L_t^j \langle \sigma^i(t), \sigma^j(t) \rangle}{1 + \alpha_j L_t^j} dt, \quad t \leq T_i, \; i = 1, \ldots, n, \tag{2.54}$$

where $W$ is a standard $d$-dimensional Brownian motion under the measure $\mathbb{F}^{n+1}$ and $\sigma^i : [0, T_i] \to \mathbb{R}^d, i = 1, \ldots, n$, are bounded measurable functions. One can show that under these conditions the system of SDEs (2.54) admits a strictly positive strong solution when the initial forward LIBORs $L_0^i, i = 1, \ldots, n$, are strictly positive (see Section 14.2 in Andersen and Piterbarg (2010) and Section 18.2 in Hunt and Kennedy (2004) for more details).

**Remark 2.19.** *It is sometimes convenient to define all forward LIBORs on the interval* $[0, T_{n+1}]$ *by setting*

$$L_t^i := L_{T_i}^i, \quad t \in (T_i, T_{n+1}], \tag{2.55}$$

*or equivalently by extending the domain of functions* $\sigma^1, \ldots, \sigma^n$ *to* $[0, T_{n+1}]$ *by setting*

$$\sigma^i(t) := 0, \quad t \in (T_i, T_{n+1}]. \tag{2.56}$$

Let us now step back and discuss the LMM in the context of our economy $\mathcal{E}$ consisting of ZCBs maturing on date $T_1, \ldots, T_{n+1}$. In particular, note that such an economy consists of $n+1$ ZCBs but the LMM specifies the dynamics of only $n$ forward rates. Consequently, the LMM does not uniquely define the dynamics of ZCB prices, however it does uniquely define their relative dynamics since

$$\frac{D_{t,T_i}}{D_{t,T_j}} = \prod_{k=i}^{j-1} (1 + \alpha_k L_t^k), \quad i < j, \tag{2.57}$$

moreover the ZCB prices are uniquely defined on the dates $T_1, \ldots, T_n$ since

$$D_{T_i,T_j} = \prod_{k=i}^{j-1} (1 + \alpha_k L_{T_k}^k)^{-1}, \quad i < j. \tag{2.58}$$

In particular, this is usually enough for pricing purposes since the LIBOR and swap derivatives typically have cashflows occurring on dates $T_1, \ldots, T_{n+1}$ (see Section 7 in Jamshidian (1997) and Chapter 20 in Joshi (2011) for more detail).

The specification of an LMM as in equation (2.54) is particularly useful from the computational perspective. For example, it allows for a straight-forward implementation via Monte Carlo methods. However, it offers little intuition about the model's dynamics. It is therefore often useful to introduce *instantaneous volatility* and *instantaneous correlation* functions. The instantaneous volatility functions $\sigma^{inst,i} : [0, T_i] \to \mathbb{R}_+, i = 1, \dots, n$, are given by

$$\sigma^{inst,i}(t) := \sqrt{\langle \sigma^i(t), \sigma^i(t) \rangle}, \tag{2.59}$$

and the instantaneous correlation functions $\rho_{i,j}^{inst} : [0, T_i \wedge T_j] \to [-1, 1], i, j = 1, \dots, n$, are given by

$$\rho_{i,j}^{inst}(t) := \frac{\langle \sigma^i(t), \sigma^j(t) \rangle}{\sigma^{inst,i}(t)\sigma^{inst,j}(t)}. \tag{2.60}$$

It is easy to see that

$$d(\log L_t^i)d(\log L_t^j) = \rho_{i,j}^{inst}(t)\sigma^{inst,i}(t)\sigma^{inst,j}(t)dt, \tag{2.61}$$

and one can show that the instantaneous volatility and correlation functions uniquely determine an LMM (see Section 3.2 inRebonato (2002)). Furthermore, the time $t \leq T_i$ implied volatility of a caplet written on $L_{T_i}^i$ is a deterministic function given by

$$\sigma^{impl,i}(t) := \frac{1}{\sqrt{T_i - t}} \left( \int_t^{T_i} \sigma^{inst,i}(s)^2 ds \right)^{\frac{1}{2}}. \tag{2.62}$$

It is often convenient to fix a calendar time $t$ and consider the time $t$ implied volatilities as a function of the maturity of the caplet, i.e.

$$T_i \mapsto \sigma^{impl,i}(t), \quad T_i > t. \tag{2.63}$$

We will refer to such function as the time $t$ *term structure of volatilities* or simply term structure of volatilities when $t$ is clear from the context.

Finally, let us briefly comment on one of the main issues user of the LMMs face when implementing them in practice namely the high dimensionality of the model. In particular, note that the drift term of the forward LIBOR $L^i$ under the terminal measure (see equation (2.54)) depends on the state of the forward rates $L^i, \dots, L^n$. Consequently, to implement the LMM it is not enough to keep track only of the current value of the Brownian motion driving the dynamics and one must keep track of all forward LIBORs. In particular, the finite difference methods are not suitable for implementing the LMM and one is usually forced to resort to Monte Carlo methods (see Section 14.6 in Andersen and Piterbarg (2010)). We will see in Chapter 3 that under certain restrictions on the functions $\sigma^1, \dots, \sigma^n$ we can overcome the so called 'curse of dimensionality and discuss the limitations of such an restriction.

### 2.3.2 Markov-functional Model

The second class of models we will be interested in this thesis are *Markov-functional models* (MFMs). They were introduced by Hunt et al. (2000) and also by Balland and Hughston (2000) based on the earlier ideas of Hunt et al. (1998). The main idea of an MFM is that we express the prices of ZCBs as functions of the state of some Markov process $x$. This allows us to implement an MFM by only keeping track of only process $x$. In this thesis we adopt the definition of an MFM from Kennedy (2010).

**Definition 2.20.** *A model of a $T^* < \infty$ time horizon economy consisting of ZCBs with maturities from a non-empty set $\mathcal{T} \subset (0, T^*]$[4] is said to be Markov-functional if there exists a numeraire pair $(N, \mathbb{N})$ and a (d-dimensional) $\{\mathcal{F}_t\}_{t \in [0,T^*]}$-adapted process $x$ such that:*

1. *$x$ is a Markov process under the measure $\mathbb{N}$;*

2. *for all $T \in \mathcal{T}$ and $t \leq T$ the time $t$ price of a $T$-maturity ZCB $D_{t,T}$ is $\sigma(x_t)$-measurable.*

Before we analyse Definition 2.20 in more detail let us make a trivial observation that the existence of a numeraire pair ensures that any MFM is arbitrage-free. Besides the numeraire pair, the central part of the definition of the MFM is a Markov process $x$ which we will refer to as the *driving process* or the *driver*. It is easy to see that for a given numeraire pair the driving process is not unique. For example, let $x$ be a driving process and $a > 0$ then the process $x'$ defined by $x'_t := ax_t$ gives rise to the same model as the process $x$ (we will discuss this in greater detail in Section 4.4). Moreover, the driving processes need not be of the same dimension.

With this in mind we say that the dimension of an MFM is $d$ if:

1. There exists a $d$-dimensional driving process $x$;

2. Dimension of any other driving process $x'$ (possibly corresponding to a different numeraire pair) is at least $d$.

However, in practical application one typically singles out a specific numeraire pair and driving process when working with an MFM and refers to the dimension of the MFM as the dimension of the chosen driving process. We will see in Chapter 4 that the 'true spirit' of an MFM is in choosing the driving process and the EMM in advance and then determining the numeraire and the functional forms of the ZCBs by calibrating the model to prices of caplets or swaptions.

#### LIBOR Market Models as Markov-functional Models

Let us conclude this section with a comment how LMMs can be seen as MFMs. In particular, let us consider an LMM under the terminal measure consisting of forward LIBORs $L^1, \ldots, L^n$ given by the System of SDEs (2.54).

[4] In the context of the economy $\mathcal{E}$ we set $\mathcal{T} = \{T_1, \ldots, T_{n+1}\}$.

In order to identify, the LMM as the MFM we then have to find a driving process $x$ and show that the prices of ZCBs can be expressed in terms of the driving process. Recall that, in the LMM the prices of ZCBs were uniquely defined only on dates $T_1, \ldots, T_{n+1}$. As a consequence, the embedding of an LMM into an MFM is not unique, however any two embedding (under the same measure) will share a common joint distribution of ZCB prices on the dates $T_1, \ldots, T_{n+1}$. In that sense, we will embed the LMM into the MFM by specifying only the functional forms of ZCBs on dates $T_1, \ldots, T_{n+1}$ which is in practice enough for pricing of the derivatives.

Recall, that the prices of ZCBs on the dates $T_1, \ldots, T_{n+1}$ can be expressed in terms of forward LIBORs as in equation (2.58). We can then define the driving process to be the process $x = (L^i)_{i=1}^n$ (where we extend each $L^i$ to $[0, T_{n+1}]$ in the sense of Remark 2.19). Note, that $x$ is an $n$-dimensional Markov process under terminal measure and that the prices of ZCBs on dates $T_1, \ldots, T_{n+1}$ can be expressed as functions of the state of the driving process as in equation (2.58).

Note, that the resulting MFM is $n$-dimensional regardless of the dimension of the Brownian motion driving the dynamics of the LMM. The connection between MFM and LMM will be further explored in 4 and 6.

# Chapter 3

# Classification of Two- and Three-Factor Time-Homogeneous Separable LIBOR Market Models

The *LIBOR market models* (LMMs), introduced in Section 2.3 are one of the most popular classes of term structure models. One of the reasons for their popularity can be attributed to the flexibility of their parameterisations. However, this flexibility comes with a major drawback, the dimension of an LMM is equal to the number of forward rates in the model. This makes them particularly cumbersome to use for pricing of derivatives with early exercise features.

To overcome the issue of high-dimensionality Pietersz et al. (2004) proposed the *separability* constraint on the volatility structure of the LMM and proved that a separable LMM has an approximation with dimension equal to the number of Brownian motions driving the model dynamics. This process came with two drawbacks. Firstly, it greatly restricted the class of available parameterisations. In particular, it was noted in Joshi (2011) that the separability condition is too restrictive to use when the instantaneous volatilities are time-homogeneous. Secondly, the approximation obtained is not arbitrage-free and is only useful for time horizons up to 15 years.

In this chapter we mainly address the first issue. We show that the separability condition can be relaxed by allowing the components of the driving Brownian motions to be correlated. Under the relaxed separability condition we characterise two- and three-factor separable LMMs with time-homogeneous instantaneous volatilities and show that they are of practical interest.

We briefly comment on the second issue, namely that the approximation considered admits arbitrage, by pointing out the ideas presented in Bennett and Kennedy (2005). In particular, we note that by defining a suitable Markov-functional model one can retain the benefits of low-dimensionality while avoiding problems with arbitrage.

The remainder of the chapter is structured as follows. In Section 3.1 we introduce the LMM driven by correlated Brownian motions and introduce the single time step approximation. The separability condition is discussed and generalised in Section 3.2. In Section 3.3 we characterise the two- and three-factor separable LMM with time-homogeneous instantaneous volatilities. In Section 3.4 we discuss the models obtained from a practical point of view. Section 3.5 concludes the chapter.

## 3.1 LIBOR Market Model

The LIBOR market model was briefly introduced in Section 2.3. In particular, we have specified the LMM under the terminal measure in equation (2.54) and introduced the instantaneous volatility functions $\sigma^{inst,i}, i = 1, \ldots, n$, and instantaneous correlation function $\rho^{inst}_{i,j}, i, j = 1, \ldots, n$ in equations (2.59) and (2.60) respectively. Moreover, we noted that the instantaneous volatility functions and correlation functions uniquely define the model dynamics.

In particular, by specifying the instantaneous volatility functions one implicitly specifies the evolution of the term structure of volatilities over time via equation (2.62). In practice one often does not have a particular view on the dynamics of the term structure of volatilties and is faced with two natural choices. Either one chooses the implied volatilities to be constant functions of time (i.e. they only depend on the maturity $T_i$ of the caplet) or so that the implied volatilities are a function of the time to maturity (i.e. they depend on the difference $T_i - t$) (see Section 6.2 in Rebonato (2002)). In this chapter we will focus on the latter choice. It is easy to see that the implied volatilities of caplets will depend on the time to maturity if the instantaneous volatility functions satisfy the *time-homogeneity* condition

$$\sigma^{inst,i}(t) = \sigma^{inst}(T_i - t), \quad t \leq T_i,\ i = 1, \ldots, n, \tag{3.1}$$

where $\sigma^{inst} : [0, T_n] \to \mathbb{R}_+$ is some bounded measurable function. In particular, $\sigma^{inst}$ is often taken to be of the form

$$\sigma^{inst}(\tau) = \big(a + b\tau\big)\exp\big(-c\tau\big) + d, \tag{3.2}$$

where $\tau$ is the time to maturity. To ensure that equation (3.2) represents a valid parameterisation the parameters must be chosen such that $\sigma^{inst}$ is non-negative and bounded on $[0, T_n]$. This parameterisation was proposed by Rebonato (1999) and remains a popular choice amongst practitioners.

Let us now turn our attention back to the specification of the LMM. Recall that we assumed that the Brownian motion driving the model dynamics has independent components. While this assumption is in general non-restrictive, it turns out to be beneficial to relax it when there are additional constraints associated with functions $\sigma^i, i = 1, \ldots, n$, in equation (2.54).

Let $\rho : [0, T_n] \to [-1, 1]^{d\times d}$ be a measurable function, such that $\rho_t$ is a correlation matrix for $t \in [0, T_n]$. We will construct an LMM driven by a $d$-dimensional Brownian motion $W$, satisfying

$$dW_t dW_t^T = \rho(t)dt, \tag{3.3}$$

and bounded measurable volatility functions $\sigma^i : [0, T_i] \to \mathbb{R}^d, i = 1, , \dots, n$.

First observe, that since $\rho(t), t \leq T_n$, is a correlation matrix, in particular a positive semidefinite matrix, there exists a unique positive semidefinite matrix $R_t$, called the principal square root, such that $R_t R_t = \rho(t)$. Then we can define the matrix valued function $R : t \mapsto R_t$, moreover one can show that taking the principal square root is a continuous operation and therefore $R_t$ is a measurable function. Now let $\tilde{W}$ be a $d$-dimensional standard Brownian motion. Then we can define a process $W = (W_t)_{t \in [0,T_n]}$ by

$$W_t := \int_0^t R_t d\tilde{W}_t \tag{3.4}$$

note that $W$ satisfies

$$dW_t dW_t^T = (R_t d\tilde{W}_t)(R_t d\tilde{W}_t)^T = R_t (d\tilde{W}_t d\tilde{W}_t^T) R_t = R_t^2 dt = \rho(t) dt. \tag{3.5}$$

Therefore, the process $W$ as defined in equation (3.4) is a Brownian motion with the correlation structure we desired. Next we define functions $\tilde{\sigma}^i : [0, T_i] \to \mathbb{R}^d, i = 1, \dots, n$, by

$$\tilde{\sigma}^i(t) := R_t \sigma^i(t). \tag{3.6}$$

It is easy to see that $\tilde{\sigma}^i, i = 1, \dots, n$ are bounded measurable functions. Then we can define an LMM driven by $\tilde{W}$ and volatility functions $\tilde{\sigma}^i, i = 1, \dots, n$ as in equation (2.54), i.e. for $i \in \{1, \dots, n\}$

$$dL_t^i = L_t^i \langle \tilde{\sigma}^i(t), d\tilde{W}_t \rangle - L_t^i \sum_{j=i+1}^n \frac{\alpha_j L_t^j \langle \tilde{\sigma}^i(t), \tilde{\sigma}^j(t) \rangle}{1 + \alpha_j L_t^j} dt, \quad t \leq T_i. \tag{3.7}$$

Note that

$$\langle \tilde{\sigma}^i(t), d\tilde{W}_t \rangle = \langle R_t \sigma^i(t), d\tilde{W}_t \rangle = \langle \sigma^i(t), R_t d\tilde{W}_t \rangle = \langle \sigma^i(t), dW_t \rangle \tag{3.8}$$

and

$$\langle \tilde{\sigma}^i(t), \tilde{\sigma}^j(t) \rangle = \langle R_t \sigma^i(t), R_t \sigma^j(t) \rangle = \langle \sigma^i(t), R_t^2 \sigma^j(t) \rangle = \langle \sigma^i(t), \rho(t) \sigma^j(t) \rangle, \tag{3.9}$$

then we can rewrite (3.7) for $i = 1, \dots, n$, as

$$dL_t^i = L_t^i \langle \sigma^i(t), dW_t \rangle - L_t^i \sum_{j=i+1}^n \frac{\alpha_j L_t^j \langle \sigma^i(t), \rho(t) \sigma^j(t) \rangle}{1 + \alpha_j L_t^j} dt, \quad t \leq T_i. \tag{3.10}$$

We will refer to the collection of functions $\{\sigma^i\}_{i=1}^n$ as the volatility structure and will say that an LMM is parametrised by the pair $(\{\sigma^i\}_{i=1}^n, \rho)$. We can express the instantaneous volatility and correlation functions in terms of functions $\sigma^1, \dots, \sigma^n$ and $\rho$ as

$$\sigma^{inst,i}(t) = \sqrt{\langle \sigma^i(t), \rho(t) \sigma^i(t) \rangle}, \quad t \leq T_i, \ i = 1, \dots, n, \tag{3.11}$$

and

$$\rho_{i,j}^{inst}(t) = \frac{\langle \sigma^i(t), \rho(t)\sigma^j(t)\rangle}{\sigma^{inst,i}(t)\sigma^{inst,j}(t)}, \quad t \le T_i \wedge T_j,\ i,j = 1,\ldots,n. \tag{3.12}$$

**Remark 3.1.** *Note that we allowed $\rho$ to be of any rank. In particular, if $\rho(t)$ is of rank $d' < d$ for $t \le T_n$, we get a $d$-factor parameterisation of a $d'$ factor LMM. This may seem suboptimal for implementation purposes, however as we will later observe this is not necessarily the case.*

Let us conclude this section by briefly discussing the implementation of the LMM. We have noted in Section 2.3 that one of the biggest challenges when implementing the LMM comes from the state dependent drifts occurring in the SDEs for the forward LIBORs (see equations (2.54) and (3.10)). In particular this ensures that the LMM is Markovian in dimension $n$ regardless of the dimension of the Brownian motion driving the dynamics. Furthermore, there are no closed form solutions for the joint distribution of the LIBORs at any date $t > 0$. Therefore, in order to implement the LMM it is necessary to use a suitable approximation. This is usually done in the log-space since

$$d\log L_t^i = \langle \sigma^i(t), dW_t\rangle - \left(\frac{1}{2}\sigma^{inst,i}(t)^2 + \sum_{j=i+1}^{n} \frac{\alpha_j L_t^j \langle \sigma^i(t), \rho(t)\sigma^j(t)\rangle}{1+\alpha_j L_t^j}\right)dt \tag{3.13}$$

and the distribution of $\int_{t_1}^{t_2} \langle \sigma^i(t), dW_t\rangle$ is known explicitly.

In this chapter we will focus on an approximation in which the forward LIBORs are evolved from time 0 to time $t$ in a single time-step. An early description of this method can be found in Hunter et al. (2001), however we will closely follow the approach and notation in Pietersz et al. (2004). Let us denote by $Z$ a vector valued process, where the $i$th component, $i = 1,\ldots,n$, $Z_i$ is given by

$$Z_i(t) := \int_0^t \langle \sigma^i(t), dW_t\rangle, \quad t \le T_i. \tag{3.14}$$

We say that $(L^{A,i})_{i=1}^n$ is a *single time-step approximation* of $(L^i)_{i=1}^n$ if

$$\log L_t^{A,i} = \log L_0^i + Z_i(t) + \mu^i(t, Z(t)), \quad t \le T_i, i = 1,\ldots,n, \tag{3.15}$$

where $\mu^i$ is defined by the drift approximation used (e.g. Euler, Brownian bridge, see Joshi and Stacey (2008)). Note that the drift approximation implicitly depends on the initial term structure. Furthermore, observe that the process $Z$ is in general an $n$-dimensional Markov process.

**Remark 3.2.** *Observe that the process $Z_i$ is only well defined for $t \le T_i$, hence the drift approximation $\mu^j$ at time $t \le T_j$ may only depend on the $i$th component of vector $Z$ if $t \le T_i$. However, this does not cause problems since the drift part of $\log L^j$ only depends on the state of the $L^{j+1},\ldots,L^n$.*

**Remark 3.3.** *Instead of approximating the LMM under the terminal measure, we could have used any $T_i$ forward measure or the spot measure.*

The single-time step approximation is a powerful computational tool, however it does come with one major drawback. Like most approximations of the LMM it is not arbitrage-free. In particular, the quality of approximation decreases with time and care must be taken when using it over long time horizons. This is typically less of a problem for the schemes that use many time steps to evolve the forward LIBORs in time. Nevertheless, the single time-step approximation is a useful method for short- and medium-term time horizons and its true power will be demonstrated in Section 3.2.

**Remark 3.4.** *In this thesis we do not explore the accuracy of the single time-step approximation. For separable LMMs (see the next section) this has been studied Pietersz et al. (2004) and Ng (2009). A comparison of drift approximations can be found in Joshi and Stacey (2008).*

## 3.2 Separability

We have noted in the previous section that a $d$-factor LMM is an $n$-dimensional model. Therefore, one typically needs to implement it by using Monte Carlo methods, which are particularly cumbersome to use when pricing derivatives with early exercise features such as Bermudan swaptions. However, it was first shown by Pietersz et al. (2004) that this single-time step approximation of a $d$-factor LMM can be expressed as a function of some $d$-dimensional Markov process if we impose the *separability* condition on the functions $\sigma^i, i = 1, \ldots, n$.

**Definition 3.5.** *A volatility structure $\{\sigma^i : [0, T_i] \to \mathbb{R}^d\}_{i=1}^n$ is separable if there exist a function $\sigma : [0, T_n] \to \mathbb{R}^d$ and vectors $v^1, \ldots, v^n \in \mathbb{R}^d$ such that*

$$\sigma^i(t) = v^i * \sigma(t), \quad t \leq T_i,\ i = 1, \ldots, n, \tag{3.16}$$

*where the operator $*$ denotes entry-by-entry multiplication of vectors.*

*We say that a d-factor LMM is separable if it can be parametrised by $(\{\sigma^i\}_{i=1}^n, \rho)$ where the volatility structure $\{\sigma^i\}_{i=1}^n$ is separable.*

Definition 3.5 is a slight generalisation of the one given by Pietersz et al. (2004), in particular Pietersz et al. (2004) only considered the separable LMMs driven by independent Brownian motions. Next we extend the result by Pietersz et al. (2004) and prove that any single time-step approximation of a $d$-factor separable LMM can be expressed as a function of some $d$-dimensional Markov process for a general correlation structure of the driving Brownian motion $W$.

**Proposition 3.6.** *Suppose that forward LIBORs $(L^i)_{i=1}^n$ are given by a d-factor separable LMM and let $(L^{A,i})_{i=1}^n$ be a single-time step approximation to $(L^i)_{i=1}^n$ of the form as in equation* (3.15). *Then there exists a d-dimensional Markov process and functions $f^i$ :*

$[0, T_i] \times \mathbb{R}^d \to \mathbb{R}^+, i = 1, \ldots, n,$ *such that*

$$L_t^{A,i} = f^i(t, x_t), t \leq T_i, \ i = 1, \ldots, n. \tag{3.17}$$

*Proof.* Since $(L^i)_{i=1}^n$ are given by a separable $d$-factor LMM, there exists a parameterisation $(\{\sigma^i\}_{i=1}^n, \rho)$ such that the volatility structure $\{\sigma^i\}_{i=1}^n$ is separable, i.e. there exists function $\sigma : [0, T_n] \to \mathbb{R}^d$ and vectors $v^1, \ldots, v^n \in \mathbb{R}^d$ satisfying equation (3.16).

Let $W$ be the $d$-dimensional Brownian motion, such that $dW_t dW_t^T = \rho(t)$, driving the dynamics of the LMM (under the terminal measure) and define the vector valued process $Z = (Z_i)_{i=1}^n$ as in equation (3.14). Now define a $d$-dimensional Markov process $x = (x_t)_{t \in [0, T_n]}$ by

$$x_t := \int_0^t \sigma(s) * dW_s, \quad t, \leq T_n, \tag{3.18}$$

and observe that

$$Z_i(t) = \langle v^i, x_t \rangle, \quad t \leq T_i, \ i = 1, \ldots, n. \tag{3.19}$$

In particular, $Z(t) = v x_t$, where $v = [v^1, \ldots, v^n]^T$ is a $d \times n$ matrix. Then any single time-step approximation $(L^{A,i})_{i=1}^n$ of $(L^i)_{i=1}^n$ is of the form

$$\log L^{A,i} = \log L_0^i + \langle v^i, x_t \rangle + \mu^i(t, v x_t), \quad t \leq T_i, \ i = 1, \ldots, n, \tag{3.20}$$

where $\mu^i$ depends on the drift approximation used. In particular there exist functions $f^i : [0, T_i] \times \mathbb{R}^d \to \mathbb{R}^+, i = 1, \ldots, n$, such that

$$L_t^{A,i} = f^i(t, x_t), t \leq T_i, \ i = 1, \ldots, n. \tag{3.21}$$

□

Proposition 3.6 is in fact independent of the equivalent martingale measure used to specify the model and the single time-step approximation. It was originally argued by Pietersz et al. (2004) that if one is to implement the single time-step approximation on a grid the terminal measure needs to be used to avoid the path-dependence of the numeraire. However, one can easily implement the single time-step approximation under the spot measure by using the same ideas as in the implementation of the Markov-functional model under the spot measure (Fries and Rott, 2004) (see also Section 4.3).

**Remark 3.7.** *Joshi (2011) provides an alternative formulation of the separability condition which he refers to as 'matrix separability'. This is briefly discussed in Section 12.8 in Joshi (2011) and can be shown to be equivalent to the Definition 3.5. The formulation presented above is more natural for the problem we consider in the next section when we consider the time-homogeneous separable LMMs.*

Since a single time-step approximation of a separable LMM can significantly reduce the computational effort needed for valuation of callable derivatives it is a natural question to ask how flexible are the separable LMMs. We will address this question in Section 3.3.

## 3.3 Time-Homogeneous and Separable LIBOR Market Model

We have pointed out in Section 3.1 that time-homogeneity of instantaneous volatilities is usually a desirable property of an LMM. In this section we will be interested which time-homogeneous instantaneous volatility functions can be obtained in a $d$-factor LMM when we also impose the separability condition on the volatility structure. In particular we will be interested in solutions of the system of functional equations

$$\sigma^{inst}(T_i - t)^2 = \big\langle v_i * \sigma(t), \rho_t(v_i * \sigma(t))\big\rangle, \quad t \le T_i,\ i = 1, \dots, n. \tag{3.22}$$

Note that (3.22) implicitly depends on the choice of the reset dates $T_1, \dots, T_n$. It is therefore reasonable to search only for the solutions that continuously depend on the reset dates. This can be simply achieved by searching for the solutions of the functional equation

$$\sigma^{inst}(T - t)^2 = \big\langle v(T) * \sigma(t), \rho_t(v(T) * \sigma(t))\big\rangle, \quad t \le T. \tag{3.23}$$

where we require $v : [0, \infty) \to \mathbb{R}^d$ to be a continuous function.

We will first consider one-factor volatility structures, that is $d = 1$. This problem has already been examined in Joshi (2011), however it is instructive to study it first as it points out some of the important aspects of the problem that will be encountered later. In the one-factor case equation (3.23) can be rewritten as

$$\sigma^{inst}(T - t)^2 = v(T)^2\sigma(t)^2, t \le T. \tag{3.24}$$

Note that if $\sigma^{inst}(x) = 0$ for some $x \ge 0$, then $\sigma^{inst} \equiv 0$ and either $v \equiv 0$ or $\sigma \equiv 0$ (or both). Clearly, such solution is not of our interest, we can therefore assume without loss of generality that $\sigma^{inst}$ is a strictly positive function.

Next we define functions $f, g, h$, by $f(x) := \sigma^{inst}(x)^2$, $g(y) := \sigma(-y)^2$, and $h(x) := v(x)^2$, where $x \ge 0$ and $y \le 0$. Then we can rewrite equation (3.24) as

$$f(x + y) = h(x)g(y), \quad x \ge 0, -x \le y \le 0. \tag{3.25}$$

Equation (3.25) is commonly known as Pexider equation. It can be shown that under the assumption that $f$ is a continuous function[1] the general solution to the Pexider equation is of the form $f(x) = ab\exp(cx)$, $g(y) = a\exp(cy)$ and $h(x) = b\exp(cx)$, where $a, b, c \in \mathbb{R}$ (see Section 3.1 in Aczél (1966)).

Note that $f, g, h$ are non-negative functions, therefore we are only interested in positive

[1] In fact it is enough to assume that $f$ is continuous at a single point.

solutions to the Pexider equation and we need to restrict the parameters to $a, b > 0$. Furthermore, each solution to equation (3.25) can be mapped to four solutions of equation (3.24):

1. $\sigma(t) = \sqrt{a}\exp(-\frac{1}{2}ct)$ and $v(T) = \sqrt{b}\exp(\frac{1}{2}cT)$;
2. $\sigma(t) = -\sqrt{a}\exp(-\frac{1}{2}ct)$ and $v(T) = \sqrt{b}\exp(\frac{1}{2}cT)$;
3. $\sigma(t) = \sqrt{a}\exp(-\frac{1}{2}ct)$ and $v(T) = -\sqrt{b}\exp(\frac{1}{2}cT)$;
4. $\sigma(t) = -\sqrt{a}\exp(-\frac{1}{2}ct)$ and $v(T) = -\sqrt{b}\exp(\frac{1}{2}cT)$.

However for all cases $\sigma^{inst}(T-t) = \sqrt{ab}\exp(\frac{1}{2}c(T-t))$. Now recall that $\sigma$ and $v$ affect the dynamics of the LMM through their product. Furthermore, the sign of the product $v(T)\sigma(t)$ can be absorbed into the Brownian motion driving the dynamics. Therefore, all four solutions lead to the same LMM and we can without loss of generality assume that one of the parameters $a$ and $b$ is equal to one.

Therefore a one-factor time-homogeneous and separable LMM can be parametrised as

$$\sigma(t) = \alpha\exp(\beta t), \tag{3.26}$$

$$v(T) = \exp(-\beta T), \tag{3.27}$$

$$\sigma^{inst} = \alpha\exp\big(-\beta(T-t)\big), \tag{3.28}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. However, only the case when $\beta \geq 0$ is of practical interest.

As mentioned earlier the one-factor time-homogeneous separable LMMs was already characterised in Joshi (2011). Neverthless, there are two important observations we can make from our thought process. Firstly, although we imposed the continuity condition on function $f$ this turned out not to be a restriction since any solution to the Pexider equation is either smooth or nowhere-continuous. Secondly, any solution to the Pexider equation corresponded to four solutions to equation (3.24) which all lead to the same dynamics of the LMM. We will see that above observations also hold in a $d$-factor setting where equation (3.23) can be transformed to a Levi-Civitá equation

$$f(x+y) = \sum_{i=1}^{k} g_i(x)h_i(y), \tag{3.29}$$

where $k = \frac{1}{2}d(d+1)$.

It can be shown that if $f, g_i, h_i, i = 1, \ldots, k$ is a continuous solution to equation (3.29) then $f, g_i, h_i \in \mathcal{C}^\infty$ and $f$ is of the form

$$f(x) = \sum_{i} P_i(x)\exp(\lambda_i x), \tag{3.30}$$

where $P_i$ is a polynomial of degree $k_i - 1$, such that $\sum_i k_i = k$, and $\lambda_i \in \mathbb{C}$ (See Section 4.2 in Aczél (1966)).

### 3.3.1 The Two-Factor Case

In the two-factor case equation (3.23) can be rewritten as

$$\begin{aligned}\sigma^{inst}(T-t)^2 = v_1(T)^2\sigma_1(t)^2 + v_2(T)^2\sigma_2(t)^2 \\ + 2v_1(T)v_2(T)\rho_{1,2}(t)\sigma_1(t)\sigma_2(t).\end{aligned} \tag{3.31}$$

To simplify the analysis of equation (3.31) we introduce functions

$$f(x) = \sigma^{inst}(x)^2, \tag{3.32}$$

$$g_i(x) = \sigma_i(x)^2, \quad i = 1, 2, \tag{3.33}$$

$$g_3(x) = 2\rho_{1,2}(x)\sigma_1(x)\sigma_2(x), \tag{3.34}$$

$$h_i(x) = v_i(x)^2, \quad i = 1, 2, \tag{3.35}$$

$$h_3(x) = v_1(x)v_2(x). \tag{3.36}$$

We can then rewrite equation (3.31) as

$$f(T-t) = \sum_{i=1}^{3} g_i(t)h_i(T). \tag{3.37}$$

Note that equation (3.37) can be easily transformed to the form of equation (3.29) by the following change of coordinates

$$(x(T,t), y(T,t)) = (T, -t). \tag{3.38}$$

Therefore, if we assume that $f, g_i, h_i$ are continuous functions, $f$ is of the form as in equation (3.30).

**Theorem 3.8.** *Let $v, \sigma : \mathbb{R}_+ \to \mathbb{R}^2$ and $\rho_{1,2} : \mathbb{R}_+ \to [-1, 1]$ be continuous functions such that equation (3.31) holds for some function $\sigma^{inst} : \mathbb{R}_+ \to \mathbb{R}_+$.*

*Then $v, \sigma$ and $\rho_{1,2}$ are parametrised up to the uniqueness of $\sigma^{inst}$ by one of the following parameterisations*

*2.1.* $\alpha_1, \alpha_2 \geq 0, \beta_1,\ \beta_2 \in \mathbb{R}$ *and* $\gamma \in [-1, 1]$

$$v(T) = \begin{bmatrix}\exp(-\beta_1 T) \\ \exp(-\beta_2 T)\end{bmatrix}, \tag{3.39}$$

$$\sigma(t) = \begin{bmatrix}\alpha_1 \exp(\beta_1 t) \\ \alpha_2 \exp(\beta_2 t)\end{bmatrix}, \tag{3.40}$$

$$\rho_{1,2}(t) = \gamma; \tag{3.41}$$

*2.2.* $\alpha > 0, \beta \in \mathbb{R}, \gamma \geq 0$ *and* $\lambda \in \mathbb{R}$

$$v(T) = \begin{bmatrix} T\exp(-\lambda T) \\ \exp(-\lambda T) \end{bmatrix}, \tag{3.42}$$

$$\sigma(t) = \begin{bmatrix} \alpha\exp(\lambda t) \\ \alpha\sqrt{(t+\beta)^2+\gamma}\exp(\lambda t) \end{bmatrix}, \tag{3.43}$$

$$\rho_{1,2}(t) = -\frac{t+\beta}{\sqrt{(t+\beta)^2+\gamma}}; \tag{3.44}$$

*2.3.* $\alpha, \beta, \theta, \lambda \in \mathbb{R}$*,* $\gamma \geq \sqrt{\alpha^2+\beta^2}$

$$v(T) = \begin{bmatrix} \operatorname{sgn}\left(\cos\frac{\theta T}{2} + \sin\frac{\theta T}{2}\right)\sqrt{1+\sin(\theta T)}\exp(-\lambda T) \\ \operatorname{sgn}\left(\cos\frac{\theta T}{2} - \sin\frac{\theta T}{2}\right)\sqrt{1-\sin(\theta T)}\exp(-\lambda T) \end{bmatrix}, \tag{3.45}$$

*(a) If* $\alpha^2+\beta^2 > \gamma^2$

$$\sigma(t) = \begin{bmatrix} \sqrt{\gamma+\alpha\cos(\theta t)+\beta\sin(\theta t)}\exp(\lambda t) \\ \sqrt{\gamma-\alpha\cos(\theta t)-\beta\sin(\theta t)}\exp(\lambda t) \end{bmatrix}, \tag{3.46}$$

$$\rho_{1,2} = \frac{\beta\cos(\theta t)-\alpha\sin(\theta t)}{\sqrt{\gamma^2-(\alpha\cos(\theta t)+\beta\sin(\theta t))^2}}; \tag{3.47}$$

*(b) If* $\alpha^2+\beta^2 = \gamma^2$

$$\sigma(t) = \begin{bmatrix} \operatorname{sgn}\left(\cos\frac{\theta t-\phi}{2}\right)\sqrt{\alpha\cos(\theta t)+\beta\sin(\theta t)+\gamma}\exp(\lambda t) \\ -\operatorname{sgn}\left(\sin\frac{\theta t-\phi}{2}\right)\sqrt{-\alpha\cos(\theta t)-\beta\sin(\theta t)+\gamma}\exp(\lambda t) \end{bmatrix}, \tag{3.48}$$

$$\rho_{1,2} = 1, \tag{3.49}$$

*where*

$$\phi = \begin{cases} \arccos\frac{\alpha}{\gamma^2}; & \beta \geq 0 \\ -\arccos\frac{\alpha}{\gamma^2}; & \beta < 0 \end{cases}. \tag{3.50}$$

The proof can be found in Appendix B.

We analyse Parameterisations 2.1 and 2.2 in Section 3.4. Parameterisation 2.3 is not of practical interest and will not be analysed further. Let us mention that none of them can capture the 'hump' and the long term level of instantaneous volatility simultaneously. For this reason we next consider the three-factor case.

### 3.3.2 The Three-Factor Case

In the three-factor case equation (3.23) can be rewritten to

$$\begin{aligned}\sigma^{inst}(T-t)^2 = {} & v_1(T)^2\sigma_1(t)^2 + v_2(T)^2\sigma_2(t)^2 + v_3(T)^2\sigma_3(t)^2 \\ & + 2v_1(T)v_2(T)\rho_{1,2}(t)\sigma_1(t)\sigma_2(t) \\ & + 2v_1(T)v_3(T)\rho_{1,3}(t)\sigma_1(t)\sigma_3(t) \\ & + 2v_2(T)v_3(T)\rho_{2,3}(t)\sigma_2(t)\sigma_3(t).\end{aligned} \tag{3.51}$$

We can now proceed similarly as in the two-factor case and we define functions $f, g_i, h_i, i = 1, \ldots, 6$ by

$$f(x) = \sigma^{inst}(x)^2, \tag{3.52}$$

$$g_i(x) = \sigma_i(x)^2, \quad i = 1, 2, 3 \tag{3.53}$$

$$g_4(x) = 2\rho_{1,2}(x)\sigma_1(x)\sigma_2(x), \tag{3.54}$$

$$g_5(x) = 2\rho_{1,3}(x)\sigma_1(x)\sigma_3(x), \tag{3.55}$$

$$g_6(x) = 2\rho_{2,3}(x)\sigma_2(x)\sigma_3(x), \tag{3.56}$$

$$h_i(x) = v_i(x)^2, \quad i = 1, 2, 3, \tag{3.57}$$

$$h_4(x) = v_1(x)v_2(x), \tag{3.58}$$

$$h_5(x) = v_1(x)v_3(x), \tag{3.59}$$

$$h_6(x) = v_2(x)v_3(x). \tag{3.60}$$

We can then rewrite equation (3.51) to

$$f(T-t) = \sum_{i=1}^{6} g_i(t)h_i(T). \tag{3.61}$$

Again we obtain an equation that can be can be easily transformed to equation (3.29) by the change of coordinates $(x(T,t), y(T,t)) = (T, -t)$. If we assume that $\sigma, v$ and $\rho$ are continuous functions then so are $g_i, h_i, i = 1, \ldots, 6$, and function $f$ has to be of the form as in equation (3.30). In the three-factor case we will only be interested in solutions where the coefficients $\lambda_i$ in equation (3.30) are real numbers.

**Theorem 3.9.** *Let* $\sigma^{inst} : \mathbb{R}_+ \to \mathbb{R}_+,\ v, \sigma : \mathbb{R}_+ \to \mathbb{R}^2$ *and* $\rho_{1,2}, \rho_{1,3}, \rho_{2,3} : \mathbb{R}_+ \to [-1, 1]$ *be continuous functions. Furthermore, assume that matrix*

$$\rho(t) = \begin{bmatrix} 1 & \rho_{1,2}(t) & \rho_{1,3}(t) \\ \rho_{1,2}(t) & 1 & \rho_{2,3}(t) \\ \rho_{1,3}(t) & \rho_{2,3}(t) & 1 \end{bmatrix} \tag{3.62}$$

*is a correlation matrix for* $t \geq 0$*.*

*Then the following parameterisations are solutions to equation* (3.51)*:*

*3.1.* $\alpha_1, \alpha_2, \alpha_3 \geq 0, \beta_1,\ \beta_2, \beta_3 \in \mathbb{R}$ *and* $\gamma \in [-1,1]^{3\times 3}$ *a correlation matrix*

$$v(T) = \begin{bmatrix} \exp(-\beta_1 T) \\ \exp(-\beta_2 T) \\ \exp(-\beta_3 T) \end{bmatrix}, \tag{3.63}$$

$$\sigma(t) = \begin{bmatrix} \alpha_1 \exp(\beta_1 t) \\ \alpha_3 \exp(\beta_2 t) \\ \alpha_2 \exp(\beta_3 t) \end{bmatrix}, \tag{3.64}$$

$$\rho(t) = \gamma; \tag{3.65}$$

*3.2.* $\alpha, \beta, \gamma, \delta, \zeta \geq 0, \lambda, \mu \in \mathbb{R}, \eta \in [-1,1]$ *and* $\varepsilon \in [\delta - \sqrt{\beta\eta^{-2} - \beta}, \delta - \sqrt{\beta\eta^{-2} - \beta}]$

$$\begin{aligned} \sigma^{inst}(T-t)^2 = {} & \alpha^2(\beta + (T - t + \gamma - \delta)^2) \exp(-2\lambda(T-t)) \\ & + 2\alpha\zeta\eta(T - t + \gamma - \varepsilon) \exp(-(\lambda + \mu)(T-t)) \\ & + \zeta^2 \exp(-2\mu(T-t)), \end{aligned} \tag{3.66}$$

$$v(T) = \begin{bmatrix} (T+\gamma)\exp(-\lambda T) \\ \exp(-\lambda T) \\ \exp(-\mu T) \end{bmatrix}, \tag{3.67}$$

$$\sigma(t) = \begin{bmatrix} \alpha \exp(\lambda t) \\ \alpha\sqrt{\beta + (t+\delta)^2} \exp(\lambda t) \\ \zeta \exp(\mu t) \end{bmatrix} \tag{3.68}$$

$$\tag{3.69}$$

*and* $\rho$ *defined by*

$$\rho_{1,2}(t) = -\frac{t+\delta}{\sqrt{\beta + (t+\delta)^2}}, \tag{3.70}$$

$$\rho_{2,3}(t) = \eta, \tag{3.71}$$

$$\rho_{1,3}(t) = -\eta\frac{t+\varepsilon}{\sqrt{\beta + (t+\delta)^2}}. \tag{3.72}$$

*3.3.* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \geq 0, \lambda \in \mathbb{R}$

$$\begin{aligned} \sigma^{inst}(T-t)^2 = {} & \Big(\zeta^2(T - t + \gamma - \delta) + \beta\zeta^2(T - t + \gamma - \varepsilon) \\ & + \alpha\zeta^2\Big) \exp(-\lambda(T-t)) \end{aligned} \tag{3.73}$$

$$v(T) = \begin{bmatrix} (T+\gamma)^2\exp(-\lambda T) \\ (T+\gamma)\exp(-\lambda T) \\ \exp(-\lambda T) \end{bmatrix}, \tag{3.74}$$

$$\sigma(t) = \begin{bmatrix} \zeta \exp(\lambda t) \\ \zeta\sqrt{\beta + 4(t+\delta)^2}\exp(\lambda t) \\ \zeta\sqrt{\alpha + \beta(t+\varepsilon)^2 + (t+\delta^4)}\exp(\lambda t) \end{bmatrix} \tag{3.75}$$

$$\tag{3.76}$$

*and $\rho$ defined by*

$$\rho_{1,2}(t) = -\frac{2(t+\delta)}{\sqrt{4(t+\delta)^2+\beta}}, \tag{3.77}$$

$$\rho_{1,3}(t) = \frac{(t+\delta)^2}{\sqrt{(t+\delta)^4+\beta(t+\varepsilon)^2+\alpha}}, \tag{3.78}$$

$$\rho_{2,3}(t) = -\frac{2(t+\delta)^2+\beta(t+\varepsilon)}{\sqrt{(4(t+\delta)^2+\beta)((t+\delta)^4+\beta(t+\varepsilon)^2+\alpha)}}. \tag{3.79}$$

The proof of Theorem 3.9, can be simply done by verifying that parameterisations presented are valid ($\rho(t)$ needs to be a correlation matrix) and satisfy the time-homogeneity condition.

**Remark 3.10.** *Theorem 3.9 does not cover all the parameterisations for which $\lambda_i$'s in* (3.30) *are real. In particular, there may be a more general solution in the case of Parameterisation 3.3 However, one can show that the theorem covers all the cases where* $\lim_{x\to\infty}\sigma^{inst}(x) > 0$ *or* $(\sigma^{inst})^2$ *is a weighted sum of exponential functions.*

## 3.4 Analysis

Recall that a separable LMM is given by vectors $v_1, \dots, v_n$, a vector valued function $\sigma$ and a matrix valued function $\rho$, such that $\rho(t), t \leq T_n$, is a correlation matrix. However, to analyse the dynamics of an LMM it is more intuitive to think in terms of the instantaneous volatility and correlation functions. For a separable LMM these can be expressed in terms vectors of $v_i, i = 1, \dots, n$, $\sigma$ and functions $\rho$ by combining equations (3.11), (3.12) and (3.16) as

$$\sigma^{inst,i}(t) = \sqrt{\langle v_i * \sigma(t), \rho(t)(v_i * \sigma(t))\rangle}, \tag{3.80}$$

$$\rho^{inst}_{i,j}(t) = \frac{\langle v_i * \sigma(t), \rho(t)(v_j * \sigma(t))\rangle}{\sigma^{inst,i}(t)\sigma^{inst,i}(t)}. \tag{3.81}$$

Recall that we have imposed the time-homogeneity condition on the instantaneous volatility functions explicitly in Theorems 3.8 and 3.9. However it turns out that in the resulting instantaneous correlation functions $\rho^{inst}_{i,j}, i, j = 1, \dots, n$, also depend on the maturities $T_i, T_j$ and the calendar time $t$ only through the times to maturity $T_i - t$ and $T_j - t$. Moreover, parameterisations obtained in Theorems 3.8 and 3.9 are independent of the choice of the setting dates $T_1, \dots, T_n$. Therefore we can think of instantaneous volatilities and correlations for the purposes of this section as functions $\sigma^{inst} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\rho^{inst} : \mathbb{R}^2_+ \to [-1, 1]$, whose arguments are times to maturity.

In the two-factor model we get the following parameterisations of the instantaneous volatility and correlation:

2.1. $\alpha_1, \alpha_2 \geq 0$, $\beta_1, \beta_2 \in \mathbb{R}$ and $\gamma \in [-1, 1]$

$$\begin{aligned}\sigma^{inst}(x)^2 &= \alpha_1^2 \exp(-2\beta_1 x) + \alpha_2^2 \exp(-2\beta_2 x) \\ &\quad + 2\alpha_1\alpha_2\gamma \exp(-(\beta_1 + \beta_2)x),\end{aligned} \tag{3.82}$$

$$\begin{aligned}\rho^{inst}(x_1, x_2) = \Big(&\alpha_1^2 \exp(-\beta_1(x_1 + x_2)) + \alpha_2^2 \exp(-\beta_2(x_1 - x_2)) \\ &+ \alpha_1\alpha_2\gamma \exp(-\beta_1 x_1 - \beta_2 x_2) \\ &+ \alpha_1\alpha_2\gamma \exp(-\beta_2 x_1 - \beta_1 x_2)\Big) \\ &\Big/ \big(\sigma^{inst}(x_1)\sigma^{inst}(x_2)\big);\end{aligned} \tag{3.83}$$

2.2. $\alpha > 0$, $\beta, \lambda \in \mathbb{R}$ and $\gamma \geq 0$

$$\sigma^{inst}(x)^2 = \alpha^2((x - \beta)^2 + \gamma) \exp(-2\lambda x), \tag{3.84}$$

$$\rho^{inst}(x_1, x_2) = \frac{(x_1 - \beta)(x_2 - \beta) + \gamma}{\sqrt{((x_1 - \beta)^2 + \gamma)((x_2 - \beta)^2 + \gamma)}}. \tag{3.85}$$

In the three-factor case we get the following parameterisations:

3.1. $\alpha_1, \alpha_2, \alpha_3 \geq 0$, $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and $\gamma_{1,2}\gamma_{1,3}, \gamma_{2,3} \in [-1, 1]$

$$\sigma^{inst}(x)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_i\alpha_j\gamma_{i,j} \exp(-(\beta_i + \beta_j)x), \tag{3.86}$$

$$\rho^{inst}(x_1, x_2) = \frac{\sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_i\alpha_j\gamma_{i,j} \exp(-\beta_i x_1 - \beta_j x_2)}{\sigma^{inst}(x_1)\sigma^{inst}(x_2)}, \tag{3.87}$$

where $\gamma_{i,j} := \gamma_{j,i}$ and $\gamma_{i,i} := 1$ and $\Gamma = (\gamma_{i,j})_{i,j=1}^3$ is a correlation matrix;

3.2. $\alpha > 0$, $\beta, \varepsilon, \lambda, \mu \in \mathbb{R}$, $\gamma, \delta \geq 0$ and $\eta \in [-1, 1]$

$$\begin{aligned}\sigma^{inst}(x)^2 &= \alpha^2((x - \beta)^2 + \gamma) \exp(-2\lambda x) \\ &\quad + 2\alpha\delta\eta(x - \varepsilon) \exp(-(\lambda + \mu)x) \\ &\quad + \delta^2 \exp(-2\mu x)\end{aligned} \tag{3.88}$$

$$\begin{aligned}\rho^{inst}(x_1, x_2) = \Big(&\alpha^2\big((x_1 - \beta)(x_2 - \beta) + \gamma\big) \exp\big(-2\lambda(x_1 + x_2)\big) \\ &+ \alpha\delta\eta(x_1 - \varepsilon) \exp(-\lambda x_1 - \mu x_2) \\ &+ \alpha\delta\eta(x_2 - \varepsilon) \exp(-\lambda x_2 - \mu x_1) \\ &+ \delta^2 \exp\big(-2\mu(x_1 + x_2)\big)\Big) \\ &\Big/ \big(\sigma^{inst}(x_1)\sigma^{inst}(x_2)\big);\end{aligned} \tag{3.89}$$

3.3. $\alpha, \gamma, \varepsilon > 0,\ \beta, \delta, \lambda \in \mathbb{R}$

$$\sigma^{inst}(x)^2 = \alpha^2\big((x-\beta)^4 + \gamma(x-\delta)^2 + \varepsilon\big)\exp(-2\lambda x) \tag{3.90}$$

$$\begin{aligned}\rho^{inst}(x_1, x_2) = &\Big((x_1-\beta)^2(x_2-\beta)^2 + \gamma(x_1-\delta)(x_2-\delta) + \varepsilon\Big)\\ &\Big/\Big(\sqrt{(x_1-\beta)^4 + \gamma(x_1-\delta)^2 + \varepsilon}\\ &\cdot\sqrt{(x_2-\beta)^4 + \gamma(x_2-\delta)^2 + \varepsilon}\Big).\end{aligned} \tag{3.91}$$

Observe that Parameterisation 2.1 can be seen as a special case of Parameterisation 3.1 by setting $\alpha_3 = 0$ and $\gamma_{1,2} = \gamma$ and that Parameterisation 2.2 can be seen as a special case of Parameterisation 3.2 by setting $\delta = 0$.

In the rest of the section we analyse the instantaneous volatility functions obtained in the above parameterisations by relating them to the implied volatilities which can be observed on the market. Then we consider the instantaneous correlations in the above parameterisations and conclude the section by pointing out some practical implications of using two- and three-factor separable and time-homogeneous LMMs.

### 3.4.1 Instantaneous Volatiltiy

We have noted in Section 3.1 that time-homogeneity of instantaneous volatilities is a desirable property of LMMs. This motivated us to characterise the two- and three-factor time-homogeneous and separable LMMs. Now we analyse the flexibility of the instantaneous volatility functions obtained.

In practice the instantaneous volatilities of forward rates cannot be observed directly but we can observe the term structure of volatilities (see equation (2.63)) for a finite set of different times to maturity. Section 6.3 in Rebonato (2002) contains analysis of historical data on term structure of volatility. In particular, he points out that the term structure of volatilities remains relatively stable over time and at each date has one of the following shapes

- *Hump shape*: the term structure of volatilities first increases in time to maturity and decreases after some time to maturity $T'$;
- *Monotonically decreasing*: the term structure monotonically decreases in time to maturity.

Furthermore, he observes that the implied volatilities do not decrease to zero as the time to maturity increases but approach some non-negative constant, which we will call the *long-term level of volatility.*

Under the assumption that the instantaneous volatilities are time-homogeneous, i.e. there exists a function $\sigma^{inst}$ such that condition in equation (3.1) holds, then it is easy to observe:

- If $\sigma^{inst}$ is hump shaped then the term structure of volatilities is hump shaped;

- If $\sigma^{inst}$ is monotonically decreasing then the term structure of volatilities is monotonically decreasing.

Moreover, if $\lim_{x\to\infty} \sigma^{inst}(x) = 0$ then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma^{inst}(x)^2 dx = 0. \tag{3.92}$$

In particular, if $\sigma^{inst}$ is a decreasing function on an interval $(a, \infty)$ for some $a \geq 0$ then the implied volatilities will converge to some non-zero long term level if and only if $\lim_{x\to\infty} \sigma^{inst}(x) \neq 0$.

Therefore, a good parameterisation of a time-homogeneous instantaneous volatility function will converge to a positive constant as time to maturity increases and will be able to represent both hump-shaped and monotonically decreasing instantaneous volatilities.

### The Two-Factor Parameterisations

We begin by analysing the instantaneous volatility functions we can obtain in the two-factor case and which are given by equations (3.82) and (3.84).

**Parameterisation 2.1** The instantaneous volatility function for the Parameterisation 2.1 is given by the parameters $\alpha_1, \alpha_2 \geq 0$, $\beta_1\beta_2 \in \mathbb{R}$, $\gamma \in [-1, 1]$ and equation (3.82). For the purpose of this discussion we will assume that $\alpha_1, \alpha_2 > 0$ and $\beta_1 \neq \beta_2$ as the instantaneous volatility function otherwise reduces to a single exponential. Furthermore we will assume that $0 \leq \beta_1 < \beta_2$ to ensure that the instantaneous volatility function is bounded on $\mathbb{R}_+$. Figure 3.1 shows plots of the instantaneous volatility function for various choices of parameter values.

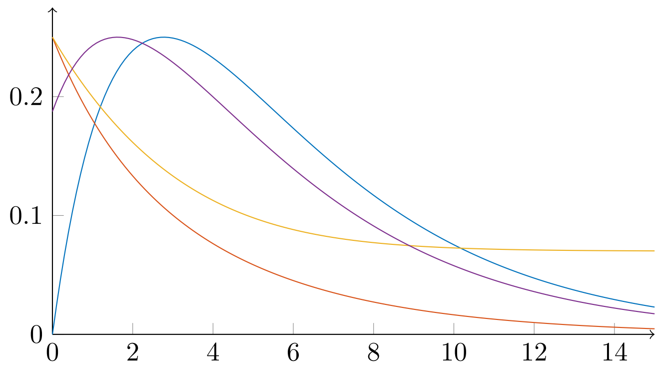



Figure 3.1: Plots of instantaneous volatility as a function of time to maturity corresponding to Parameterisation 2.1 (equation (3.82)) for various different choices of parameter values.

Clearly this parameterisation can capture the long-term level of volatility when $\beta_1 = 0$ in this case $\lim_{x\to\infty} \sigma^{inst}(x) = \alpha_1$. Moreover, when $\gamma \in [0, 1]$ the function $\sigma^{inst}$ is strictly decreasing. On the other hand if $\gamma \in [-1, 0)$ the instantaneous volatility function has a local minimum at $x' = \frac{1}{\beta_2}\log\frac{\alpha_1}{-\alpha_2\gamma}$. When $x' \leq 0$ the instantaneous volatility function is strictly increasing (on $\mathbb{R}_+$) and when $x' > 0$ the instantaneous volatility function is strictly decreasing on $[0, x')$ and strictly increasing on $(x', \infty)$. In particular when $\beta_1 = 0$ the instantaneous volatility function cannot capture the hump, but it can capture the monotonically decreasing instantaneous volatilities and the long-term level of volatility.

Let us now consider the case when $\beta_1 > 0$. In this case it is obvious that $\lim_{x\to\infty}\sigma^{inst} = 0$ and the instantaneous volatility cannot capture the long-term level of volatility. Furthermore, when $\gamma \geq 0$ it is easy to observe that the instantaneous volatility function is strictly decreasing. One can show that $\sigma^{inst}$ has two local extrema $x_1'$ and $x_2'$ (on $\mathbb{R}$) if and only if

$$\gamma < -2\frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2}. \tag{3.93}$$

In particular when $\gamma = -1$ the local extrema occur at

$$x_1' = \frac{1}{\beta_2-\beta_1}\log\frac{\alpha_2}{\alpha_1}, \qquad x_2' = \frac{1}{\beta_2-\beta_1}\log\frac{\alpha_2\beta_2}{\alpha_1\beta_1}. \tag{3.94}$$

Since $\beta_1 < \beta_2$ it follows $x_1' < x_2'$ and the local minimum is attained at $x_1'$ and the local maximum is attained at $x_2'$. Note that when $\alpha_1 \geq \alpha_2$ then $x_1' \leq 0$ and $\sigma^{inst}$ is strictly increasing on $(0, x_2')$ and strictly decreasing towards zero on $(x_2, \infty)$ and is therefore hump shaped.

To summarise, the instantaneous volatility function given by Parameterisation 2.1 cannot capture the hump and the long-term level simultaneously. However, it can capture monotonically decreasing volatilities together with the long-term level of volatility.

**Parameterisation 2.2** Next we analyse the instantaneous volatility function corresponding to Parameterisation 2.2 given in equation (3.84). Figure 3.2 shows plots of the instantaneous volatility function for various choices of parameter values.

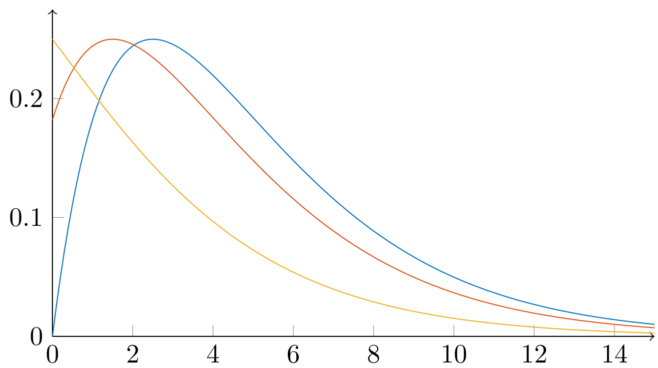



Figure 3.2: Plots of instantaneous volatility as a function of time to maturity corresponding to Parameterisation 2.2 (equation (3.84)) for various different choices of parameter values.

First observe that $\sigma^{inst}$ will be bounded (on $\mathbb{R}_+$) if and only if $\lambda > 0$, which we will assume throughout the analysis. In this case it is clear that $\lim_{x\to\infty} \sigma^{inst}(x) = 0$ and the instantaneous volatility function cannot capture the long-term level of volatility.

Secondly note that the parameter $\alpha$ is a scale parameter and does not affect the shape of the instantaneous volatility function, which is affected only by the parameters $\beta, \gamma$ and $\lambda$. Parameter $\lambda$ controls the speed of decay of instantaneous volatility function and one can think of $\beta$ and $\gamma$ as a shfit along $x$ and $y$ axis respectively. Note however that the shift will be non-linear and affected by the decay, i.e. the effect of varying $\beta$ and $\gamma$ on the instantaneous volatility will decrease as time to maturity increases.

It is then easy to observe that $\sigma^{inst}$ has local extrema (on $\mathbb{R}$) if and only if

$$\gamma < \frac{1}{4\lambda^2}, \tag{3.95}$$

which is in practice a relatively mild constraint. The local extrema are then attained at

$$x_1' = \beta + \frac{1 - \sqrt{1 - 4\gamma\lambda^2}}{2\lambda}, \qquad x_2' = \beta + \frac{1 + \sqrt{1 - 4\gamma\lambda^2}}{2\lambda}. \tag{3.96}$$

In particular, $x_1'$ is a local minimum and $x_2'$ is a local maximum.[2] Note that $x_1' < x_2'$ and that changing the parameter $\beta$ will shift the location of the local extrema, which is in line with the intuitive interpretation of the parameter $\beta$. When $x_1' \leq 0 < x_2'$ the instantaneous volatility function is strictly increasing on $(0, x_2')$ and strictly decreasing on $(x_2', \infty)$ and can therefore capture the hump. Furthermore, when $x_2' \leq 0$ the instantaneous volatility function is strictly decreasing on $R_+$. Note that in both cases $\beta < 0$.

To summarise, Parameterisation 2.2 can represent both monotonically decreasing and hump shaped volatilities. However it cannot capture the long-term level of volatility.

### The Three-Factor Parameterisations

We have seen that the two-factor parameterisations cannot capture the hump and the long-term level of volatility simultaneously. We will show that introducing the third factor leads to significantly more flexible instantaneous volatility parameterisations, given by equations (3.86), (3.88) and (3.90), that can capture the hump and the long-term level of volatility simultaneously.

**Parameterisation 3.1** First we consider the instantaneous volatility function give by equation (3.86). Figure 3.3 shows plots of the volatility function for various choices of parameter values.

Note that, by setting $\alpha_3 = 0$ the instantaneous volatility function reduces to the one we get in Parameterisation 2.1. Therefore we can assume that $\alpha_1, \alpha_2, \alpha_3 > 0$. Furthermore, in order for the instantaneous volatility function to be bounded we will additionally require $\beta_1, \beta_2, \beta_3 \geq 0$.

[2] When $\gamma = \frac{1}{4\lambda^2}$ then $x_1' = x_2'$ is a saddle point.

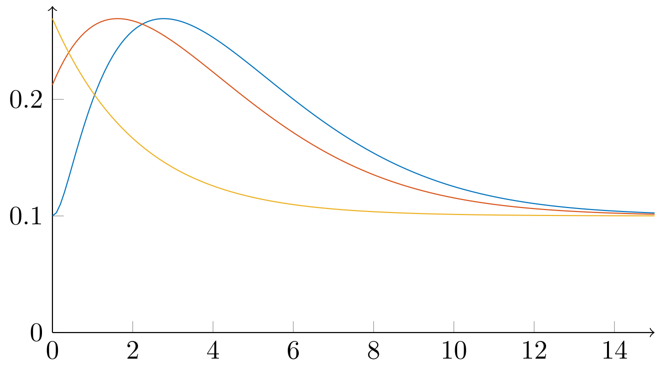



Figure 3.3: Plots of instantaneous volatility as a function of time to maturity corresponding to Parameterisation 3.1 (equation (3.86)) for various different choices of parameter values.

Recall that the main weakness of the Parameterisation 2.1 is its inability to capture the hump and the long-term level of volatility simultaneously. We will therefore only concentrate on the case when $\beta_3 = 0$ and $\beta_1 \neq \beta_2$. In this case we can interpret the parameter $\alpha_3$ as the long-term level of volatility.

For the Parameterisation 3.1 to be valid, the matrix value function $\rho(t)$ describing the time $t$ correlation structure of the Brownian motion driving the model needs to be a correlation matrix. In the case of Parameterisation 3.1 $\rho$ is given by

$$\rho(t) = \begin{bmatrix} 1 & \gamma_{1,2} & \gamma_{1,3} \\ \gamma_{1,2} & 1 & \gamma_{2,3} \\ \gamma_{1,3} & \gamma_{3,3} & 1 \end{bmatrix} \tag{3.97}$$

and is a correlation matrix if and only if $\gamma_{1,2}, \gamma_{1,3}, \gamma_{2,3} \in [-1, 1]$ and

$$\det \rho(t) = 1 - (\gamma_{1,2}^2 + \gamma_{1,3}^2 + \gamma_{2,3}^2) + 2\gamma_{1,2}\gamma_{1,3}\gamma_{2,3} \geq 0. \tag{3.98}$$

When the third factor is independent of the first two (i.e. $\gamma_{1,3} = \gamma_{2,3} = 0$), equation (3.98) is satisfied for any $\gamma_{1,2} \in [-1, 1]$ and $\sigma^{inst}$ has local extrema (on $\mathbb{R}$) if and only if

$$\gamma_{1,2} < -2\frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2}. \tag{3.99}$$

Note, that this is essentially the same condition as in the Parameterisation 2.1. Moreover, it is easy to verify that the local extrema are attained at the same points as for the Parameterisation 2.1.

When the third factor is correlated with the first two, one cannot in general explicitly find the local extrema, due to the first derivative being highly non-linear. However, allowing the third factor to be correlated with the first two clearly introduces additional flexibility to the instantaneous volatility parameterisation. In particular, this flexibility is necessary when the implied volatilities of caplets with short times to maturity are below the long-term level of volatility.

To summarise, Parameterisation 3.1 can capture both the hump and monotonically decreasing volatilities while it also captures the long term level of volatility. Its main downside is that it becomes less intuitive (but remains analytically tractable) when the factor representing the long-term level of volatility is correlated with the other two factors.

**Parameterisation 3.2** The instantaneous volatility Parameterisation 3.2 given by equation (3.88) is perhaps the most interesting parameterisation we can achieve in a three-factor separable and time-homogeneous model. Figure 3.4 shows the plots of the volatility function for various choices of parameter values.

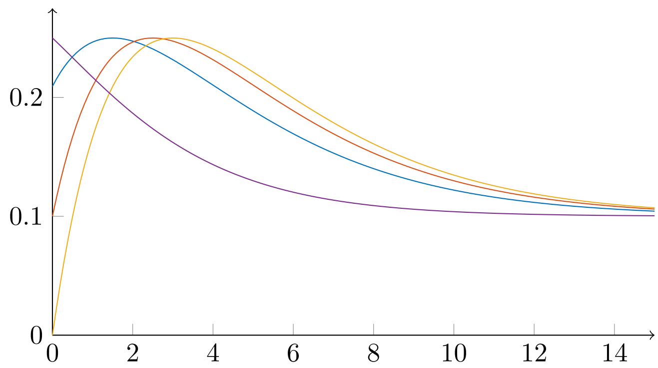



Figure 3.4: Plots of instantaneous volatility as a function of time to maturity corresponding to Parameterisation 3.2 (equation (3.88)) for various different choices of parameter values.

Note that setting the parameter $\delta = 0$ reduces the instantaneous volatility function to the one obtained in Parameterisation 2.1. In particular, we noted that the main drawback of Parameterisation 2.1 is its inability to capture the long term level of volatility.

Parameterisation 3.2 can capture the long-term level of volatility simply by setting $\mu = 0$ in which case $\delta$ can be interpreted as the long-term level of volatility. In particular, by setting $\alpha = |b|$, $\beta = -\frac{a}{b}$, $\gamma = 0$, $\delta = d$, $\varepsilon = -\frac{a}{b}$, $\eta = \operatorname{sgn} b$ and $\lambda = c$ the volatility function corresponds to the Rebonato's *abcd* instantaneous volatility parameterisation given by equation (3.2). In particular, the Parameterisation 3.2 can capture both hump and long term-level of volatility.

Clearly, we can get extra flexibility by also varying the parameters $\gamma$, $\eta$, however it is often sensible to set $\varepsilon = \beta$ as its effect on the volatility function is relatively limited.

**Parameterisation 3.3** Finally let us briefly discuss the instantaneous volatility function given by equation (3.90) corresponding to Parameterisation 3.3. Recall that the main reason for considering the three-factor models was the inability of the two-factor parameterisations to capture the hump and the long-term level of volatility simultaneously. However, note that Parameterisation 3.3 cannot capture the long-term level of volatility. Therefore it will in most case perform only marginally better over the Parameterisation 2.1 and 2.2 which does not justify the increase in the number of factors used.

### 3.4.2 Instantaneous Correlation

Let us now turn our attention to the instantaneous correlations. Recall that we are interested only in the time-homogeneous instantaneous correlations parameterisations, which can be represented by a function $\rho^{inst} : \mathbb{R}_+^2 \to [-1, 1]$ where $\rho^{inst}(x, y)$ is the instantaneous correlation between two forward rates with times to maturity $x$ and $y$ respectively.

Ideally one would take a similar approach as for instantaneous volatilities and determine the desirable properties of instantaneous correlations by relating them to prices of European swaptions. However, this turns out to be a difficult task as in general one cannot separate the effects of the instantaneous correlations from the effects of instantaneous volatilities on the European swaption prices (see Section 7.1 in Rebonato (2002)).

One therefore needs to take a different route and estimate the instantaneous correlations from historical data (see Section 7.2 in Rebonato (2002) and Section 14.3 in Andersen and Piterbarg (2010)). By doing so one usually observes that the resulting instantaneous correlation matrix satisfies the following stylised facts (see Section 7.2 in Rebonato (2002), Section 23.8 in Joshi (2011))

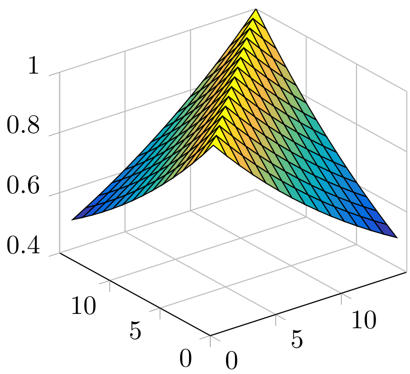
1. Instantaneous correlations are positive
$$\rho^{inst}(x, y) > 0; \tag{3.100}$$
2. Instantaneous correlations decrease as the absolute value of the difference between the two times to maturity increases
$$|x - y| < |x - z| \Rightarrow \rho^{inst}(x, y) > \rho^{inst}(x, z); \tag{3.101}$$
3. Instantaneous correlation between forward rates with the difference between their times to maturity increases as the time to maturity of the forward rate expiring earlier increases
$$x < x' \Rightarrow \rho^{inst}(x, x + y) < \rho^{inst}(x', x' + y); \tag{3.102}$$

The most basic example of an instantaneous correlation function satisfying the first two stylised facts is the *exponential instantaneous correlation function* given by parameter $\beta > 0$ and equation
$$\rho^{inst}(x, y) = \exp\bigl(-\beta|x - y|\bigr), \tag{3.103}$$

Note that the exponential instantaneous correlation violates the stylised fact 3. To correct for this violation one can introduce the *square-root exponential instantaneous correlation function* given by parameter $\beta' > 0$ and equation
$$\rho^{inst}(x, y) = \exp\bigl(-\beta'|\sqrt{x} - \sqrt{y}|\bigr). \tag{3.104}$$

Figure 3.5 shows plots of the exponential and square-root exponential instantaneous correlation functions. We used $\beta = 0.05$ to specify the exponential instantaneous correlation function and chose $\beta'$ so that instantaneous correlation functions agree for the pair of forward rates with times to maturity 1 and 15 years. Observe that for both functions the correlations rapidly decrease as the difference between the times to maturity increases.

We will later observe that the instantaneous correlations in the two- and three-factor separable and time-homogeneous LMM cannot achieve such a rapid decrease in instantaneous correlations. This is not only the case for the separable LMMs but will be true for low-factor LMMs in general and is a necessary compromise one needs to make when using a low-factor LMM.



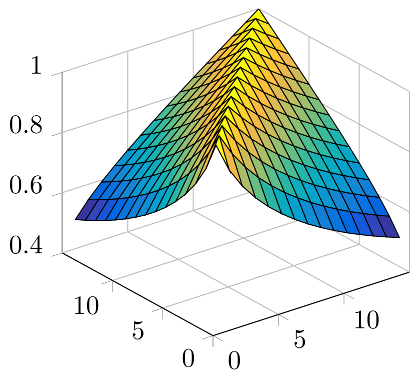




Figure 3.5: Plots of the exponential instantaneous correlation function (left) for $\beta = 0.05$ and the square-root exponential instantaneous correlation function (right) for $\beta' = 0.2436$.

Another way of comparing the instantaneous volatility functions is by performing a principal component analysis on the $n \times n$ matrix of instantaneous correlations between the rates with times to maturity $T_1, \ldots, T_n$. Such a matrix is by definition positive-semidefinite and can therefore be diagonalised. Moreover, in this case the principal values correspond to the eigenvalues $\lambda_1 > \ldots > \lambda_n$ and the $i$th principal component corresponds to the normalised eigenvector $v_i$ associated with the eigenvalue $\lambda_i$. Performing the principal component analysis on the empirical data gives the following stylised facts about the instantaneous correlation matrix (see Lord and Pelsser (2007))

1. The principal components $v_1, v_2, v_3$ corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3$ explain more than 95% of the observed correlation, with the first principal component being at least an order of magnitude more significant than the others;

2. The first principal component $v_1$ is comprised of positive elements of similar value (i.e. approximately $1/n$);

3. The first and last element of the second principal component $v_2$ are of opposite sign and monotonically decreasing (increasing) if the first element is positive (negative);

4. The first and the last element of the third principal component $v_3$ are of the same sign

but there is an element of the opposite sign which splits the elements of the principal component $v_3$ into two monotonic sequences.

The first, second and third principal components are commonly referred to as the *level, slope* and *curvature*.

In particular for the exponential and square-root exponential volatility functions for a set of annual forward rates with maturities 1 to 15 years, we find that the first three eigenvalues explain approximately 94% of the variability and Figure 3.6 shows that the first three principal components can be interpreted as level, slope and curvature.

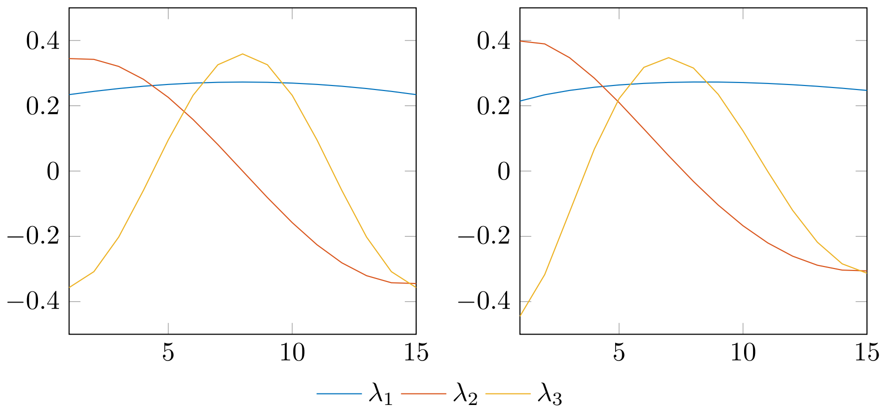



Figure 3.6: Plots of first three principal components of the exponential instantaneous correlation matrix(left) for $\beta = 0.05$ and for the square-root exponential instantaneous correlation matrix (right) for $beta' = 0.2436$.

### The Two-Factor Parameterisations

We now analyse the two-factor instantaneous correlation functions we obtained in Parameterisations 2.1 and 2.2. Note that in the two-factor case the instantaneous correlation matrix is of rank two or less and will therefore have at most two non-zero eigenvectors, which we would like to interpret as level and slope.

**Parameterisation 2.1** We begin by considering the instantaneous correlation function given by equation (3.83). Without loss of generality we can assume that $\alpha_1, \alpha_2 > 0$, $\beta_1 \neq \beta_2$. Now recall that the parameter $\gamma$ is the correlation between two components of the Brownian motions driving the separable LMM.

In particular when $\gamma \in \{-1, 1\}$ the components of the Brownian motion are perfectly (inversely) correlated. In this case the LMM is essentially a one-factor model and the forward rates are perfectly correlated. Note that when $\gamma \in \{-1, 1\}$ the resulting LMM is essentially driven by a single factor (see Remark 3.1), however it is separable in the dimension two and cannot be represented by a one-factor separable LMM.

On the other hand when $\gamma \in (-1, 1)$ the instantaneous correlation function is not identically equal to one and the resulting correlation matrix is of rank two. Moreover, the

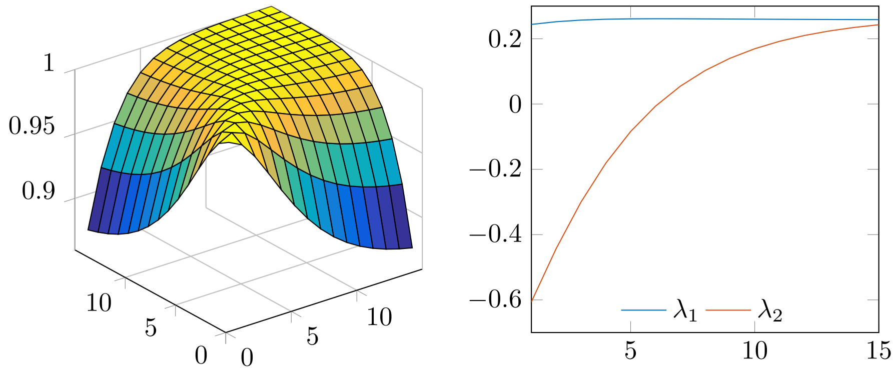



Figure 3.7: Plot of an instantaneous correlation function (left) and the first two principal components of the associated instantaneous correlation matrix for annual forward rates with times to maturity 1 to 15 years (right) corresponding to Parameterisation 2.1.

instantaneous correlations are strictly positive for every choice of parameters. However, it is in general difficult to analyse its dependence on the parameters due to complex interplay amongst them. Nevertheless, for a sensible choice of parameter values the correlation function results in mild-decorrelation between forward rates with short and long time to maturity and near perfect correlations between rates with longer times to maturity.

Figure 3.7 shows plots of a typical instantaneous correlation function (3.83) for a reasonable choice of parameter values and the first and second eigenvectors of the associated correlation matrix. Note that the forward rates with long maturities are nearly perfectly correlated, however there is some decorrelation between the rates of short to medium maturities and other rates. Moreover, the first two principal components of the correlation matrix can be interpreted as level and slope.

**Parameterisation 2.2** We now turn our attention to the instantaneous correlation function given by equation (3.85). First observe that it only depends on the parameters $\beta$ and $\gamma$.

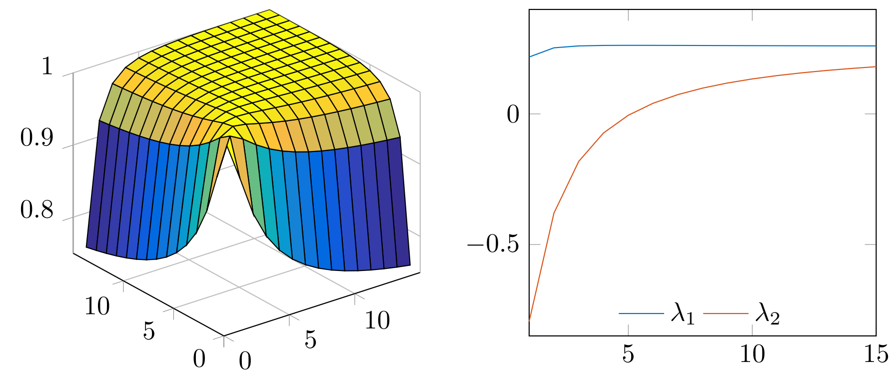



Figure 3.8: Plot of an instantaneous correlation function (left) and the first two principal components of the associated instantaneous correlation matrix for annual forward rates with times to maturity 1 to 15 years (right) corresponding to Parameterisation 2.2.

First note that when $\gamma = 0$ the instantaneous correlation function can be written as $\rho^{inst}(x_1, x_2) = \text{sgn}((x_1 - \beta)(x_2 - \beta))$, in particular the model is effectively driven by a single factor. However when $\gamma > 0$ the instantaneous correlation function results in non-perfect correlations among forward rates.

On the other hand when $\beta > 0$ the instantaneous correlation function may attain negative values when one of the forward rates has time to maturity less then $\beta$ and the other has time to maturity sufficiently greater than $\beta$. However, this turns out not to cause any problems from a practical perspective as $\beta > 0$ results in an unrealistic shape of the instantaneous volatility function. The more interesting scenario occurs when $\beta \leq 0$; and the instantaneous correlations are strictly positive. In this case increasing $\gamma$ will decrease the correlations and decreasing $\beta$ will increase the correlations amongst forward rates.

Figure 3.8 shows plots of a typical instantaneous correlation function for a reasonable choice of parameter values and the first and second principal component of a corresponding correlation matrix. Note the instantaneous correlations for rates of long-maturities are nearly perfect and there is some decorrelation between the forward rates of short and other times to maturity. Furthermore, the first two principal components can be interpreted as level and slope.

### The Three-Factor Parameterisations

Having analysed the two-factor parameterisations let us now consider the three-factor parameterisations 3.1 and 3.2. In the three-factor case we expect to observe higher levels of deceleration and also the curvature in principal component analysis of the correlation.

**Parameterisation 3.1** First consider the instantaneous correlation function given by equation (3.87). Recall that the matrix valued function $\rho$ as defined in equation (3.97) is a correlation matrix describing the correlations amongst the components of driving Brownian motion. Therefore, the instantaneous correlation function will result in non-perfect instantaneous correlations only when the rank of matrix $\rho(t)$ is strictly greater than one.

Recall that from practical standpoint fixing $\gamma_{1,2} = -1$ is often desirable as it results in hump-shaped volatilities. It is then easy to see that the matrix $\rho(t)$ is a correlation matrix if and only if $\gamma_{1,3} = \gamma_{2,3} = 0$ and we have a three-factor separable parameterisation of a two-factor LMM. Nevertheless, as demonstrated by Figure 3.9, the instantaneous correlations obtained in such model are reasonable. In fact the decorrelation achieved is much greater than the ones observed in the two-factor separable models. Furthermore, the first two principal components of the correlation matrix can be interpreted as level and slope.

On the other hand when $\rho(t)$ is a full rank matrix, the resulting model will be a proper three-factor LMM and the instantaneous correlation matrix will have three principal components corresponding to non-zero eigenvalues. Figure 3.10 shows plots of an instantaneous correlation function and the first three principal components of the associated instantaneous correlation matrix for a full rank $\rho(t)$. Note that the principal components can be interpreted as level, slope and curvature.

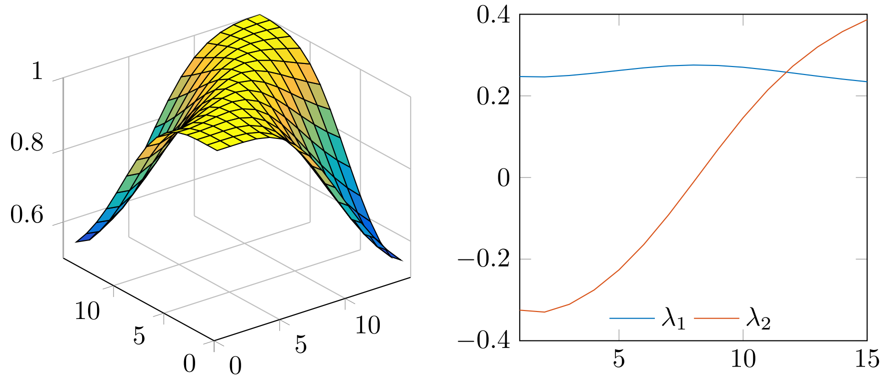



Figure 3.9: Plot of an instantaneous correlation function (left) and the first two principal components of the associated instantaneous correlation matrix for annual forward rates with times to maturity 1 to 15 years (right) corresponding to Parameterisation 3.1 when $\gamma_{1,2} = -1$.

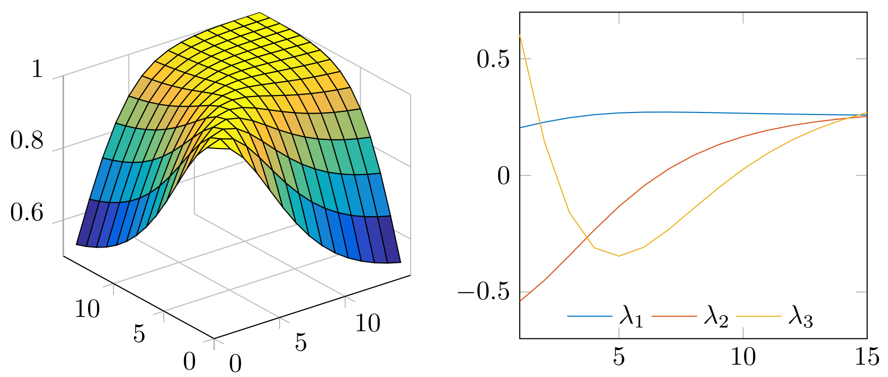



Figure 3.10: Plot of an instantaneous correlation function (left) and the first three principal components of the associated instantaneous correlation matrix for annual forward rates with times to maturity 1 to 15 years (right) corresponding to Parameterisation 3.1 when $\gamma_{1,2} = 0$.

Observe the correlation functions in Figures 3.9 and 3.10 have significantly different shapes demonstrating the flexibility of the instantaneous correlation function (3.89).

**Parameterisation 3.2** Finally let us consider the instantaneous correlation function given by equation (3.89) corresponding to perhaps the most interesting parameterisation of the three-factor separable and time-homogeneous LMM.

We begin by noting that in the special case when the parameters are chosen so that the instantaneous volatility function corresponds to the Reobnato's *abcd* parameterisation the resulting model is one-factor but it is represented by a three-factor separable parameterisation.

However, for a general parameterisation the instantaneous correlations will be non-perfect. Figure 3.11 shows plots of an instantaneous correlation function and the first three principal components of the associated correlation matrix for reasonable parameter values. Note that the instantaneous correlation function has shape similar to the one presented in Figure 3.9.

Furthermore, observe that the principal components can be interpreted as the level, slope and curvature.

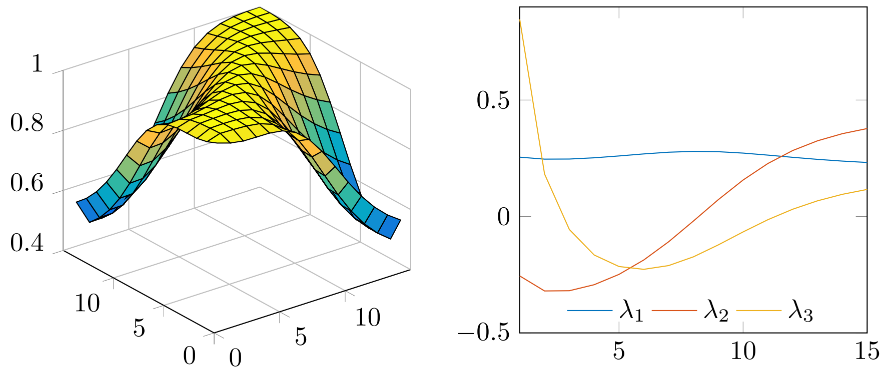



Figure 3.11: Plot of an instantaneous correlation function (left) and the first three principal components of the associated instantaneous correlation matrix for annual forward rates with times to maturity 1 to 15 years (right) corresponding to Parameterisation 3.2.

### 3.4.3 Remarks on Calibration and Implementation

Let us conclude this section by pointing out some practical remarks about the two- and three-factor separable parameterisations discussed in this section. Recall, that in all cases the instantaneous volatility and correlation function were determined by the same set of parameters. As a consequence, one has to simultaneously calibrate to the caplet and swaption prices.

In particular, to calibrate to caplet and swaption prices in the LMM one needs to be able to efficiently evaluate the *terminal covariance* elements between forward rates

$$C_{i,j}(T_k) = \int_0^{T_k} \rho^{inst}(T_i - t, T_j - t)\sigma^{inst}(T_i - t)\sigma^{inst}(T_j - t)dt, \quad k \leq i \wedge j. \tag{3.105}$$

It turns out that the terminal covariance elements have a closed form representation, thus allowing for efficient calibration.

Furthermore, one could exploit the fact that the terminal covariance elements can be determined explicitly and perform a global calibration of a full-factor LMM and then calibrate the separable parameterisation to the terminal covariance elements that capture the dynamics of the forward rates relevant for pricing a particular instrument.

## 3.5 Conclusion

In this chapter we have addressed one of the main issues of the separable LMMs, their flexibility. We have generalised the separability condition and characterised the two- and three-factor separable LMM with time-homogeneous instantaneous volatilities. We then demonstrated that parameterisations obtained are of practical interest by analysing their instantaneous volatilities and correlations. In particular, we have shown that a two-factor model can capture the long-term level of volatility when the instantaneous volatility function is decreasing or it can capture the hump shape in instantaneous volatility function but not the long-term level of volatility. To capture the hump and the long term level of volatility we introduced the third factor, which can capture the popular Rebonato's *abcd* volatility function as a special case.

For all parameterisations we observed that the corresponding instantaneous correlation is also time-homogeneous, which was not directly imposed in the formulation of the problem. Furthermore, for reasonable parameterisations the instantaneous correlation functions were qualitatively in line with the stylised facts.

For the ease of presentation we restricted our attention to the basic log-normal version of the LMM. However, one can generalise the separability condition further to the local-volatility LMM and reduce the dimension of the single time-step approximation, (see Section 4.5 for more details).

To conclude let us touch on the issue of the single time-step approximation admitting arbitrage and being useful only for time horizons up to 15 years. One way to avoid this issue is to use an appropriately specified *Markov-functional model* (MFM) instead of the single-time step approximation of a separable LMM. The main idea of the MFMs is to express forward rates at any given time as a function of some low-dimensional Markov process and is by construction arbitrage-free and efficient to implement.

Recall, that in proving that single-time step approximation of a separable LMM has dimension equal to the dimension of Brownian motion driving the dynamics we have explicitly defined a Markov process $x$ in equation (3.18). One can then use this driving process to drive the dynamics of an MFM calibrated to the caplet prices from the separable LMM. Bennett and Kennedy (2005) have shown that in the case of a one-factor separable LMM the MFM specified as above has similar dynamics as the LMM. They believed that this observation also holds for separable LMM with higher number of factors.

Therefore, Theorems 3.8 and 3.9 are not only useful for characterising the two- and three-factor separable (log-normal) LMMs with time homogeneous instantaneous volatilities, but can be used in a more general local-volatility setting or to define two- and three-dimensional MFMs with dynamics similar to the separable LMM.

We will discuss the link between the separable LMM and the MFM and how the results from this chapter can be applied beyond the (log-normal) LMM in Chapter 6.

# Chapter 4

# One-dimensional Markov-functional Models driven by Non-Gaussian Processes

Markov-functional models (MFMs), introduced by Hunt et al. (2000), provide a framework that allows one to define an arbitrage-free interest-rate model of any (finite) dimension that can be calibrated to any arbitrage-free formula for caplet or swaption prices. While there exist high-dimensional MFMs (see for example Kaisajuntti and Kennedy (2013)),[1] low-dimensional MFMs are of particular interest as they allow for efficient implementation on a low-dimensional grid.

The cost one needs to pay for the low-dimensionality is often the absence of a system of SDEs describing the dynamics of forward rates or ZCBs. This makes the MFMs less intuitive to use than their high-dimensional alternatives. Bennett and Kennedy (2005) acknowledged this problem and showed that an MFM under the terminal measure calibrated to Black formula for caplet prices (Black, 1976) and driven by a one-dimensional Gaussian process has dynamics similar to the one-factor separable LIBOR market model (LMM). Calibrating to Black formula results in a model with constant implied volatilities of caplet prices and cannot capture the skew or smile shaped implied volatilities typically observed on the market.

It is possible to calibrate the MFM to a skew or smile in implied volatilities, but using a Gaussian driving process might not lead to a stable or desired evolution of the implied volatility surface. However, the current literature on the MFMs has been exclusively focused on the models driven by Gaussian processes. This is most likely the consequence of the absence of numerical algorithms that implement MFMs driven by non-Gaussian processes.

The aim of this chapter is therefore twofold. Firstly, we provide new and efficient algorithms that can be used to implement an MFM driven by a (not-necessarily Gaussian) one-dimensional Markov process under the terminal and the spot measure. Secondly, we describe a systematic approach that can be used to specify MFMs under the terminal measure

[1] In fact we can view any LMM as a high-dimensional MFM.

with a stable evolution of the implied volatility surface by exploring the link between the one-factor separable local-volatility LMMs and the MFMs under the terminal measure.

The remainder of the chapter is organised as follows. In Section 4.1 recap the definition of a Markov-functional model and discus some of its basic properties. In Sections 4.2 and 4.3 we recall the constructions of MFMs under the terminal and the spot measure respectively and provide new algorithms that can be used to implement them for a general one-dimensional (not-necessarily Gaussian) driving process. In Section 4.4 we discuss the importance of choosing the right combination of caplet prices and driving process when specifying an MFM and give a simple copula based criterion that can be used to distinguish between driving processes. In Section 4.5 we explore the link between the one-factor separable local-volatility LMMs and the one-dimensional MFMs under the terminal measure. We then use these ideas in Section 4.6 where we propose a new combination of a one-dimensional driving process and caplet prices that exhibit skew in implied volatilities. In Section 4.7 we give a numerical example of the MFM specified by the aforementioned combination and show that it exhibits stable evolution of caplet implied volatilities. Section 4.8 concludes.

## 4.1 Definition and Sufficient Conditions

We have formally defined Markov-functional models of a $T^* < \infty$ time-horizon economy consisting of ZCB with maturities in a set $\mathcal{T} \subset (0, \infty)$ in Subsection 2.3.2. Essentially, a model of such an economy is called Markov-functional if there exists a numeraire pair $(N, \mathbb{N})$ such that the prices of ZCBs can be expressed as functions of some driving process $x$, which is a Markov process under the measure $\mathbb{N}$.

One way to specify the dynamics of an MFM is therefore, to specify the dynamics of the ZCBs and show that they satisfy the conditions of Definition 2.20. An example of such an approach is the LMM (albeit indirectly via the forward LIBORs).

Rather than by specifying the dynamics of the ZCBs another way to define an arbitrage-free model of a term structure is by specifying the numeraire pair $(N, \mathbb{N})$. In this case we can determine the prices of ZCBs by applying the fundamental pricing formula (see equation (2.20)) to a unit claim at some time $T \leq T^*$

$$D_{t,T} = N_t \mathbb{E}_{\mathbb{N}}\big[N_T^{-1}\big|\mathcal{F}_t\big], \quad t \leq T \leq T^*.[2] \tag{4.1}$$

Such an approach results in $D_{\cdot,T}$ being $\{\mathcal{F}_t\}_{t \leq T}$-adapted.[3] Therefore, a model specified by a numeraire pair is in general not Markov-functional. Next we give sufficient conditions when a numeraire pair defines a Markov-functional model.

**Proposition 4.1.** *Consider a model of a $T^* < \infty$ horizon economy consisting of ZCBs maturing on dates in set $\mathcal{T} \subset (0, T^*]$ defined by the numeraire pair $(N, \mathbb{N})$ and assume there*

[2]Reader familiar with the short-rate models will notice that in a short rate model one essentially specifies the dynamics of the continuously compounded rolling bank account under the associated risk neutral measure and then uses the fundamental pricing formula to determine the prices of ZCBs.

[3]Note, that by definition the ZCBs will also be adapted to the augmented filtration generated by them.

*exists a process $x$ such that*

*1'. $x$ is Markov process under the measure $\mathbb{N}$;*

*2'. For $0 \le t \le T \in \mathcal{T}$ the quotient $\frac{N_t}{N_T} \in \mathcal{L}^1(\mathbb{N})$ and $\frac{N_t}{N_T}$ is $\sigma(x_s, s \in [t,T])$-measurable.*

*Then the model is Markov-functional.*

*Furthermore, suppose that $V_T$ is a replicable European claim*[4] *expiring at time $T \le T^*$ that depends only on time $T$ prices of traded ZCBs. If $\frac{N_t}{N_T} V_T \in \mathcal{L}^1(\mathbb{N})$ then the time $t$ price $V_t$ of $V_T$ is given by*

$$V_t = \mathbb{E}_{\mathbb{N}}\left[\frac{N_t}{N_T} V_T \middle| x_t\right]. \tag{4.2}$$

**Lemma 4.2.** *Let $x$ be a Markov process under the measure $\mathbb{N}$ and let $0 \le t < T$. Suppose $V \in \mathcal{L}^1(\mathbb{N})$ is $\sigma(x_s; s \in [t,T])$-measurable. Then*

$$\mathbb{E}_{\mathbb{N}}[V|\mathcal{F}_t] = \mathbb{E}_{\mathbb{N}}[V|x_t]. \tag{4.3}$$

Proof of Lemma 4.2 can be found in Appendix B.

*Proof of Proposition 4.1.* To prove that the model defined by the numeraire pair $(N, \mathbb{N})$ satisfying conditions 1' and 2' in Proposition 4.1 it is enough to prove that it satisfies the measurability condition 2 in Definition 2.20.

Let $0 \le t \le T \in \mathcal{T}$, then by assumption 2' $\frac{N_t}{N_T} \in \mathcal{L}^1(\mathbb{N})$ and therefore

$$D_{t,T} = N_t \mathbb{E}_{\mathbb{N}}\left[\frac{1}{N_T} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\frac{N_t}{N_T} \middle| \mathcal{F}_t\right]. \tag{4.4}$$

Furthermore, by assumption 2' $\frac{N_t}{N_T}$ is also $\sigma(x_s; s \in [t,T])$-measurable and we can apply Lemma 4.2 to prove

$$D_{t,T} = \mathbb{E}_{\mathbb{N}}\left[\frac{N_t}{N_T} \middle| x_t\right], \tag{4.5}$$

therefore $D_{t,T}$ is $\sigma(x_t)$-measurable and can be expressed as a function of $x_t$. Consequently, condition 2 in Definition 2.20 is fulfilled and the model is Markov-functional.

Now suppose that $V_T$ is a replicable European claim expiring at time $T \le T^*$ that depends only on time $T$ prices of traded ZCBs. Since $D_{T,S}$ is $\sigma(x_T)$-measurable for $S \in [T, T^*] \cap \mathcal{T}$ then $V_T$ is $\sigma(x_T)$ measurable and $\frac{N_t}{N_T} V_T$ is $\sigma(x_s, s \in [t,T])$-measurable. Furthermore, we assumed that $\frac{N_t}{N_T} V_T \in \mathcal{L}^1(\mathbb{N})$ and therefore

$$V_t = \mathbb{N}_t \mathbb{E}_{\mathbb{N}}\left[\frac{V_T}{N_T} \middle| \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{N}}\left[\frac{N_t}{N_T} V_T \middle| \mathcal{F}_t\right]. \tag{4.6}$$

We can now apply Lemma 4.2 to show

$$V_t = \mathbb{E}_{\mathbb{N}}\left[\frac{N_t}{N_T} V_T \middle| x_t\right]. \tag{4.7}$$

[4] When specifying an MFM via the numeraire pair one usually implicitly assumes the uniqueness of the EMM and thus the completeness of the economy.

□

Condition 2' imposed in Proposition 4.1 is a mild one. In particular, it allows for the three most common numeraire processes: $T^*$-maturity ZCB, the discretely compounded bank account and the continuously compounded bank account.

Proposition 4.1 has another important corollary. Since it ensures that the prices of European claims are functions of the state of the driving process the prices of derivatives with finitely many early exercise features (e.g. Bermudan swaptions) can be calculated using backward induction by only keeping track of the state of the driving process.

In the remainder of the chapter we focus on the economy consisting of $n+1$ ZCBs maturing on dates $T_1 < \ldots < T_{n+1}$ and MFMs driven by one-dimensional Markov processes under the two popular choices of EMMs: terminal and spot measure. Unlike the traditional literature on MFM we do not assume that the driving process is Gaussian,[5] but allow for a general diffusion process $x$ with continuous marginal distributions (see Remark 4.5 for more detail on this restriction).

## 4.2 Markov-functional Model under the Terminal Measure

We now fix the numeraire pair to $(D_{\cdot,T_{n+1}}, \mathbb{F}^{n+1})$, i.e. we take the $T_{n+1}$-maturity ZCB as the numeraire and consider the model under the associated EMM $\mathbb{F}^{n+1}$, which is commonly referred to as the *terminal measure*. In Subsection 4.2.1 we review the theoretical construction of an MFM under the terminal measure calibrated to swaption prices as presented originally in Hunt et al. (2000). We then introduce a new algorithm that implements the model on a grid for a general diffusion process $x$ with continuous marginal distributions in Subsection 4.2.2.

First let us describe the main assumption behind the construction. For each $i \in \{1, \ldots, n\}$ we fix an index $j_i \in \{i+1, \ldots, n+1\}$. The pair $(i, j_i), i \in \{1, \ldots, n\}$, then identifies a forward swap rate process $y^{i \times j_i}$ associated with a payers (fixed-for-floating) interest rate swap (see Subsection 2.2.2) starting at time $T_i$ and with last payment date at time $T_{j_i}$. Our aim is to calibrate the model to swaptions written on swap rates $y^{1\times j_1}, \ldots, y^{n \times j_n}$.

**Remark 4.3.** *The choice of indices $j_1, \ldots, j_n$ in practice depends on the exotic derivative one wishes to price. Two common choices are:*

1. *$j_i = i+1, i = 1, \ldots, n$, in this case the chosen forward swap rate is in fact a forward LIBOR, i.e. $y^{i\times j_i} = L^i$, and the resulting model is an alternative to the LMM.*
2. *$j_i = n+1, i = 1, \ldots, n$, in this case $y^{i\times j_i} = y^{i \times n+1}$, i.e. the forward swap rates have a common last payment date, and the resulting model is an alternative to the Swap market model of Jamshidian (1997).*

[5]However, the algorithms presented in the following two sections can be also applied to a Gaussian driving process $x$.

For the construction to work we need to make two additional assumptions:

1. In our model $y_{T_i}^{i\times j_i}, i = 1, \dots, n$, can be written as an increasing càdlàg[6] function of $x_{T_i}$;

2. We are given the initial value of the numeraire $D_{0,T_{n+1}}$ and the prices of digital swaptions (see Subsection 2.2.3) written on $y_{T_i}^{i\times j_i}, i = 1, \dots, n$, for strikes $K \geq 0$ which are represented by a decreasing càdlàg function $V_0^{\text{dswaption},i\times j_i} : [0, \infty) \to \mathbb{R}$, i.e. for $K \geq 0$
$$V_0^{\text{dswaption},i\times j_i}(K) = D_{0,T_{n+1}} \mathbb{E}_{\mathbb{F}^{n+1}} \left[ \frac{P_{T_i}^{i\times j_i}}{D_{T_i,T_{n+1}}} \mathbf{1}_{\{y_{T_i}^{i\times j_i} > K\}} \right]. \tag{4.8}$$

Assumption 1 is an appropriate and non-restrictive modelling assumption ensuring that the driving process $x$ is capturing the level of rates in the economy. On the other hand assumption 2 provides us with the data needed to calibrate the model. In particular, recall that knowing the prices of digital swaptions on $y_{T_i}^{i\times j_i}$ is equivalent to knowing the prices of European swaptions on $y_{T_i}^{i\times j_i}$.

### 4.2.1 Recovery of Functional Forms

We now describe how to use assumptions 1 and 2 to construct an MFM under the terminal measure driven by a diffusion process $x$ with continuous marginal distributions and calibrated to the prices of digital swaptions $V_0^{\text{dswaption},i\times j_i}, i = 1, \dots, n$. Recall that a model of term structure is essentially determined by the dynamics of the numeraire, in our case the ZCB maturing at time $T_{n+1}$. As for the LMMs the construction presented in this section uniquely determines the joint distribution of numeraire only on the dates $T_1, T_2, \dots, T_{n+1}$, however this is in general non-restrictive as the numeraire discounted prices of ZCBs will be well-defined (see also discussion in Subsection 2.3.1).

We recover the functional forms of the numeraire iteratively working from time $T_{n+1}$ backwards. At time $T_{n+1}$ there is nothing to be determined as $D_{T_{n+1},T_{n+1}} \equiv 1$.

Next suppose that at time $T_i, i \in \{1, \dots, n\}$, the functional forms of $D_{T_{i+1},T_{n+1}}, \dots, D_{T_{n+1},T_{n+1}}$, have already been determined. We now wish to infer the functional form of $D_{T_i,T_{n+1}}$ from the market prices of digital swaptions written on $y_{T_i}^{i\times j_i}$.

Recall that $y_{T_i}^{i\times j_i}$ satisfies (see equation (2.20))
$$y_{T_i}^{i\times j_i} = \frac{1 - D_{T_i,T_{j_i}}}{P_{T_i}^{i\times j_i}}. \tag{4.9}$$

Dividing the numerator and the denominator on the right-hand side of equation (4.9) by $D_{T_i,T_{n+1}}$ yields
$$y_{T_i}^{i\times j_i} = \frac{D_{T_i,T_{n+1}}^{-1} - D_{T_i,T_{j_i}} D_{T_i,T_{n+1}}^{-1}}{P_{T_i}^{i\times j_i} D_{T_i,T_{n+1}}^{-1}}. \tag{4.10}$$

[6] Right continuous with limits from left.

Next recall that $P_{T_i}^{i\times j_i}$ is a linear combination of ZCB prices $D_{T_i,T_{i+1}},\dots,D_{T_i,T_{j_i}}$. We can therefore evaluate $P_{T_i}^{i\times j_i}D_{T_i,T_{n+1}}^{-1}$ using the martingale property[7] as

$$P_{T_i}^{i\times j_i}D_{T_i,T_{n+1}}^{-1} = \mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{P_{T_{i+1}}^{i\times j_i}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})}\middle|\mathcal{F}_{T_i}\right] \tag{4.11}$$

$$= \mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{P_{T_{i+1}}^{i\times j_i}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})}\middle|x_{T_i}\right], \tag{4.12}$$

where we also use the Markov property. Similarly we can express $D_{T_i,T_{j_i}}D_{T_i,T_{n+1}}^{-1}$ as

$$D_{T_i,T_{j_i}}D_{T_i,T_{n+1}}^{-1} = \mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{D_{T_{i+1},T_{j_i}}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})}\middle|x_{T_i}\right]. \tag{4.13}$$

Since we already know the functional form of the numeraire at times $T_{i+1},\dots,T_{j_i}$, we also know the functional forms of time $T_{i+1}$ prices of ZCBs maturing at times $T_{i+1},\dots,T_{j_i}$, from equation (4.1). In particular, $P_{T_i}^{i\times j_i}D_{T_i,T_{n+1}}^{-1}$ and $D_{T_i,T_{j_i}}D_{T_i,T_{n+1}}^{-1}$ are known functions of $x_{T_i}$. We can then use equation (4.10) to express $D_{T_i,T_{n+1}}$ in terms of known functions of $x_{T_i}$ and yet unknown functional form $y_{T_i}^{i\times j_i}(x_{T_i})$ as

$$D_{T_i,T_{n+1}}(x_{T_i}) = \left(\left(\frac{D_{T_i,T_{j_i}}}{D_{T_i,T_{n+1}}}\right)(x_{T_i}) + y^{i\times j_i}(x_{T_i})\left(\frac{P_{T_i}^{i\times j_i}}{D_{T_i,T_{n+1}}}\right)(x_{T_i})\right)^{-1},^{8} \tag{4.14}$$

where we pointed out the dependence on $x_{T_i}$ explicitly. To determine the functional form of $D_{T_i,T_{n+1}}$ it is therefore enough to recover the functional form of $y_{T_i}^{i\times j_i}$.

We determine the dependence of $y_{T_i}^{i\times j_i}$ on $x_{T_i}$ by calibrating the model to the prices of digital swaptions written on $y_{T_i}^{i\times j_i}$. We define a function $J^i$ on the range of $x_{T_i}$ by

$$J^i(x^*) := D_{0,T_{n+1}}\mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{P_{T_i}^{i\times j_i}}{D_{T_i,T_{n+1}}}\mathbf{1}_{\{x_{T_i}>x^*\}}\right]. \tag{4.15}$$

Note that the functions $J^i$ and $V_0^{\text{dswaption},i\times j_i}$ differ only in the indicator function, i.e. the set over which the expectation is taken. Secondly, note that $P_{T_i}^{i\times j_i}D_{T_i,T_{n+1}}^{-1}$ is a known function of $x_{T_i}$ and the function $J^i$ is well defined. By assumption 1, $y_{T_i}^{i\times j_i}$ is an increasing càdlàg function of $x_{T_i}$. Therefore, for each $K$ in the range of $y_{T_i}^{i\times j_i}$ there exists a unique $x^*$ such that the set identity

$$\{y_{T_i}^{i\times j_i} > K\} = \{x_{T_i} > x^*\} \tag{4.16}$$

holds and therefore $V_0^{\text{dswaption},i\times j_i}(K) = J^i(x^*)$. In particular, if $V_0^{\text{dswaption},i\times j_i}$ is a strictly

[7] $P^{i\times j_i}$ is the price process of a self-financing trading strategy in which we buy at time 0 the amount $\alpha_k$ of ZCBs with maturity $T_k$ for $k=i+1,\dots,j_i$, and sell them at time $T_{i+1}$.

[8] When $j_i=i+1$, i.e. $y^{i\times j_i}=L^i$, equation (4.14) can be expressed as

$$D_{T_i,T_{n+1}}(x_{T_i}) = \left(\left(1+\alpha_i L_{T_i}^i(x_{T_i})\right)\mathbb{E}_{\mathbb{F}^{n+1}}\left[D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})^{-1}\middle|x_{T_i}\right]\right)^{-1}.$$

decreasing function of strike then

$$y_{T_i}^{i\times j_i}(x^*) = \big(V_0^{\text{dswaption},i\times j_i}\big)^{-1}\big(J^i(x^*)\big). \tag{4.17}$$

In general the functional form of $y_{T_i}^{i\times j_i}$ is given by

$$y_{T_i}^{i\times j_i}(x^*) = \sup\big\{K \geq 0; V_0^{\text{dswaption},i\times j_i}(K) \geq J^i(x^*)\big\}. \tag{4.18}$$

**Remark 4.4.** *Note that defining $y_{T_i}^{i\times j}$ as in* (4.18) *(and taking the supremum of an empty set to be 0) will ensure that the MFM is arbitrage free even when the digital caplet prices will admit arbitrage (e.g. when the caplet prices are not strictly decreasing.).*

Having recovered the functional form of the $y_{T_i}^{i\times j_i}$ we can determine the functional form of $D_{T_i,T_{n+1}}$ from equation (4.14).

Finally, we can determine the functional forms of other ZCBs by using the fact that the price process of any numeraire discounted ZCB is a martingale under the EMM associated with the numeraire, in our case the terminal measure. In particular, for $j > i$

$$D_{T_i,T_j}(x_{T_i}) = D_{T_i,T_{n+1}}(x_{T_i})\mathbb{E}_{\mathbb{F}^{n+1}}\big[D_{T_j,T_{n+1}}(x_{T_j})^{-1}\big|x_{T_i}\big]. \tag{4.19}$$

Let us conclude the theoretical construction of the MFM under the terminal measure with a trivial but important observation. For a given set of indices $\{j_1, \ldots, j_n\}$, an initial value of the numeraire $D_{0,T_{n+1}}$, a set of digital swaption prices $\{V_0^{\text{dswaption},i\times j_i}\}_{i=1}^n$, and a diffusion process $x$ with continuous marginal distributions, there exists a unique MFM under the terminal measure driven by $x$ and calibrated to the digital swaption prices $\{V_0^{\text{dswaption},i\times j_i}\}_{i=1}^n$ that satisfies assumption 1 (see also Theorem (Uniqueness) in Bennett and Kennedy (2005)).

**Remark 4.5.** *Recall that we have assumed that the driving process $x$ is a diffusion process with continuous marginal distributions. However, in the construction we actually only needed that $x$ has continuous marginal distributions during the calibration of the MFM,.*

*To illustrate the need for such an assumption suppose that distribution of $x_{T_i}$ has an atom at $x^*$. Then by assumption 1 distribution $y_{T_i}^{i\times j_i}$ has an atom at value $K = y_{T_i}^{i\times j_i}(x^*)$. It is then easy to see that $V_0^{\text{dswaption},i\times j_i}$ has to have a jump at strike $K$. However, function $V_0^{\text{dswaption},i\times j_i}$ was chosen independently of the driving process and it may be the case that it does not have a jump at strike $K$, e.g. $V_0^{\text{dswaption},i\times j_i}$ might be a continuous function. In such a case the construction will result in a model that is not calibrated to digital caplet prices $V_0^{\text{dswaption},i\times j_i}$.*

*In general, when the marginal distribution of $x_{T_i}$ has atoms the construction can be carried out if and only if the image of function $V_i$ is a subset of the image of function $J^i$.*

### 4.2.2 Numerical Implementation

We now present a new implementation of the above construction on a grid. In particular, the use of piecewise polynomial function is a novel approach that allows for efficient implementation of the MFM when the driving process is a general (not-necessarily Gaussian) diffusion process with continuous marginal distributions.

For each time $T_i, i \in \{1, \ldots, n\}$, we assume we have $m$ grid-points

$$-\infty < h_{i,1} < \ldots < h_{i,m} < \infty. \tag{4.20}$$

For notational convenience it is beneficial to think of plus and minus infinity as grid-points $h_{i,0} = -\infty$ and $h_{i,m+1} = \infty, i = 1, \ldots, n$. Moreover, we will denote by $h_{0,1} = x_0$ the initial value of the driving process.

The information about the distribution of the driving process is given to us via one time-step conditional moments. More precisely, for some integer $q \geq 1$ we are given

$$\Delta^i_{j,k,l} := \mathbb{E}_{\mathbb{F}^{n+1}}\left[x^l_{T_i}\mathbf{1}_{\{h_{i,k} < x_{T_i} \leq h_{i,k+1}\}}\middle| x_{T_{i-1}} = h_{i-1,j}\right], \tag{4.21}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 0, \ldots, m$, and $l = 0, \ldots, q$.

**Remark 4.6.** *Since $x$ is a diffusion process we know that its transition density is a solution to the corresponding Kolmogorov forward equation. When the transition density is known explicitly the $\Delta$'s can be well approximated using numerical integration or even calculated exactly. However, even when the Kolmogorov forward equation does not have a closed form solution one can still approximate $\Delta$'s efficiently, for example by discretising the SDE for the driving process in time.*

Our aim is to determine at each time step the functional form of the numeraire on the grid-points via equations (4.15) and (4.18) and then recover the functional forms of other ZCBs on the grid-points via equation (4.19).

Observe that the most computationally demanding part of the construction is evaluating the function $J^i$, given in equation (4.15), since it involves integration over the distribution of $x_{T_i}$ which is not known a priori. One could evaluate $J^i$ directly by integrating $i$ times over the one time step conditional distributions, however such an approach is computationally intensive and would defeat the aim of using a low-dimensional MFM, which is computational efficiency. We avoid the direct evaluation of $J^i$ by first 'building' the distribution of $x_{T_i}$ forward in time. To do so we introduce 'basis functions' that will allow us quick and simple approximation of smooth (and with minor modifications piecewise smooth) functions (see Appendix A for more detail).

We consider basis functions $b_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m$, that are continuous and of the form

$$b_{i,j}(x) := \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l} x^l \mathbf{1}_{\{h_{i,k} < x \leq h_{i,k+1}\}}, \quad x \in \mathbb{R} \tag{4.22}$$

for some coefficients $b^i_{j,k,l} \in \mathbb{R}$. Moreover we impose the condition

$$b_{i,j}(h_{i,k}) = \delta_{j,k}, \quad j, k = 1, \ldots, m. \tag{4.23}$$

In our algorithm we can approximate any smooth function $f:\mathbb{R}\to\mathbb{R}$ by a piecewise polynomial function $\tilde{f}$ defined by

$$\tilde{f}(x) := \sum_{j=1}^{m} f(h_{i,j}) b_{i,j}(x), \quad x \in \mathbb{R}. \tag{4.24}$$

Note that $f$ and $\tilde{f}$ will have the same value for the grid-points $h_{i,k}, k = 1,\dots,m$, since

$$\tilde{f}(h_{i,k}) = \sum_{j=1}^{m} f(h_{i,j}) \delta_{j,k} = f(h_{i,k}). \tag{4.25}$$

In Appendix A we propose a specific choice of basis functions and show that for that choice $\tilde{f}$ is a piecewise polynomial approximation of $f$ on the interval $[h_{i,1}, h_{i,m}]$.

Furthermore, if the expectation of $f(x_{T_i})$ is finite, we can approximate it as

$$\mathbb{E}_{\mathbb{F}^{n+1}}\big[f(x_{T_i})\big] \approx \sum_{j=1}^{m} f(h_{i,j}) E_{i,j}, \tag{4.26}$$

where $E_{i,j} := \mathbb{E}_{\mathbb{F}^{n+1}}[b_{i,j}(x_{T_i})]$.

We now show how to determine the expectations $E_{i,j}, i = 1,\dots,n, j = 1,\dots,m$, iteratively forward in time and then use this information to evaluate the function $J^i$ on the grid-points. First note that at time $T_1$ the conditional and unconditional distributions coincide and for $j = 1,\dots,m$,

$$\begin{aligned} E_{1,j} &= \sum_{k=0}^{m}\sum_{l=0}^{q} b^1_{j,k,l} \mathbb{E}_{\mathbb{F}^{n+1}}\big[x^l_{T_1} \mathbf{1}_{\{h_{1,k} < x \le h_{1,k+1}\}}\big] && (4.27)\\ &= \sum_{k=0}^{m}\sum_{l=0}^{q} b^1_{j,k,l} \mathbb{E}_{\mathbb{F}^{n+1}}\big[x^l_{T_1} \mathbf{1}_{\{h_{1,k} < x \le h_{1,k+1}\}} \big| x_{T_0} = h_{0,1}\big] && (4.28)\\ &= \sum_{k=0}^{m}\sum_{l=0}^{q} b^1_{j,k,l} \Delta^1_{1,k,l}. && (4.29) \end{aligned}$$

Now we can proceed iteratively forward in time. Let $i \in \{2,\dots,n\}$ and assume that $E_{i-1,j}, j = 1,\dots,m$, have already been determined. Then for $j \in \{1,\dots,m\}$

$$\begin{aligned} E_{i,j} &= \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l} \mathbb{E}_{\mathbb{F}^{n+1}}\big[x^l_{T_i} \mathbf{1}_{\{h_{i,k} < x \le h_{i,k+1}\}}\big] && (4.30)\\ &= \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l} \mathbb{E}_{\mathbb{F}^{n+1}}\Big[\mathbb{E}_{\mathbb{F}^{n+1}}\big[x^l_{T_i} \mathbf{1}_{\{h_{i,k} < x \le h_{i,k+1}\}} \big| x_{T_{i-1}}\big]\Big]. && (4.31) \end{aligned}$$

Note that since $x_{T_i}$ is a diffusion process the conditional expectation in (4.31) is a smooth function of $x_{T_{i-1}}$. Therefore, we can approximate it using the basis functions $b_{i-1,j}, j =$

$1,\dots,m$, and approximate $E_{i,j}$ as

$$E_{i,j} \approx \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l} \sum_{p=1}^{m} \mathbb{E}_{\mathbb{F}^{n+1}}\left[x^l_{T_i}\mathbf{1}_{\{h_{i,k}<x\le h_{i,k+1}\}}\middle| x_{T_{i-1}} = h_{i-1,p}\right] E_{i-1,p} \tag{4.32}$$

$$\approx \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l} \sum_{p=1}^{m} \Delta^i_{p,k,l} E_{i-1,p}. \tag{4.33}$$

Observe that the innermost summation does not depend on index $j$. Therefore, it is computationally advantageous to evaluate it in advance. In particular if we define $\Gamma^i_{k,l}, k \in \{0,\dots,m\}, l \in \{0,\dots,q\}$ by

$$\Gamma^i_{k,l} := \mathbb{E}_{\mathbb{F}^{n+1}}\left[x^l_{T_i}\mathbf{1}_{\{h_{i,k}<x\le h_{i,k+1}\}}\right] \tag{4.34}$$

and use the same trick as in (4.31) we can approximate it by

$$\Gamma^i_{k,l} \approx \sum_{p=1}^{m} \mathbb{E}_{\mathbb{F}^{n+1}}\left[x^l_{T_i}\mathbf{1}_{\{h_{i,k}<x\le h_{i,k+1}\}}\middle| x_{T_{i-1}} = h_{i-1,p}\right] E_{i-1,p} \tag{4.35}$$

$$= \sum_{p=1}^{m} \Delta^i_{p,k,l} E_{i-1,p}. \tag{4.36}$$

Then we can express $E_{i,j}$ as

$$E_{i,j} \approx \sum_{k=0}^{m}\sum_{l=0}^{q} b^i_{j,k,l}\Gamma^i_{k,l}. \tag{4.37}$$

The number of operations needed to determine $E_{i,j}, j = 1,\dots,m$ and $\Gamma^i_{k,l}, k = 0,\dots,m$, $l = 0,\dots,q$ is then of order $O(m^2 q)$.

**Remark 4.7.** *The construction of $E_{i,j}$'s using forward iteration as described above preserves the tower property of expectation in the sense that approximating the expectation of $f(x_{T_i})$ as in* (4.26) *yields the same result as approximating the conditional expectation $\mathbb{E}[f(x_{T_i})|x_{T_{i-1}}]$ using basis functions and then approximating the expectation. This is crucial for the implementation of a MFM to be arbitrage free as it ensures that the prices of derivatives do not depend on the way expectation was evaluated in the model.*

Having determined $E_{i,j}$'s and $\Gamma^i_{k,l}$'s we have the information needed about the marginal distributions of the driving process to recover the functional form of the numeraire on the grid-points $h_{i,j}, i = 1,\dots,n, j = 1,\dots,m$, as described in Subsection 4.2.1. We work iteratively backwards in time. For each time $T_i, i \in 1,\dots,n$ we first evaluate function $J^i$ as defined in equation (4.15) and determine the functional form of $y^{i\times j_i}_{T_i}$ on the grid-points $h_{i,j}, i = 1,\dots,n$. Then we recover the functional-form of $D_{T_i,T_{n+1}}$ on the grid-points $h_{i,j}, i = 1,\dots,n$ via equation (4.14).

The value of function $J^i$ on the grid-points $h_{i,j}, j = 1, \dots, n$ is given by

$$J^i(h_{i,j}) = D_{0,T_{n+1}} \mathbb{E}_{\mathbb{F}^{n+1}} \left[ \frac{P_{T_i}^{i \times j_i}(x_{T_i})}{D_{T_i,T_{n+1}}(x_{T_i})} \mathbf{1}_{\{x_{T_i} > h_{i,j}\}} \right]. \tag{4.38}$$

Recall that the fraction $\frac{P_{T_i}^{i \times j_i}(x_{T_i})}{D_{T_i,T_{n+1}}(x_{T_i})}$ can be evaluated as in equation (4.12). Then we can recover its value on the grid-points $h_{i,j}, j = 1, \dots, m$, by approximating it using basis functions. In particular,

$$\begin{aligned}
\frac{P_{T_i}^{i \times j_i}(h_{i,j})}{D_{T_i,T_{n+1}}(h_{i,j})} &= \mathbb{E}_{\mathbb{F}^{n+1}} \left[ \frac{P_{T_{i+1}}^{i \times j_i}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})} \middle| x_{T_i} \right] && (4.39) \\
&\approx \sum_{k=1}^{m} \frac{P_{T_{i+1}}^{i \times j_i}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})} \mathbb{E}_{\mathbb{F}^{n+1}} \left[ b_{i+1,k}(x_{T_{i+1}}) \middle| x_{T_i} = h_{i,j} \right] && (4.40) \\
&= \sum_{k=1}^{m} \frac{P_{T_{i+1}}^{i \times j_i}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})} \sum_{p=0}^{m} \sum_{l=0}^{q} b_{k,p,l}^{i+1} \Delta_{j,p,l}^{i+1} && (4.41) \\
&= \sum_{p=0}^{m} \sum_{l=0}^{q} \Delta_{j,p,l}^{i+1} \sum_{k=1}^{m} b_{k,p,l}^{i+1} \frac{P_{T_{i+1}}^{i \times j_i}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})}. && (4.42)
\end{aligned}$$

It is computationally advantageous to calculate the innermost summation in advance as it does not depend on index $j$. Then, the number of operations needed to determine the fraction $\frac{P_{T_i}^{i \times j_i}(h_{i,j})}{D_{T_i,T_{n+1}}(h_{i,j})}$ for all $j \in \{1, \dots, m\}$, is of order $O(m^2 q)$.

**Remark 4.8.** *Note that we have evaluated the time $T_{i+1}$ functional forms of ZCB in the previous step. Therefore, evaluating $P_{T_{i+1}}^{i \times j_i}(h_{i+1,k})$ in advance will involve $O(n)$, operations and $O(mn)$ for all $k \in 1, \dots, m$, which will be in practice negligible. In particular, when $j_i = i + 1$ then $P_{T_{i+1}}^{i \times i+1}(h_{i+1,k}) = \alpha_i$ is trivial.*

Next observe that evaluating $J^i$ involves integrating over a discontinuous function, therefore we can not directly apply the approximation from equation (4.26). We first simplify the problem by observing the following relationship

$$J^i(h_{i,j}) = J^i(h_{i,j+1}) + D_{0,T_{n+1}} \bar{J}_{i,j}, \quad j = 1, \dots, m, \tag{4.43}$$

where $J^i(h_{m+1,i}) = 0$ (recall that $h_{m+1,i} = \infty$) and

$$\bar{J}_{i,j} := \mathbb{E}_{\mathbb{F}^{n+1}} \left[ \frac{P_{T_i}^{i \times j_i}(x_{T_i})}{D_{T_i,T_{n+1}}(x_{T_i})} \mathbf{1}_{\{h_{i,j} < x_{T_i} \le h_{i,j+1}\}} \right], \quad j = 1, \dots, m. \tag{4.44}$$

Although evaluating $\bar{J}_{i,j}$ involves integration over a discontinuous function, we can still evaluate it efficiently as we can approximate its continuous part first and then multiply the indicator function with the approximation. In particular

$$\bar{J}_{i,j} \approx \mathbb{E}_{\mathbb{F}^{n+1}}\left[\left(\sum_{k=1}^{m} \frac{P_{T_i}^{i\times j_i}(h_{i,k})}{D_{T_i,T_{n+1}}(h_{i,k})} b_{i,k}(x_{T_i})\right)\mathbf{1}_{\{h_{i,j}<x_{T_i}\leq h_{i,j+1}\}}\right] \tag{4.45}$$

$$= \sum_{k=1}^{m} \frac{P_{T_i}^{i\times j_i}(h_{i,k})}{D_{T_i,T_{n+1}}(h_{i,k})} \mathbb{E}_{\mathbb{F}^{n+1}}\left[b_{i,k}(x_{T_i})\mathbf{1}_{\{h_{i,j}\leq x_{T_i}\leq h_{i,j+1}\}}\right] \tag{4.46}$$

$$= \sum_{k=1}^{m} \frac{P_{T_i}^{i\times j_i}(h_{i,k})}{D_{T_i,T_{n+1}}(h_{i,k})} \sum_{l=0}^{q} b_{k,j,l}^{i}\Gamma_{j,l}^{i}. \tag{4.47}$$

Evaluating $\bar{J}_{i,j}$ for all $j \in \{1,\ldots,m\}$, then requires $O(m^2q)$ operations. Afterwards we can determine the value of $J^i$ on the grid-points iteratively in $O(m)$ operations and the total number of operations required to evaluate $J^i$ on the grid-points $h_{i,j}, j = 1,\ldots,m$, is $O(m^2q)$.

Next we determine the functional form of $y_{T_i}^{i\times j_i}$ on the grid-points $h_{i,j}, j = 1,\ldots,m$, as in equation (4.18)[9]. To recover the functional form of the numeraire on the grid-points $h_{i,j}, j = 1,\ldots,m$, by using equation (4.14), we first need to determine the functional form of $D_{T_i,T_{j_i}}D_{T_i,T_{n+1}}^{-1}$ via equation (4.13)

$$\frac{D_{T_i,T_{j_i}}(h_{i,j})}{D_{T_i,T_{n+1}}(h_{i,j})} = \mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{D_{T_{i+1},T_{j_i}}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})}\middle| x_{T_i} = h_{i,j}\right] \tag{4.48}$$

$$\approx \sum_{k=1}^{m} \frac{D_{T_{i+1},T_{j_i}}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})} \mathbb{E}_{\mathbb{F}^{n+1}}\left[b_{i+1,k}(x_{T_{i+1}})\middle| x_{T_i} = h_{i,j}\right] \tag{4.49}$$

$$= \sum_{k=1}^{m} \frac{D_{T_{i+1},T_{j_i}}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})} \sum_{p=0}^{m}\sum_{l=0}^{q} b_{k,p,l}^{i+1}\Delta_{j,p,l}^{i+1} \tag{4.50}$$

$$= \sum_{p=0}^{m}\sum_{l=0}^{q} \Delta_{j,p,l}^{i+1} \sum_{k=1}^{m} b_{k,p,l}^{i+1} \frac{D_{T_{i+1},T_{j_i}}(h_{i+1,k})}{D_{T_{i+1},T_{n+1}}(h_{i+1,k})}, \tag{4.51}$$

similarly as for the ratio $\frac{P_{T_i}^{i\times j_i}(h_{i,j})}{D_{T_i,T_{n+1}}(h_{i,j})}$ we should evaluate the innermost summation in advance. Then we can evaluate $\frac{D_{T_i,T_{j_i}}(h_{i,j})}{D_{T_i,T_{n+1}}(h_{i,j})}$ for all $j \in \{1,\ldots,m\}$ using $O(m^2q)$ operations.

Finally, we can determine the functional forms of the remaining ZCBs on the grid-points

[9]The complexity of this step is usually negligible in comparison to the $O(m^2q)$.

via equation (4.19),

$$D_{T_i,T_k}(h_{i,j}) = D_{T_i,T_{n+1}}(h_{i,j})\mathbb{E}_{\mathbb{F}^{n+1}}\left[\frac{D_{T_{i+1},T_k}(x_{T_{i+1}})}{D_{T_{i+1},T_{n+1}}(x_{T_{i+1}})}\middle| x_{T_i} = h_{i,j}\right] \tag{4.52}$$

$$\approx D_{T_i,T_{n+1}}(h_{i,j})\sum_{r=1}^{m}\frac{D_{T_i,T_k}(h_{i+1,r})}{D_{T_i,T_{n+1}}(h_{i+1,r})}\mathbb{E}_{\mathbb{F}^{n+1}}\left[b_{i+1,r}(x_{T_{i+1}})\middle| x_{T_i} = h_{i,j}\right] \tag{4.53}$$

$$= D_{T_i,T_{n+1}}(h_{i,j})\sum_{r=1}^{m}\frac{D_{T_i,T_k}(h_{i+1,r})}{D_{T_i,T_{n+1}}(h_{i+1,r})}\sum_{p=0}^{m}\sum_{l=0}^{q} b^{i+1}_{r,p,l}\Delta^{i+1}_{j,p,l} \tag{4.54}$$

$$= D_{T_i,T_{n+1}}(h_{i,j})\sum_{p=0}^{m}\sum_{l=0}^{q}\Delta^{i+1}_{j,p,l}\sum_{r=1}^{m}\frac{D_{T_i,T_k}(h_{i+1,r})}{D_{T_i,T_{n+1}}(h_{i+1,r})}b^{i+1}_{r,p,l}. \tag{4.55}$$

Again, note that evaluating the innermost conditional expectation separately is computationally advantageous. In such case we then require $O(m^2q)$ operations to determine the functional form of a ZCB on all grid-points at each time.

#### Computational Complexity of the Algorithm

Let us conclude this section by briefly commenting on the computational complexity of the above algorithm. The algorithm can be naturally divided into three parts. In the first part the information about marginal distributions of the driving process – that is $E_{i,j}$'s and $\Gamma^i_{j,k}$'s – is constructed from the $\Delta$'s. The second part is the calibration to digital swaption prices and determining the functional forms of the numeraire. In the last part remaining functional forms of ZCBs are recovered from the martingale property.

The number of operations performed in the first two parts is of order $O(m^2q)$ for each time step and therefore of order $O(m^2nq)$ for all time steps. In the last part we need to perform $O(m^2q)$ operations to determine the functional form of a single ZCB per time step. However, the total number of functional forms we need to determine is of order $O(n^2)$, therefore in the third part we perform $O(m^2n^2q)$ operations in total, making the total number of operations of order $O(m^2n^2q)$.

In most practical applications, the order of the basis function $q$ will take values 1, 3 or 5 and will be considerably smaller than $m$ and (usually) $n$. Furthermore, the basis function will typically be non-zero only on $q+1$ intervals. Modifying the algorithm accordingly (by only summing over non-zero elements) can significantly reduce the number of operations, but not the overall complexity.

## 4.3 Markov-functional Model under the Spot Measure

Let us now fix the numeraire pair $(B, \mathbb{F}^0)$, that is we take the discretely compounded or rolling bank account $B$ as the numeraire and the associated EMM $\mathbb{F}^0$ called the *spot measure* (see Subsection 2.2.2). Markov-functional models under the spot measure were first introduced by Fries and Rott (2004) and we will outline their construction in Subsection 4.3.1. In

Subsection 4.3.2 we will show how ideas from the previous section can be used to implement efficiently an MFM under the spot measure driven by a diffusion process $x$ with continuous marginal distributions.

Recall from Subsection 2.2.2 that the discretely compounded rolling bank account represents the price process of a unit deposited at time 0 that is reinvested or rolled over at the prevailing LIBOR rate between on the dates $T_1, \ldots, T_n$. In particular, at time $T_i, i = 0, \ldots, n$, we deposit the value $B_{T_i}$ for the period $[T_i, T_{i+1}]$ at the LIBOR rate $L^i_{T_i}$, at time $T_{i+1}$ we receive the deposited value back with interest, i.e. $B_{T_{i+1}} = B_{T_i}(1 + \alpha_i L^i_{T_i})$, and reinvest the whole amount for another period. In particular, the value of the rolling bank account on setting dates $T_1, \ldots, T_{n+1}$ is given by

$$B_{T_i} = \prod_{j=0}^{i-1}(1 + \alpha_j L^j_{T_j}). \tag{4.56}$$

**Remark 4.9.** *Note, that the process $B$ is a path-dependent process and it is not obvious that it can be used as a numeraire for an MFM. However, one can check that for any MFM (regardless of the choice of the numeraire pair) the rolling bank account satisfies the measurability condition from Proposition 4.1. Let $0 < t \leq T \leq T_{n+1}$ then $t \in (T_{i-1}, T_i]$ and $T \in (T_{j-1}, T_j]$ for some $i \leq j$ and the ratio*

$$\frac{B_t}{B_T} = \frac{D_{t,T_i}}{D_{T,T_j}} \prod_{k=i}^{j-1}(1 + \alpha_k L^k_{T_k}) \tag{4.57}$$

*is $\sigma(x_s; s \in [t, T])$-measurable. Hence, the rolling bank account is a suitable numeraire for defining an MFM.*

We will calibrate the MFM under the spot measure to the prices of digital caplets in-arrears (see Subsection 2.2.3). The reason for choosing the digital digital caplets in-arrears over the digital swaptions, will become apparent in the next subsection when we present the construction. First, we need to impose two additional assumptions

1. In our model $L^i_{T_i}, i = 1, \ldots, n$, can be written as an increasing càdlàg function of $x_{T_i}$;
2. We are given the initial value of the $T_1$-maturity ZCB $D_{0,T_1}$ and the prices of the *digital caplets in-arrears* written on $L^i_{T_i}, i = 1, \ldots, n$, for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$V_0^{\mathrm{dca},i}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{L^i_{T_i} > K\}}\right]. \tag{4.58}$$

Assumptions 1 and 2 can be interpreted in the same way as for the MFM under the terminal measure, namely they provide us with the data needed for calibration and ensure that the driving process $x$ represents the level of interest rates in the economy.

**Remark 4.10.** *Because the MFM under the spot measure is effectively calibrated to the* *prices of European caplets, it is particularly suitable for valuation of exotic derivatives that depend only on the forward LIBORs opposed to general forward swap rates.*

### 4.3.1 Recovery of Functional Forms

We now describe, how assumptions 1 and 2 can be used to construct an MFM under the spot measure calibrated to prices of digital caplets in-arrears $\{V_0^{\mathrm{dca},i}\}_{i=1}^n$ and driven by a diffusion process $x$ with continuous marginal distributions. As in the case of the MFM under the terminal measure, we will determine the functional form of the numeraire, the rolling bank account $B$, on dates $T_1, \ldots, T_{n+1}$.

Recall that by assumption 2 the value of numeraire at time $T_1$ namely $B_{T_1} = 1 + \alpha_0 L^0_{T_0} = D^{-1}_{0,T_1}$ is already known. Next note that for $i \geq 2$ $B_{T_i}$ is a function of $(x_{T_1}, \ldots, x_{T_{i-1}})$ and is uniquely determined by the functional forms of $L^j_{T_j}, j = 1, \ldots, i-1$, as in equation (4.56).

We will recover the functional forms of the LIBORs at their setting date – and therefore the functional form of the numeraire – forwards in time. Suppose that we have already determined the functional form of the numeraire $B_{T_i}$ for some $i \in \{1, \ldots, n\}$. Since $B_{T_{i+1}} = (1 + \alpha_i L^i_{T_i}) B_{T_i}$ we need to determine the functional form of $L^i_{T_i}$ in order to recover the functional form of $B_{T_{i+1}}$. This is done by calibrating to digital caplets in-arrears written on $L^i_{T_i}$.

Similarly as in the previous section we can define function $J^i$ by

$$J^i(x^*) := \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}} \mathbf{1}_{\{x_{T_i} > x^*\}}\right]. \tag{4.59}$$

Recall that the functional form of $B_{T_i}$ has already been determined in the previous step and therefore $J^i$ is well defined. Now observe that the functions $V^i$ and $J^i$ differ only in the indicator function. We can now use the same argument as in the construction of the MFM under the terminal measure to argue that assumption 1 implies that for each $K$ in the range of $L^i_{T_i}$ there exists a unique $x^*$ such that the set identity

$$\left\{L^i_{T_i} > K\right\} = \left\{x_{T_i} > x^*\right\} \tag{4.60}$$

holds and consequently

$$L^i_{T_i}(x^*) = \sup\left\{K \geq 0; V_0^{\mathrm{dca},i}(K) \geq J^i(x^*)\right\}. \tag{4.61}$$

Having determined the functional form of the numeraire for all times $T_i, i = 1, \ldots, n+1$, we can recover the functional forms of the ZCBs. In particular for $j > i$

$$D_{T_i,T_j} = \mathbb{E}_{\mathbb{F}^0}\left[\frac{B_{T_i}}{B_{T_j}} \middle| x_{T_i}\right]. \tag{4.62}$$

Let us finish the construction by pointing out that the construction of the MFM under the spot measure is unique in the sense that for a given set $\{V_0^{\mathrm{dca},i}\}_{i=1}^n$ of prices of digital caplets in-arrears, an initial value of the $T_1$-maturity ZCB $D_{0,T_1}$, and a diffusion process $x$ with continuous marginal distributions there exists a unique MFM under the spot measure calibrated to prices of digital caplets in-arrears $\{V_0^{\mathrm{dca},i}\}_{i=1}^n$ and the initial condition $D_{0,T_1}$ that satisfies assumption 1.

### 4.3.2 Numerical Implementation

Before presenting an algorithm that implements the construction of an MFM under the spot measure on a grid let us point out some computational issues arising in the model. As in the case of the MFM under the terminal measure, it appears that evaluating the function $J^i$ will be a crucial step from the implementation standpoint. This is because evaluating $J^i$ involves integrating over the joint distribution of $(x_{T_j})_{j=1}^i$.

Fries and Rott (2004) solve this problem when the driving process is Gaussian by introducing linear 'pricing functionals' that can be constructed iteratively. The algorithm we describe in this section extends their idea to a more general class of diffusion processes with continuous marginal distributions. Furthermore, Fries and Rott (2004) do not give full detail on how the pricing functionals are constructed and do not give the set of contingent claims on which they are acting (see Remark 4.11).

The inputs to our algorithm are the same as for the MFM under the terminal measure. For each time $T_i, i = 1, \dots, n$, we assume we are given $m$ grid-points

$$-\infty < h_{i,1} < \dots < h_{i,m} < \infty \tag{4.63}$$

and denote for notational convenience $h_{0,i} = -\infty$ and $h_{i,m+1} = \infty$. Furthermore, we denote the initial value of the driving process by $h_{0,1} = x_0$.

The information about the driving process is given to us by one time step conditional moments. More precisely, for some integer $q \geq 1$ we are given

$$\Delta^i_{j,k,l} = \mathbb{E}_{\mathbb{F}^0}\big[x^l_{T_i}\mathbf{1}_{\{h_{i,k}<x_{T_i}\leq h_{i,k+1}\}}\big|x_{T_{i-1}} = h_{i-1,j}\big], \tag{4.64}$$

for $i = 1, \dots, n, j = 1, \dots, m$ and $l = 0, \dots, q$.

Our aim is to provide an implementation, that allows for efficient calibration of the model (i.e. efficient evaluation of function $J^i$ at grid-points) and also efficient valuation of derivatives.

As in the implementation of the MFM under the terminal measure, we introduce continuous piecewise polynomial 'basis functions' $b_{i,j}, i = 1, \dots, n, j = 1, \dots, m$, of the form

$$b_{i,j}(x) = \sum_{k=1}^{m}\sum_{l=0}^{q} b^i_{j,k,l}x^l\mathbf{1}_{\{h_{i,k}<x\leq h_{i,k+1}\}}, \quad x \in \mathbb{R}, \tag{4.65}$$

such that

$$b_{i,j}(h_{i,k}) = \delta_{j,k}, \quad j,k = 1, \dots, m. \tag{4.66}$$

We can then approximate any smooth function $f$ of $x_{T_i}$ by a piecewise polynomial function $\tilde{f}$ (see Appendix A for more detail) defined by

$$\tilde{f}(x) = \sum_{j=1}^{m} f(h_{i,j}) b_{i,j}(x), \quad x \in \mathbb{R}. \tag{4.67}$$

Now we deviate from the implementation of the MFM under the terminal measure and define constants $E_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m$ by

$$E_{i,j} = \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(x_{T_i})}{B_{T_i}}\right]. \tag{4.68}$$

This allows us to approximate any expectation of the form $\mathbb{E}[f(x_{T_i})B_{T_i}^{-1}]$, where $f$ is a 'smooth enough' function, by

$$\mathbb{E}_{\mathbb{F}^0}\left[\frac{f(x_{T_i})}{B_{T_i}}\right] \approx \sum_{j=1}^{m} f(h_{i,j}) E_{i,j}. \tag{4.69}$$

**Remark 4.11.** *The $E_{i,j}$'s defined above are analogous to the pricing functional used in Fries and Rott (2004). In particular, we can define a linear pricing functional $V^i$ acting on time $T_i$ claims of the form $f(x_{T_i})$, where $f$ is a 'smooth enough' function, by*

$$V^i\big[f(x_{T_i})\big] := \sum_{i=1}^{m} f(h_{i,j}) E_{i,j}. \tag{4.70}$$

*While we have not specified the smoothness condition completely as it depends on a particular choice of basis functions (see Appendix A), we will always need the $f$ to be at least continuous. Therefore, we can not apply $V^i$ to the claim $\mathbf{1}_{\{x_{T_i} > x^*\}}$ and use it to evaluate $J^i$ directly.*

Next we show how to determine the $E_{i,j}$'s using forward iteration and calibrate the model to the prices of digital caplets in-arrears at the each step of iteration. At time $T_1$ the marginal distribution of $x_{T_1}$ and one step conditional distribution $x_{T_1}|x_0$ coincide. Furthermore $B_{T_1} = 1 + \alpha_0 L_0^0$ is a known constant. Then we can express $E_{1,j}, j = 1, \ldots, m$ as

$$E_{1,j} = \frac{1}{B_{T_1}} \sum_{k=0}^{m} \sum_{l=0}^{q} b_{j,k,l}^1 \Delta_{1,k,l}^1. \tag{4.71}$$

Moreover, we can evaluate $J^1$ on the grid-points $h_{1,j}, j = 1, \ldots, m$, directly

$$J_1(h_{1,j}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_1}} \mathbf{1}_{\{x_{T_1} > h_{i,j}\}}\right] \tag{4.72}$$

$$= \frac{1}{B_{T_1}} \sum_{k=j}^{m} \Delta_{1,k,0}^1 \tag{4.73}$$

and the functional form of $L_{T_1}^1$ on the grid-points $h_{1,i}, i = 1, \ldots, n$, can be determined from equation (4.61).

Now we describe a step of the forward iteration. Let $i \in \{2, \dots, n\}$ and assume that we have already determined $E_{i-1,j}, j = 1, \dots, m$ and the functional form of $L^{i-1}_{T_{i-1}}$. Then for $j = 1, \dots, m$,

$$
\begin{aligned}
E_{i,j} &= \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(x_{T_i})}{B_{T_i}}\right] && (4.74)\\
&= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(x_{T_i})}{1+\alpha^i L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\middle|\mathcal{F}_{T_{i-1}}\right]\right] && (4.75)\\
&= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(x_{T_i})}{1+\alpha^i L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\middle|x_{T_{i-1}}\right]\right] && (4.76)\\
&\approx \sum_{k=1}^{m} E_{i-1,k}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(x_{T_i})}{1+\alpha_{i-1} L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\middle|x_{T_{i-1}} = h_{i-1,k}\right] && (4.77)\\
&= \sum_{k=1}^{m} \frac{E_{i-1,k}}{1+\alpha_{i-1} L^{i-1}_{T_{i-1}}(h_{i-i,k})}\sum_{p=0}^{m}\sum_{l=0}^{q} b^i_{j,p,l}\Delta^i_{k,p,l} && (4.78)\\
&= \sum_{p=0}^{m}\sum_{l=0}^{q} b^i_{j,p,l}\sum_{k=1}^{m}\frac{E_{i-1,k}\Delta^i_{k,p,l}}{1+\alpha_{i-1} L^{i-1}_{T_{i-1}}(h_{i-i,k})}. && (4.79)
\end{aligned}
$$

Observe that, the innermost summation does not depend on the index $j$, therefore it is computationally more efficient to calculate it in advance. Define $\Gamma^i_{p,l}, p = 0, \dots, m, l = 0, \dots, q$ by

$$
\Gamma^i_{p,l} := \sum_{k=1}^{m}\frac{E_{i-1,k}\Delta^i_{k,p,l}}{1+\alpha_{i-1} L^{i-1}_{T_{i-1}}(h_{i-i,k})}, \tag{4.80}
$$

then we can express $E_{i,j}, j = 1, \dots, m$, as

$$
E_{i,j} = \sum_{p=1}^{m}\sum_{l=0}^{q} b^i_{j,p,l}\Gamma^i_{p,l}. \tag{4.81}
$$

Note that determining all the $\Gamma^i_{p,l}$'s requires $O(m^2 q)$ operations per time step and computing all $E_{i,j}$'s requires $O(m^2 q)$ operations per time step.

Next we describe the calibration part of the iteration step. To calibrate the model to prices of digital caplets in-arrears and determine the functional form of $L^i_{T_i}$ on the grid-points we need to evaluate function $J^i$ for grid-points $h_{i,j}, j = 1, \dots, m$,

$$
J^i(h_{i,j}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{x_{T_i} > h_{i,j}\}}\right]. \tag{4.82}
$$

Note that evaluating $J^i$ involves integrating over a discontinuous function and therefore equation (4.69) can not be applied directly. As in the case of MFM under the terminal measure we simplify the problem by observing the recursive relationship for $j = 1, \dots, m$,

$$J^i(h_{i,j}) = J^i(h_{i,j+1}) + \bar{J}_{i,j}, \tag{4.83}$$

where $J^i(h_{i,m+1}) = 0$ and

$$\bar{J}_{i,j} = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{h_{i,j} < x_{T_i} \leq h_{i,j+1}\}}\right]. \tag{4.84}$$

Next we condition on the value of $x_{T_{i-1}}$ to smooth the discontinuity thus allowing us to apply equation (4.69)

$$\begin{aligned}
\bar{J}_{i,j} &= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{\mathbf{1}_{\{h_{i,j} < x_{T_i} \leq h_{i,j+1}\}}}{1 + \alpha_{i-1}L^{i-1}_{T_i}}\middle| x_{T_{i-1}}\right]\right] && (4.85)\\
&\approx \sum_{k=1}^{m} E_{i-1,k}\mathbb{E}_{\mathbb{F}^0}\left[\frac{\mathbf{1}_{\{h_{i,j} < x_{T_i} \leq h_{i,j+1}\}}}{1 + \alpha_{i-1}L^{i-1}_{T_i}}\middle| x_{T_{i-1}} = h_{i-1,k}\right] && (4.86)\\
&= \sum_{k=1}^{m} \frac{E_{i-1,k}\Delta^i_{k,j,0}}{1 + \alpha_{i-1}L^i_{T_{i-1}}(h_{i-1,k})} && (4.87)\\
&= \Gamma^i_{j,0}. && (4.88)
\end{aligned}$$

Note that we have already calculated $\Gamma^i_{j,0}$, therefore we can then recover $J^i(h_{i,j}), j = 1, \dots, m$, iteratively in $O(m)$ operations and consequently determine the functional form of $L^i_{T_i}$ on the grid-points via equation (4.61).

Once we have recovered the functional forms of all forward LIBOR rates we can recover the functional forms of ZCBs iteratively backwards in time notting that

$$\begin{aligned}
D_{T_i,T_j} &= B_{T_i}\mathbb{E}_{\mathbb{F}^0}\left[\frac{D_{T_{i+1},T_j}}{B_{T_{i+1}}}\middle| \mathcal{F}_{T_i}\right] && (4.89)\\
&= \mathbb{E}_{\mathbb{F}^0}\left[\frac{D_{T_{i+1},T_j}}{1 + \alpha^i L^i_{T_i}}\middle| \mathcal{F}_{T_i}\right] && (4.90)\\
&= \mathbb{E}_{\mathbb{F}^0}\left[\frac{D_{T_{i+1},T_j}}{1 + \alpha^i L^i_{T_i}}\middle| x_{T_i}\right] && (4.91)
\end{aligned}$$

and therefore

$$D_{T_i,T_j}(h_{i,k}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{D_{T_{i+1},T_j}(x_{T_{i+1}})}{1+\alpha^i L^i_{T_i}(x_{T_i})}\middle| x_{T_i} = h_{i,k}\right] \tag{4.92}$$

$$\approx \sum_{p=1}^{m} \frac{D_{T_{i+1},T_j}(h_{i+1,p})}{1+\alpha_i L^i_{T_i}(h_{i,k})}\mathbb{E}_{\mathbb{F}^0}\left[b_{i+1,p}(x_{T_{i+1}})\middle| x_{T_i} = h_{i,k}\right] \tag{4.93}$$

$$= \frac{1}{1+\alpha_i L^i_{T_i}(h_{i,k})}\sum_{p=1}^{m} D_{T_{i+1},T_j}(h_{i+1,p})\sum_{r=0}^{m}\sum_{l=0}^{q} b^{i+1}_{p,r,l}\Delta^{i+1}_{k,r,l} \tag{4.94}$$

$$= \frac{1}{1+\alpha_i L^i_{T_i}(h_{i,k})}\sum_{r=0}^{m}\sum_{l=0}^{q}\Delta^{i+1}_{k,r,l}\sum_{p=1}^{m} D_{T_{i+1},T_j}(h_{i+1,p})b^{i+1}_{p,r,l}. \tag{4.95}$$

Note that we can evaluate the innermost summation in advance which requires $O(m^2q)$ operations per functional form of a ZCB for each time step. Then the total number of operation to evaluate the actual functional form of a ZCB on the grid points is of order $O(m^2q)$ per time step. In total there are $O(n^2)$ functional forms to be determined and therefore the total number of operations of the determining the functional forms of ZCBs is of order $O(m^2n^2q)$.

#### Computational Complexity of the Algorithm

Let us conclude this section by briefly commenting on the computational complexity of the above algorithm. The algorithm can be naturally divided into two parts. In the first part the information about marginal distributions of the driving process is constructed from the $\Delta$'s and the functional forms of the forward LIBORs at their setting dates are determined by calibrating to digital caplets in-arrears. In the second part the functional forms of ZCBs are recovered from the martingale property.

The number of operations performed in the first part is of order $O(m^2q)$ for each time step and therefore of order $O(m^2nq)$ for all time steps. In the second part we need to perform $O(m^2q)$ operations to determine the functional form of a single ZCB per time step. However, the total number of functional forms we need to determine is of order $O(n^2)$, therefore in the third part we perform $O(m^2n^2q)$ operations in total, making the total number of operations of order $O(m^2n^2q)$.

## 4.4 Choice of The Driving Process

In this section we focus on the Markov-functional models driven by a one-dimensional process as described in Sections 4.2 and 4.3. We will refer to the MFM as described in Section 4.2 as the MFM under the terminal measure and to the MFM as described in Section 4.3 as the MFM under the spot measure, omitting the dimension of the driving process.

For simplicity we will restrict ourselves to the LIBOR version of the MFM under the terminal measure. This corresponds to the choice $j_i = i+1, i = 1, \ldots, n$ (see Remark 4.3), in this case the prices of digital swaptions and digital caplets on $y_{T_i}^{i \times i+1} = L_{T_i}^i$ will differ by a multiplicative constant only and we will therefore say that the model is calibrated to caplet prices.

For a given set of caplet prices choosing the driving process is arguably the most important step in specifying a Markov-functional model. A suitable driving process will ensure that the dynamics of the model are reasonable. In particular, it will ensure a stable evolution of the caplet implied volatility surface over time (we will demonstrate this with an example in Section 4.7). Moreover, the choice of the driving process will also influence the correlations between rates and thus the prices of swaptions.

In general it is difficult to understand how changing the driving process will influence the dynamics of the MFM. One of the reasons for this is the complex interplay between the swaption or caplet prices and the driving process (especially for the MFM under the terminal measure, see equation (4.14)). However, the role of the driving process is well understood for the MFM under the terminal measure driven by a Gaussian process and calibrated caplet prices given by the Black formula.[10]

This case was studied by Bennett and Kennedy (2005) who compared the dynamics of a one-factor separable LMM (see Section 3.2) with the dynamics of an MFM under the terminal measure calibrated to the same set of caplet prices and driven by a one-dimensional Gaussian process. They showed that under a wide range of parameters the LMM and the MFM have similar dynamics and Bermudan swaption prices. Therefore, one can see the MFM as an arbitrage-free one-dimensional approximation to the one-factor separable LMM. This has two important consequences. Firstly, one can transfer the intuition from the one-factor separable LMM to the MFM. Secondly, one can use a computationally more efficient MFM instead of the LMM.

While the connection between the separable LMM and the MFM is strong, the assumption that the caplet prices are given by the Black formula is often too restrictive. However, we will show in the next section that the concept of separability can be extended beyond the log-normal LMM. Based on these findings, we will propose a combination of a one-dimensional driving process and caplet prices that can be used to specify an MFM with similar dynamics to the CEV-LMM.

Another way to gain an insight into the role of the driving process is by using copula theory. The basic principle of the copula theory is that any $d$-dimensional distribution $F$ can be decomposed into an $n$-dimensional copula[11] $C$ and its marginal distributions $F_i, i = 1, \dots, n$, so that

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)), \quad x_1, \dots, x_n \in \mathbb{R}. \tag{4.96}$$

Moreover, the decomposition is unique if the distribution $F$ is continuous (see Nelsen (1999)

[10] Recall that the Black formula assumes that $L_{T_i}^i$ is log-normally distributed under the measure $\mathbb{F}^{i+1}$, see also Remark 2.16.

[11] $C$ is a $d$-dimensional copula if it is an $n$-dimensional distribution function on $[0,1]^d$ with uniform marginals.

for more detail). The 'decomposition' of a Markov-functional model into a driving process and caplet prices seems to be to a certain extent analogous to this. However, for a given choice of caplet prices there are infinitely many driving processes that result in 'the same' MFM. For example driving processes $x$ and $ax + b$, where $a > 0$ and $b \in \mathbb{R}$, clearly result in 'the same' MFM.

Let us now be more precise and define what we mean by saying that two MFMs under the terminal (respectively spot) measure are 'the same'. Remember that an MFM is fully determined by the dynamics of the numeraire and underlying filtration $\{\mathcal{F}_t\}_{t\geq 0}$. On the other hand, the construction of the MFM under the terminal (respectively spot) measure uniquely determines the numeraire only on the tenor structure dates. In-between the tenor structure dates one can still recover numeraire discounted prices of ZCBs and the LIBOR rates. However this is not a major drawback and a similar observation holds for other market models including the LMM. For this reason we will compare MFMs under the terminal (respectively spot) measure only on the tenor structure dates.

Remember that in the MFM under the terminal (respectively spot) measure the LIBORs at their setting dates are increasing functions of the driving process. In particular, when they are strictly increasing the $\sigma$-algebras generated by the LIBOR at its setting date and the state of the driving process will be the same. In this case the MFM under the terminal (respectively spot) measure is uniquely determined (on the tenor structure dates) by the joint distribution of LIBORs at their setting date and the initial term structure (see equations (4.14) and (4.56)).

We will then say that two MFMs under the terminal (respectively spot) measure calibrated to the same initial term structure are equivalent if the joint distribution of the LIBORs $L^i_{T_i}, i = 1, \ldots, n$, is the same in both models. Next we prove that under mild technical conditions two MFMs under the terminal (respectively spot) measure driven by one-dimensional Markov processes $x$ and $y$ are the equivalent if and only if the copulae of the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ are the same.

**Theorem 4.12.** *Let $x$ and $y$ be one-dimensional Markov processes with absolutely continuous marginal distributions. Let $D_{0,T_1} \leq 1$; and assume that for $i \in \{1, \ldots, n\}$ $D_{0,T_i} > D_{0,T_{i+1}}$ and $V^i : [0, \infty) \to [0, 1]$ is a differentiable function. Moreover, assume that for $i \in \{1, \ldots, n\}$ $V_i$ is strictly decreasing on the subdomain $\{K; V^i(K) > 0\}$ and satisfies*

$$\lim_{K\to\infty} V^i(K) = 0. \tag{4.97}$$

1. *Suppose that $V^i(0) = D_{0,T_{i+1}}$. Then MFMs under the terminal measure, calibrated to digital caplet prices $\{V^i\}_{i=1}^n$ and initial term structure $(D_{0,T_i})_{i=1}^{n+1}$, driven by processes $x$ and $y$ are equivalent if and only if the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ have the same copula.*

2. *Suppose that $V^i(0) = D_{0,T_i}$. Then MFMs under the spot measure, calibrated to digital caplet in-arrears prices $\{V^i\}_{i=1}^n$ and initial term structure $(D_{0,T_i})_{i=1}^{n+1}$, driven by*

*processes $x$ and $y$ are equivalent if and only if the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ have the same copula.*

*Proof.* First observe that conditions on the functions $\{V^i\}_{i=1}^n$ ensure that they are an arbitrage-free system[12] of digital caplet (respectively digital caplets in-arrears) prices compatible with the initial term structure. Therefore MFMs under the terminal (respectively spot) measure, calibrated to the digital caplet (respectively digital caplets in-arrears) prices $\{V^i\}_{i=1}^n$ and initial term structure $(D_{0,T_i})_{i=1}^{n+1}$, driven by processes $x$ and $y$ are well defined. Denote by $L^{i,x}$ and $L^{i,y}$ the $i$th LIBOR rate in the MFM driven by process $x$ and $y$ respectively, $i = 1, \dots, n$.

Recall that when the LIBORs at their setting dates are strictly increasing functions of the state of the driving process, then the MFM under the terminal (respectively spot) measure is uniquely determined (on the tenor structure dates) by the initial term structure and the joint distribution of the LIBORs at their setting date under the terminal (respectively spot) measure. Therefore, the MFMs driven by $x$ and $y$ respectively will be equivalent if and only if the distributions of the vectors $(L_{T_i}^{i,x})_{i=1}^n$ and $(L_{T_i}^{i,y})_{i=1}^n$ are the same.

Since the function $V^i$ is differentiable $L_{T_i}^{i,x}$ and $L_{T_i}^{i,y}$ are absolutely continuous random variables under the $T_{i+1}$ (respectively the $T_i$) forward measure and therefore also under the terminal (respectively spot) measure. Therefore, $L_{T_i}^{i,x}$ and $L_{T_i}^{i,y}$ are strictly increasing functions of $x_{T_i}$ and $y_{T_i}$ respectively. By the invariance of the copula under increasing transformations, the copulae of vectors $(L_{T_i}^{i,x})_{i=1}^n$ and $(x_{T_i})_{i=1}^n$ are the same. Similarly, the copulae of vectors $(L_{T_i}^{i,y})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ are the same.

Now suppose that the MFMs driven by $x$ and $y$ are equivalent. Then the copulae of the vectors $(L_{T_i}^{i,x})_{i=1}^n$ and $(L_{T_i}^{i,y})_{i=1}^n$ are the same. Consequently the copulae of the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ are the same.

We now prove the opposite implication. Suppose that the copulae of the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ are the same. Define processes $\tilde{x}$ and $\tilde{y}$ by

$$\tilde{x}_t = F_t^x(x_t) \qquad \text{and} \qquad \tilde{y}_t = F_t^y(x_t), \tag{4.98}$$

where $F_t^x$ and $F_t^y$ are the distribution functions of $x_t$ and $y_t$ respectively. Note that $\tilde{x}$ and $\tilde{y}$ are one-dimensional Markov processes with marginals distributed uniformly on interval $(0, 1)$. Furthermore, for $t > 0$, $F_t^x$ and $F_t^y$ are increasing functions on the state space of $x_t$ and $y_t$ respectively. Therefore, the distributions of vectors $(\tilde{x}_{T_i})_{i=1}^n$ and $(\tilde{y}_{T_i})_{i=1}^n$ are also the same. Furthermore, we can define LIBORs

$$L_{T_i}^{i,\tilde{x}} = L_{T_i}^{i,x}\big((F_{T_i}^x)^{-1}(\tilde{x}_{T_i})\big) \qquad \text{and} \qquad L_{T_i}^{i,\tilde{y}} = L_{T_i}^{i,y}\big((F_{T_i}^y)^{-1}(\tilde{y}_{T_i})\big). \tag{4.99}$$

Then clearly

$$\big(L_{T_i}^{i,\tilde{x}}\big)_{i=1}^n \overset{d}{=} \big(L_{T_i}^{i,x}\big)_{i=1}^n \qquad \text{and} \qquad \big(L_{T_i}^{i,\tilde{y}}\big)_{i=1}^n \overset{d}{=} \big(L_{T_i}^{i,y}\big)_{i=1}^n. \tag{4.100}$$

Therefore, it follows from the previous discussion that the models driven by $x$ and $\tilde{x}$ are

[12] There is no model independent arbitrage strategy.

equivalent and so are the models driven by $y$ and $\tilde{y}$. But since $(\tilde{x}_{T_i})_{i=1}^n \stackrel{d}{=} (\tilde{y}_{T_i})_{i=1}^n$ the models driven by $\tilde{x}$ and  have to be equivalent as well, hence the models driven by $x$ and $y$ are equivalent. $\square$

**Remark 4.13.** *One can weaken the assumptions of Theorem 4.12 to allow for discontinuities in digital caplet (respectively digital caplet in-arrears) prices. In this case one can still prove that two MFM under the terminal (respectively spot) measure driven by processes $x$ and $y$ are equivalent if the copulae of the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ are the same.*

In what follows we will say that two driving processes are equivalent if they lead to equivalent MFMs. In particular, we will use Theorem 4.12 to provide a simple criterion on when two driving processes are equivalent. Suppose, that Markov process $x$ is given by a strong solution to the SDE

$$dx_t = \mu(t, x_t)dt + \sigma(t, x_t)dW_t, \quad x_0 \in \mathbb{R}, \tag{4.101}$$

and let function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \in \mathcal{C}^{1,2}$ satisfy

$$\left.\frac{\partial f}{\partial x}\right|_{(t,x)} > 0, \quad t \geq 0, x \in \mathbb{R}. \tag{4.102}$$

We can then define a process $y$ by

$$y_t = f(t, x_t), \quad t \geq 0. \tag{4.103}$$

Since $f(t, .)$ is increasing function for $t \geq 0$, the vectors $(x_{T_i})_{i=1}^n$ and $(y_{T_i})_{i=1}^n$ have the same copula. Theorem 4.12 then implies that MFMs driven by $x$ and $y$ calibrated to the same set of caplet prices are equivalent.

We are especially in interested when a driving process $x$ can be transformed as in equation (4.103) to a process $y$ of the form

$$dy_t = \tilde{\mu}(t, y_t)dt + dW_t. \tag{4.104}$$

Applying Itô's Lemma to equation (4.103) yields

$$dy_t = \left.\left(\frac{\partial f}{\partial t} + \mu(t, x_t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(t, x_t)^2\frac{\partial^2 f}{\partial x^2}\right)\right|_{(t,x_t)} dt + \sigma(t, x_t)\frac{\partial f}{\partial x}dW_t. \tag{4.105}$$

For equation (4.104) to hold $f$ has to be a solution to the PDE

$$\sigma(t, x)\frac{\partial f}{\partial x} = 1. \tag{4.106}$$

When $\sigma \in \mathcal{C}^{1,1}$ is a strictly positive function then any solution to the PDE (4.106) is given by

$$f(t,x) = \int \frac{1}{\sigma(t,x)} dx + \varphi(t), \tag{4.107}$$

for some function $\varphi \in \mathcal{C}^1$. We can then chose the function $\varphi$ so that $y_0 = 0$ and the drift part of the SDE for $y$ has no additive deterministic part.

Transforming driving processes to the form given by the SDE (4.103) is not only useful to determine whether two driving processes are equivalent. It can be also used in the numerical implementation of a driving process for which the transition densities are not know. In such a case one typically needs to discretise the driving process in time. Therefore, moving the state dependence from the local volatility to the drift can simplify the discretisation.

In practice it might be beneficial to allow for a deterministic local volatility part when implementing the driver. This will still allow for a simple discretisation of the process in time and will additionally allow the user to remove the part of the drift that is affine in the state of the driving process.

**Example 4.14.** *Let $x$ be an Ornstein–Uhlenbeck process satisfying the SDE*

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t, \quad x_0 \in \mathbb{R}, \tag{4.108}$$

*where $\theta, \sigma > 0$ and $\mu \in \mathbb{R}$. Define the function $f$ by*

$$f(t,x) = e^{\theta t}(x_t - \mu) - x_0 + \mu. \tag{4.109}$$

*Then the process $y$ defined as $y_t = f(t, x_t)$ is a deterministic time change of a Brownian motion satisfying the SDE*

$$dy_t = \sigma e^{\theta t} dW_t, \quad y_0 = 0. \tag{4.110}$$

*Since $f$ satisfies condition* (4.102)*, the Ornstein–Uhlenbeck process $x$ given by the SDE* (4.108) *and the deterministic time change of a Brownian motion $y$ given by the SDE* (4.110) *are equivalent driving processes for the MFM under the terminal or spot measure.*

**Example 4.15.** *Let $x$ be a non-degenerate displaced diffusion satisfying the SDE*

$$dx_t = (x_t - \theta)\sigma dW_t, \quad x_0 > \theta, \tag{4.111}$$

*where $\sigma > 0$ and $\theta \in \mathbb{R}$. Define the function $f$ by*

$$f(t,x) = \log(x_t - \theta) + \frac{1}{2}\sigma^2 t - \log(x_0 - \theta). \tag{4.112}$$

*Then the process $y$ defined by $y_t = f(t, x_t)$ satisfies the SDE*

$$dy_t = \sigma dW_t, \quad y_0 = 0. \tag{4.113}$$

*Since $f$ satisfies condition* (4.102)*, the displaced diffusion $x$ given by the SDE* (4.111) *and* *the deterministic time change of a Brownian motion y given by SDE* (4.113) *are equivalent driving processes for the MFM under the terminal or spot measure.*

The implications of Theorem 4.12 are not limited only to identifying drivers that result in equivalent MFMs. Dynamic copula theory (see Darsow et al. (1992) and Nelsen (1999, Sec. 6.4)) can be used to define processes that cannot be transformed to a deterministic time change of a Brownian motion using strictly increasing transformations. Furthermore, the transition densities of such processes will typically be known explicitly, allowing for efficient implementation.

## 4.5 Separable Local-volatility LIBOR Market Model

We have mentioned in the previous section the importance of choosing an appropriate combination of driving process and caplet (or swaption) prices when specifying an MFM model. In this section we describe a systematic approach for finding such combinations by exploring the link between separable LMMs and MFMs.

In the context of the log-normal LMM, separability was first introduced by Pietersz et al. (2004). They proved that a single time-step approximation to the separable $d$-factor log-normal LMM can be represented as a function of a $d$-dimensional Gaussian process (see Proposition 3.6). Moreover, they found that such an approximation is highly accurate for LMMs with time horizon up to 10 years. However, for LMMs with longer time horizons the single time-step approximation admits noticeable arbitrage.

It turns out that the MFM under the terminal measure driven by a one-dimensional Gaussian process has similar dynamics to the one-factor separable log-normal LMM (Bennett and Kennedy, 2005).[13] Bennett and Kennedy (2005) found that the two models are numerically virtually indistinguishable for time up to 10 years. For longer time horizons they observed slight differences between the two models but the qualitative behaviour remained similar.

The results by Bennett and Kennedy (2005) have two important implications. Firstly, the MFM under the terminal measure provide a computationally efficient and arbitrage-free alternative to the one-factor separable log-normal LMM. Secondly, the intuition behind the well-understood behaviour of the LMM can be transferred to the MFM, in particular the evolution of caplet implied volatility surface in the MFM will be approximately the same as in the LMM.

These ideas can be extended beyond the log-normal LMM to more general local-volatility LMMs as introduced by Andersen and Andreasen (2000). This allows us to find combinations of caplet prices and driving processes that exhibit a stable and well understood evolution of the caplet implied volatility surface. Here we will only explore the connection between one-factor models.

[13] We implicitly assume that the two models are calibrated to the same set of caplet prices and an appropriate Gaussian process is driving the MFM.

### 4.5.1 Local-volatility LIBOR Market Model

A one-factor local-volatility LMM under the terminal measure is given by a local volatility function $\phi : A \subset \mathbb{R} \to [0, \infty)$, bounded piecewise continuous functions $\sigma^i : [0, T_i] \to \mathbb{R}, i = 1, \ldots, n$, and a system of SDEs

$$dL_t^i = \sigma^i(t)\phi(L_t^i)\left(dW_t - \sum_{j=i+1}^{n} \frac{\alpha_j \sigma^j(t)\phi(L_t^j)}{1 + \alpha_j L_t^j} dt\right), \quad t \le T_i, i = 1, \ldots, n. \tag{4.114}$$

Andersen and Andreasen (2000) proved that when $\phi : [0, \infty) \to \mathbb{R}$ satisfies:

AA1. $\phi(0) = 0$;

AA2. Local Lipschitz continuity:

$$\forall n \in \mathbb{N},\ \exists C_n > 0 : (x \vee \tilde{x}) \le C_n \Rightarrow |\phi(x) - \phi(\tilde{x})| \le C_n |x - \tilde{x}|; \tag{4.115}$$

AA3. Linear growth:

$$\exists C > 0,\ \forall x \ge 0 : \phi(x)^2 \le C(1 + x^2); \tag{4.116}$$

then the System of SDEs (4.114) admits a pathwise-unique non-negative solution when the initial term structure is non-negative ($L_0^i \ge 0, i = 1, \ldots, n$). Moreover, if the initial term structure is strictly positive ($L_0^i > 0, i = 1, \ldots, n$) the solution is also strictly positive.

The most popular choices for the local volatility function found in the literature are $\phi^{\mathrm{LN}}(x) = x$ leading to log-normal LMM (Brace et al., 1997); $\phi^{\mathrm{DD}}(x) = x - \theta, \theta > 0$, leading to shifted log-normal LMM; and $\phi^{\mathrm{CEV}}(x) = x^\beta, \beta \in (0, 1)$, leading to constant elasticity of variance (CEV) LMM (Andersen and Andreasen, 2000). Note that only $\phi^{\mathrm{LN}}$ satisfies conditions AA1–AA3, $\phi^{\mathrm{DD}}$ fails condition AA1 and $\phi^{\mathrm{CEV}}$ fails condition AA2. Nevertheless, all three choices lead to a unique solution to the System of SDEs (4.114).[14] Finally let us mention the hyperbolic local-volatility function (Jäckel, 2008)

$$\phi^{\mathrm{Hyp}}(x) = \frac{1 - \beta + \beta^2}{\beta} \frac{x}{\theta} + \frac{\left(\sqrt{(x/\theta)^2 + \beta^2(1 - x/\theta)^2} - \beta\right)}{\beta}, \tag{4.117}$$

where $\beta \in (0, 1), \theta > 0$, which satisfies conditions AA1–AA3.

### 4.5.2 Separability

We adopt the definition of separability for the one-factor LMMs from Pietersz et al. (2004).[15]

**Definition 4.16.** *(Separability) A one-factor LMM given by functions $\phi$, $\sigma^i, i = 1, \ldots n$, and a system of SDEs* (4.114) *is separable if there exists a function $\sigma : [0, T_n] \to \mathbb{R}$ and*

[14] Although one needs to impose an absorbing boundary at the origin in the CEV-LMM when $\beta < \frac{1}{2}$.

[15] Pietersz et al. (2004) define separability in the context of a $d$-factor log-normal LMM ($\phi(x) = x$). However, their definition does not depend on their choice of the local-volatility function and can be therefore adopted for any local-volatility function $\phi$.

*constants* $v_1, \dots, v_n \in \mathbb{R}$ *such that*

$$\sigma^i(t) = v_i \sigma(t), \quad t \le T_i, i = 1, \dots, n. \tag{4.118}$$

In practice one is only interested in separable LMMs with $\sigma^i \not\equiv 0, i = 1, \dots, n$. We can therefore assume without loss of generality that $\sigma \equiv \sigma^n$ and $v_n = 1$.

We now give a heuristic argument as to why a one-factor separable local-volatility LMM has dynamics similar to an MFM under the terminal measure driven by some one-dimensional Markov process. First we follow the idea of Pietersz et al. (2004) to remove the state dependence from the $dW$ part of the SDE (4.114). Then we will use separability to rewrite the SDE in terms of process $L^n$ as 'the driver' of the LMM.

Suppose that $\phi : A \to \mathbb{R}$ is a strictly positive and continuously differentiable function and assume that there exists a solution to the System of SDEs (4.114) for some initial term structure (note that the state space for each LIBOR rate is the set $A$). Then we can define a function $f : A \to \mathbb{R}$, by

$$f(x) := \int \frac{1}{\phi(x)} dx. \tag{4.119}$$

**Remark 4.17.** *In particular when* $A = (0, \infty)$, $\phi$ *satisfies conditions AA2 and AA3, and additionally* $\lim_{x \downarrow 0} \phi(x) = 0$, *the System of SDEs* (4.114) *will have a unique strictly positive solution if the initial term structure is strictly positive.*

Note that $f$ is twice continuously differentiable and applying Itô's Lemma to $f(L_t^i)$ yields

$$df(L_t^i) = v_i \sigma^n(t) \left( dW_t - \sigma^n(t) \sum_{j=i+1}^{n} \frac{\alpha_j v_j \phi(L_t^j)}{1 + \alpha_j L_t^j} dt \right) - \frac{1}{2} v_i^2 \sigma^n(t)^2 \phi'(L_t^i) dt, \tag{4.120}$$

in particular for $i = n$

$$df(L_t^n) = \sigma^n(t) dW_t - \frac{1}{2} \sigma^n(t)^2 \phi'(L_t^n) dt. \tag{4.121}$$

Then we can rewrite (4.120) as

$$\frac{df(L_t^i)}{v_i} = df(L_t^n) + \frac{\sigma^n(t)^2}{2} \big( \phi'(L_t^n) - v_i \phi'(L_t^i) \big) dt - \sigma^n(t)^2 \sum_{j=i+1}^{n} \frac{\alpha_j v_j \phi(L_t^j)}{1 + \alpha_j L_t^j} dt. \tag{4.122}$$

Equation (4.122) suggests that as long as the sum of the finite variation parts will be close to some deterministic function of time, we will observe the following relationship between the LIBORs:

$$f(L_t^i) \approx v_i f(L_t^n) + c^i(t), \quad t \le T_i, i = 1, \dots n, \tag{4.123}$$

for some deterministic functions $c^i, i = 1, \dots, n$. This should be understood in the sense that the value of $L_t^n$ has a strong predictive power about the value of $L_t^i$ according to the relationship in equation (4.123). This has been already observed by Bennett and Kennedy (2005) in the case of the log-normal separable LMM ($\phi(x) = x$ and $f(x) = \log(x)$). However, such a relationship also holds for more general local-volatility functions $\phi$.

In this case we can define a one-dimensional Markov process $x$ by $x_t = f(L_t^n), t \in [0, T_n]$, and rewrite equation (4.123) to

$$L_{T_i}^i \approx f^{-1}\big(v_i x_t + c^i(T_i)\big), \quad i = 1, \ldots, n. \tag{4.124}$$

In particular since $f^{-1}$ is an increasing function each LIBOR rate at its setting date will be an 'increasing function' of $x_{T_i}$.

Recall that there exists a unique MFM under the terminal measure driven a one-dimensional process $x$, calibrated to the same caplet prices as the separable LMM and such that the LIBOR rates at their setting dates are increasing functions of the state of the driving process. Therefore, when the relationship in equation (4.123) holds for the LMM we expect to observe a similar relationship between the functional forms of the LIBORs in the MFM under the terminal measure driven by $x = f(L_\cdot^n)$ and calibrated to the caplet prices from the LMM. Therefore, we can expect that the dynamics of the two models will be similar. In particular, we expect that behaviour of the future implied volatility surface to be similar in both models. This in turn allows us, to use the MFM as a computationally efficient alternative to the separable one-factor LMM and we can transfer the intuition about the LMM to the MFM.

Next, let us examine the drift part of SDE (4.122) more carefully. First we consider the term

$$\sigma^n(t)^2 \sum_{j=i+1}^{n} \frac{\alpha_j v_j \phi(L_t^j)}{1 + \alpha_j L_t^j}. \tag{4.125}$$

From the findings of Bennett and Kennedy (2005) we know that this term is 'well-behaved' for $\phi(x) = x$. In this case LIBORs take values in interval $(0, \infty)$ and each component of the sum is bounded and satisfies

$$\lim_{x \downarrow 0} \frac{\alpha_j v_j x}{1 + \alpha_j x} = 0 \qquad \text{and} \qquad \lim_{x \to \infty} \frac{\alpha_j v_j x}{1 + \alpha_j x} = v_j. \tag{4.126}$$

This suggests that for a general local-volatility function $\phi$ the drift part (4.125) will be well-behaved when

$$\lim_{x \downarrow \inf(A)} \frac{\alpha_j v_j \phi(x)}{1 + \alpha_j x} = 0 \qquad \text{and} \qquad \lim_{x \uparrow \sup(A)} \frac{\alpha_j v_j \phi(x)}{1 + \alpha_j x} < \infty. \tag{4.127}$$

Observe that when LIBORs take values in $(0, \infty)$ or $[0, \infty)$ and $\phi$ satisfies conditions AA1 and AA3 then the conditions given in equation (4.127) will be satisfied.

Let us now turn our attention to the second part of the drift term,

$$\frac{\sigma^n(t)^2}{2}\big(\phi'(L_t^n) - v_i \phi'(L_t^i)\big). \tag{4.128}$$

Note that in the log-normal case this term is deterministic, since $\phi(x) = x$ and $\phi'(x) = 1$ and therefore is not problematic. However, for a general local-volatility function it is much more difficult to analyse. Nevertheless, provided that $\phi$ has bounded derivative the drift term (4.128) will also be bounded. This requirement is stronger then conditions AA2 and AA3.

To verify our heuristic argument we have tested numerically the separable one-factor CEV and hyperbolic local-volatility LMMs for the presence of the relationship (4.123). It turns out that the relationship (4.123) has strong predictive power for both models under a wide range of scenarios. This is not surprising in the case of hyperbolic local-volatility since it satisfies conditions AA1–AA3 and has bounded derivative. On the other hand the CEV local-volatility function does not have a bounded derivative in the neighbourhood of zero and the drift term (4.128) could potentially be unbounded. Nevertheless, it turns out that as long the probability of absorption is low (which is the case for reasonable parameterisations) the relationship (4.123) has a strong predictive power.

## 4.6 LCEV Driving Process

In Section 4.5 we provided a heuristic argument to explain why a one-factor separable local-volatility LMM is close to some MFM under the terminal measure. In this section we apply these ideas to the one-factor separable CEV-LMM.

The CEV-LMM was proposed by Andersen and Andreasen (2000) as an extension of the LMM framework that can capture the skew in caplet implied volatilities typically observed in the market. A one-factor CEV-LMM under the terminal measure is given by the system of SDEs (4.114) and the local volatility function $\phi(x) = x^{\beta}, \beta \in (0, 1)$. Like the log-normal LMM, the CEV-LMM offers a closed form solution for the caplet-prices and good approximations for swaption prices, thus allowing for fast and efficient calibration. However, the CEV-LMM comes with a range of shortcomings. Firstly, it is a high dimensional model. Secondly, the LIBORs can get absorbed at the origin with a positive probability. This in combination with the fact that $\lim_{x\downarrow 0} \phi'(0) = \infty$ implies that the discussion from Section 4.5 cannot be applied directly to CEV-LMM.

One way to overcome the technical difficulties is by defining a stopping time $\tau$ to be the first time any of the LIBOR rates hits some level $\varepsilon > 0$, where $0 < \varepsilon < \max_i L_0^i$ and consider the stopped version of the CEV-LMM. Note that the state space of stopped LIBORs is the interval $[\varepsilon, \infty)$ on which $\phi$ has bounded derivative. The discussion from Section 4.5 then suggests that we will observe the relationship between the rates as described in equation (4.123) in the stopped version of the one-factor separable CEV-LMM.

On the other hand if $\varepsilon$ and the probability of rates absorbing at the origin are small enough the one-factor separable CEV-LMM and its stopped version will have similar dynamics. Thus suggesting that, the relationship in equation (4.123) will also have strong predictive power for the CEV-LMM. In this case we can rewrite equation (4.123) as

$$(L_t^i)^{\gamma} \approx v_i (L_t^n)^{\gamma} + c^i(t), \quad t \leq T_i, i = 1, \ldots, n, \tag{4.129}$$

where $\gamma = 1 - \beta$.

This gives a heuristic explanation for the numerical observation we made in Section 4.5 that the one-factor separable CEV-LMM is also close to some one-dimensional model in the sense of equation (4.129). However using $L^n$, a CEV process, as the driver of the approximating MFM under the terminal measure comes with a range of difficulties. Firstly, one needs to modify the algorithm to allow for atoms in the driving process. Secondly, if $\mathbb{F}^{n+1}(L^i_{T_i} = 0) < \mathbb{F}^{n+1}(L^n_{T_i} = 0)$ the model cannot calibrate to the caplet prices.

To overcome this, we go back to Andersen and Andreasen (2000) who acknowledged that rates absorbing at the origin is a qualitatively undesirable feature. As an alternative they proposed a 'limited' CEV (LCEV) local-volatility function $\tilde{\phi}$ defined by

$$\phi^{\mathrm{LCEV}}(x) = x\big(x^{\beta-1} \wedge \varepsilon^{\beta-1}\big), \tag{4.130}$$

where $\varepsilon > 0$. Observe that the CEV and LCEV local-volatility functions are the same when $x > \varepsilon$. Therefore, the CEV-LMM and the LCEV-LMM will at least locally have the same dynamics when rates are larger than $\varepsilon$. Andersen and Andreasen (2000) compared the swaption prices in the two models and found that the closed form formulae for caplet prices and approximations for swaption prices from the CEV-LMM can be used in the LCEV-LMM.

Based on these observations we propose to use the LCEV process $x$ given by

$$dx_t = \sigma^n_t x_t\big(x_t^{\beta-1} \wedge \varepsilon^{\beta-1}\big)dW_t, \quad x_0 = L^n_0, \tag{4.131}$$

as the driving process for an MFM under the terminal measure calibrated to the caplet prices from the one-factor separable CEV-LMM.

In the next section we demonstrate with an example that such an MFM is indeed close to the CEV-LMM and that the behaviour of future implied volatilities is similar in both models.

## 4.7 Numerical results

In this section we give a numerical example demonstrating the ideas from Sections 4.5 and 4.6. We first consider a one-factor separable CEV-LMM and show by considering equation (4.129) that it can be approximated well by some one-dimensional model. We then define an MFM under the terminal measure driven by an LCEV process and calibrated to caplet prices from the CEV-LMM as suggested in Section 4.6. We show that the relationship in equation (4.129) holds to a good approximation in the MFM. Next, we compare the future implied volatilities of caplets in both models and demonstrate that they behave similarly. Finally, we give an example where a 'bad' combination of caplet prices and driving process leads to undesirable dynamics of future implied volatilities.

In the following example we use a one-factor separable CEV-LMM with parameters as summarised in Table 4.1 as the reference model. For our approach to work equation (4.129)

Table 4.1: Parameters used to define the one-factor separable CEV-LMM.

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $n$ | 10 | $L_0^i$ | 0.06 |
| $T_i$ | $i$ | $\sigma_t^i$ | 0.05 |
| $\beta$ | 0.4 | $\gamma$ | 0.6 |

needs to have strong predictive power in the CEV-LMM. To test for this we simulated 100,000 paths of the CEV-LMM using the Euler scheme with time step 0.02 and performed a linear regression to estimate parameters $a_i$ and $b_i$ in

$$(L_{T_i}^i)^\gamma = a_i(L_{T_i}^n)^\gamma + b_i + \epsilon_i, \quad i = 1, \ldots, n-1, \tag{4.132}$$

where $\epsilon_i$ represents the error term.

Table 4.2 summarises the results. Note that the coefficient of determination $R^2$ is above 99% in all cases. This indicates that the CEV-LMM is close to some one-dimensional model. Now, let us turn our attention to the coefficient $a_i$. Recall that we have predicted in equation (4.129) that $a_i$ takes the value of the separability constant $v_i$, which is in our case equal to 1 since $\sigma_t^i = \sigma_t^n, t \leq T_i$. Observe that for all $i \in \{1, \ldots, n-1\}$ the coefficient $a_i$ is indeed close to 1. The fact that it is systematically below 1 and that the coefficients $b_i, i = 1, \ldots, n$, are systematically below zero is a consequence of the drift in the SDE (4.114) being negative on average. Most importantly, as discussed in Section 4.5, these results suggest that the CEV-LMM is a good guide for an MFM under the terminal measure.

The above results come with a small caveat. The parameterisation chosen in Table 4.1 favours the problem in the sense that the probability of forward rates absorbing at the origin is low. In particular, we found similar results to hold for increasing and decreasing initial term structures and for other one-factor separable volatility functions provided that the probability of absorption remained low. When this is not the case the relationship breaks down, however in such case CEV-LMM might not be the most appropriate model.

Table 4.2: Summary of results of the regressions $(L_{T_i}^i)^\gamma = a_i(L_{T_i}^n)^\gamma + b_i + \epsilon_i$.

| $i$ | $a_i$ | $b_i$ | $R^2$ |
|---|---|---|---|
| 1 | 0.9941 | -0.0030 | 0.9999 |
| 2 | 0.9894 | -0.0054 | 0.9999 |
| 3 | 0.9863 | -0.0071 | 0.9999 |
| 4 | 0.9844 | -0.0082 | 0.9996 |
| 5 | 0.9832 | -0.0085 | 0.9992 |
| 6 | 0.9828 | -0.0080 | 0.9988 |
| 7 | 0.9838 | -0.0068 | 0.9985 |
| 8 | 0.9866 | -0.0050 | 0.9988 |
| 9 | 0.9920 | -0.0028 | 0.9994 |

Following the discussion from Section 4.6 we wish to define an MFM under the terminal measure calibrated to caplet prices from the CEV-LMM and driven by a process $x$, satisfying the SDE

$$dx_t = \sigma_t^n x_t \big(x_t^{\beta-1} \wedge \varepsilon^{\beta-1}\big) dW_t, \quad x_0 = L_0^n,\ \varepsilon = 0.001.^{16} \tag{4.133}$$

Since the LCEV process is not a Gaussian process, we cannot use the standard numerical implementation of the MFM under the terminal measure, but we can use the algorithm introduced in Section 4.2. Recall that the algorithm requires us to choose the grid-points, basis functions and evaluate the conditional expectations in equation (4.21). The LCEV process does not have a closed form expression for the transition densities, but as discussed in Remark 4.6 this can be overcome by discretising the driving process in time.

We have chosen to discretise the driving process by using a log-Euler scheme with time steps of the size $\Delta t = 1/16$. Such an approach leads to a model driven by an approximation of a LCEV process and the size of the time step $\Delta t$ effectively controls the quality of the approximation. For this reason choosing $\Delta t = 1$ can (and as we will later observe) lead to different dynamics of the MFM.

The grid-points were chosen so that the probability of the process $x$ taking value in-between the smallest and the largest valued grid-point at each step was approximately 99.9999% and we used 150 grid-points at each time step. We choose to use the basis functions of order 5 as described in Appendix A with constant extrapolation in the upper and lower tail.

**Remark 4.18.** *Clearly there are other choices one can make to obtain the building blocks needed to implement the algorithm. In particular, one could potentially use a higher order scheme to reduce the number of time steps in-between the setting dates.*

With these choices the MFM can be calibrated to the caplet prices from the CEV-LMM using the algorithm from Section 4.2. We have already shown that the relationship (4.129) has a strong predictive power in the CEV-LMM, if the MFM has indeed similar dynamics as the CEV-LMM we expect to observe similar relationship between the functional forms in the MFM.

Figure 4.1 shows the plots of the functional form of $(L^i_{T_i})^\gamma$ as a function of $(L^n_{T_i})^\gamma$ in the MFM and a scatter plot (100 paths) of $(L^i_{T_i})^\gamma$ versus $(L^n_{T_i})^\gamma$ in the CEV-LMM for $i \in \{3, 7\}$. Observe that the scatter plots almost exactly overlay the functional forms obtained from the MFM ($\Delta t = 1/16$) and a similar observation holds for other times and rates. This indicates that the joint distribution of the LIBORs at a given date $(L^i_{T_i}, \ldots, L^n_{T_n})$ is close in the two models.

Note also that the relationship between the forward rates $(L^7_{T_7})^\gamma$ and $(L^n_{T_7})^\gamma$ for the MFM in Figure 4.1 is non-linear near the origin. This is because $\mathbb{F}^{n+1}(L^i_{T_i} = 0) > 0$ but the atom at zero disappears for $L^i_{T_j}, j < i$, since the LCEV driving process has continuous transition densities.

Having compared the models' joint distributions at a given setting date, we now wish to compare their dynamics. We do so by comparing their future implied volatilities of caplet

[16] It turns out that choice of $\varepsilon$ does not affect the model significantly.

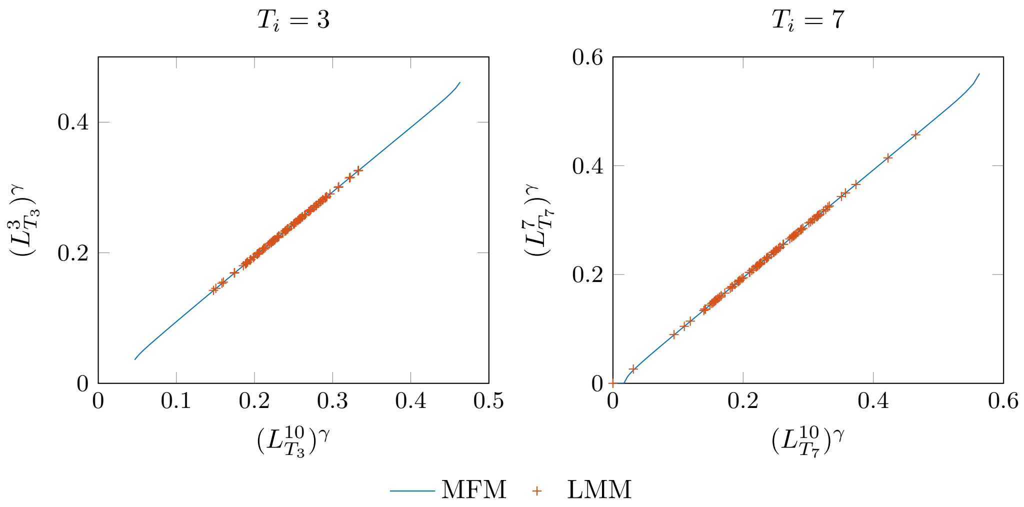



Figure 4.1: $(L^i_{T_i})^\gamma$ as a function of $(L^{10}_{T_i})^\gamma$ in the MFM for $\Delta t = 1/16$ (solid line) and a scatter plot of $(L^i_{T_i})^\gamma$ versus $(L^{10}_{T_i})^\gamma$ in the CEV-LMM (crosses).

prices. Note that a time $T_j$ price of a caplet on the $L^i_{T_i}, i > j$ with a strike $K$ is a random variable

$$V^i_{T_j}(K) = D_{T_j,T_{n+1}}\left[\frac{D_{T_i,T_{i+1}}}{D_{T_i,T_{n+1}}}(L^i_{T_i} - K)_+\middle|\mathcal{F}_{T_j}\right]. \tag{4.134}$$

In particular, in the MFM it is enough to condition on the state of the driving process $x_{T_j}$. On the other hand, in the CEV-LMM one needs to condition on the state of the vector $(L^j_{T_j}, \ldots, L^n_{T_j})$. Therefore, $V^i_{T_j}$ is a function of $x_{T_j}$ in the MFM and a function of $(L^j_{T_j}, \ldots, L^n_{T_j})$ in the CEV-LMM.

This allows us to compare the two models in the following way. First, we can evaluate the value of $V^i_{T_j}$ for a given value of $x_{T_j} = x^*$ in the MFM. Recall, that the state of the driving process in the MFM determines the state of the economy and in particular the values of the LIBORs. We can then determine the value of $V^i_{T_j}$ in the CEV-LMM by conditioning on the values of the LIBORs from the MFM $L^k_{T_j}(x^*), k = j, \ldots, n$. Since we have observed the relationship (4.129) that the CEV-LMM is essentially a one-dimensional model and such a choice ensures that the state of the economy in the CEV-LMM is the same as in the MFM we can view the differences in the forward caplet prices as only coming from the differences in models' dynamics.

Figure 4.2 shows the plots of time $T_2$ and $T_5$ prices versus strike of a caplet written on LIBOR $L^7_{T_7}$ expressed as implied volatilities in the CEV-LMM, MFM with $\Delta t = 1/16$ and MFM with $\Delta t = 1$. We have chosen the values of the $x_{T_j}$ so that $L^7_{T_j}$ took the values: 0.03 (top three lines), 0.06 (middle three lines) and 0.09 (bottom three lines). Observe that the implied volatilities in the MFM with $\Delta t = 1/16$ and the CEV-LMM are very close and there are only minor differences when the caplet approaches maturity. In particular, the skew in the implied volatilities in the MFM has been preserved. On the other hand the implied volatilities in the MFM with $\Delta t = 1$ are not only shifted away from what we observe in the

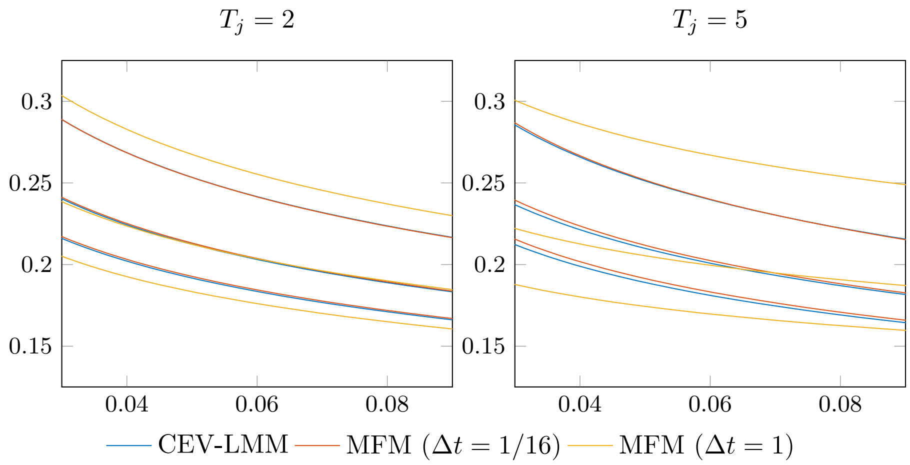



Figure 4.2: Time $T_j, j \in \{2, 5\}$ forward implied volatilities versus strike of the caplet on $L^7_{T_7}$ in the CEV-LMM (blue), MFM with $\Delta t = 1/16$ (red) and $\Delta t = 1$ (yellow) for the levels of $L^7_{T_j}$: 0.03 (top three lines), 0.06 (middle three lines) and 0.09 (bottom three lines).

CEV-LMM but also become less skewed as the caplet approaches maturity.

Similar results hold for other caplets. The implied volatilities in the MFM with $\Delta t = 1/16$ are almost indistinguishable from the ones in the CEV-LMM when the time to maturity of a caplet is large and we can observe minor differences when the caplet is close (one and two setting dates) to maturity. On the other hand in the MFM with $\Delta t = 1$, the differences between the two models occur much sooner. This indicates that simply discretising the LCEV process in-between the setting dates leads to an approximation with significantly different dynamics (transition probabilities).

We can draw two conclusions from Figure 4.2. Firstly, it is the combination of the choice of a driving process and caplet prices which determines the joint distribution of LIBORs at their setting date and the features of the MFM. In this section we have an example where a different choice of the driving process for given caplet prices has a significant effect on the forward implied volatilities. Secondly, in the MFM driven by the LCEV process with $\Delta t = 1/16$ the evolution of the forward implied volatilities was stable and similar to the one observed in the CEV-LMM and therefore the MFM represents an arbitrage-free and computationally efficient alternative to the CEV-LMM.

## 4.8 Conclusion

In this chapter we addressed two important issues related to one-dimensional MFMs driven. Firstly, we described two new algorithms that can be used to implement the MFM under the terminal and the spot measure for a one-dimensional (not-necessarily Gaussian) driving process.

Secondly, we have shown how the driving processes can be analysed using the copula theory and shown that it is only the dependence structure of the driving process that influences the model dynamics.

Finally, by exploring the link between the one-factor separable local-volatility LMM and the MFM under the terminal measure, we described a systematic approach that can be used to specify MFMs with stable evolution of future caplet implied volatility surface. Moreover, the resulting MFM can be viewed as a one-dimensional arbitrage-free approximation to the LMM. We have demonstrated our approach by approximating a one-factor separable CEV-LMM. This is arguably the most challenging model to approximate amongst the popular local-volatility LMMs due to LIBORs absorbing at the origin with positive probability.

To conclude let us point out some interesting possible applications of our findings and some ideas for future research.

The use of copula theory in the context of MFMs provides us with exciting new possibilities both for defining the MFMs and implementing them. From the implementation standpoint it seems that using processes with uniform marginal distributions would be desirable as it would allow for the grid-points to be 'evenly spaced' in the sense of the probability mass between two consecutive grid-points. From the theoretical perspective it gives us a simple way to move away from processes with Gaussian dependence structure as there exist several non-Gaussian parametric families of copulae that are consistent with the Markov property (see Darsow et al. (1992)).

Another interesting question worth pursuing is the relationship between the MFM under the spot measure and the LMM. Pietersz et al. (2004) claimed that the one time step approximation to the separable LMM can only be used to implement the LMM under the terminal measure as it avoids the problems occurring from numeraire being path-dependent. However, as we have seen in Section 4.3 the path-dependence of the rolling bank account numeraire can be dealt with efficiently and similar ideas can be used to implement a one-factor separable LMM.

# Chapter 5

# A Two-currency Markov-functional Model under the Spot Measure

In the previous chapters we have been interested in the economies consisting of ZCBs denominated in the same currency. In this chapter we extend the economy of our interest by adding to it ZCBs denominated in a different currency.

In the context of LIBOR market model the two-currency extension have been introduced by Schlögl (2002) and subsequently studied Benner et al. (2009), Mikkelsen (2002) and others. While the LIBOR market models offer a theoretically appealing approach for modelling the two-currency economy they suffer from the high-dimensionality as outlined in Section 2.3 and later in Chapter 3.

To avoid the problem of high-dimensionality we will be interested in the two-currency extensions of the Markov-functional model. In the context of MFMs a two-currency model was first introduced by Fries and Rott (2004). A special case where the dynamics of the foreign currency are deterministic was also presented by Fries and Eckstaedt (2011). The models introduced in the two papers use the Markov-functional approach to calibrate to the domestic and foreign interest rate markets and a parametric approach to model the foreign exchange market.

The aim of this chapter is to present a Markov-functional model of a two-currency economy that uses the 'Markov-functional sweep' – uses the option prices to determine the functional forms – to calibrate to both interest rate markets and to the foreign exchange market.

The remainder of the chapter is structured as follows. In Section 5.1 we formally define the two-currency economy and discuss how it fits in the arbitrage pricing theory framework from Section 2.1. In Section 5.2 we define a two-currency MFM and review the approach taken by Fries and Rott (2004). In Section 5.3 we propose a new version of a two-currency MFM under the spot measure. Section 5.4 discusses its numerical implementation on a grid and Section 5.5 concludes.

## 5.1 Two-currency Economy

Let $0 = T_0 < T_1 < \ldots < T_{n+1}$ be an increasing sequence of dates. Of our interest will be an economy consisting of ZCB denominated in two different currencies – referred to as the domestic and the foreign currency – maturing on the dates $T_i, i = 1, \ldots, n+1$. More precisely, for each $i = 1, \ldots, n+1$, there exist two ZCBs maturing at time $T_i$, one denominated in the domestic currency and one denominated in the foreign currency. We will refer to them as the domestic ($T_i$-maturity) ZCB and the foreign ($T_i$-maturity) ZCB and denote their time $t \leq T_i$ values by $D_{t,T_i}$ and $\tilde{D}_{t,T_i}$ respectively. Note that the prices of domestic ZCBs are denominated in the domestic currency and the prices of foreign ZCBs are denominated in the foreign currency.

We will assume that the two currencies can be exchanged at any time without frictions and will denote by $FX = (FX_t)_{t\in[0,T_{n+1}]}$ the (spot) foreign exchange rate process in the direct quotation. In particular, $FX_t, t \leq T_{n+1}$, is the price of a unit of the foreign currency denominated in the domestic currency. Furthermore, we will assume that the foreign exchange process is strictly positive and finite-valued $\mathbb{P}$-almost surely.[1]

Note that the arbitrage pricing theory as presented in the Section 2.1 assumed that the economy consists of assets denominated in the same currency and cannot be applied directly to the two-currency setting. We will overcome this issue by taking the foreign ZCBs denominated in the domestic currency, $FX_{\cdot}\tilde{D}_{\cdot,T_i} = (FX_t\tilde{D}_{t,T_i})_{t\in[0,T_i]}, i = 1, \ldots, n+1$, as the fundamental assets alongside the domestic ZCBs.[2]

This allows us to use all the tools presented in Section 2.1, in particular we know that if there exists a numeraire pair, such an economy is arbitrage-free (in the sense of Definition 2.5) and the fundamental pricing formula (2.9) holds for the replicable contingent claims. However, it is beneficial to introduce more 'natural' terminology when considering the foreign denominated claims.

We will say that $\tilde{V}_T$ is an attainable foreign claim expiring at time $T \leq T_{n+1}$ if $V_T := FX_T\tilde{V}_T$ is an attainable claim in the sense of Definition 2.7. Note that for the claim $V_T$ its time $t \leq T$ price $V_t$ and can be obtained from the fundamental pricing formula. This allows us to define the foreign currency denominated time $t \leq T$ price $\tilde{V}_t$ of $\tilde{V}_T$ by $\tilde{V}_t := \frac{V_t}{FX_t}$. Moreover, it is easy to see that for any numeraire pair $(N, \mathbb{N})$

$$\tilde{V}_t = \frac{N_t}{FX_t}\mathbb{E}_{\mathbb{N}}\left[\frac{FX_T\tilde{V}_T}{N_T}\middle|\mathcal{F}_t^A\right]. \tag{5.1}$$

We will refer to equation (5.1) as the fundamental pricing formula for foreign claims.

Now we can define the concept of foreign numeraire. We will say that a strictly positive process $\tilde{N} = (\tilde{N}_t)_{t\in[0,T_{n+1}]}$ is a foreign numeraire if $\tilde{N}_{T_{n+1}}$ is an attainable foreign claim and $\tilde{N}$ is its foreign denominated price process.

---

[1] Recall, we are working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

[2] Of course we need to extend the time domain of the ZCBs to $[0, T_{n+1}]$ in the sense of Remark 2.13 or 2.14.

Note that if $\tilde{N}$ is a foreign numeraire then $FX\tilde{N} := (FX_t\tilde{N}_t)_{t\in[0,T_{n+1}]}$ is a numeraire in the sense of Definition 2.1. Moreover, if the economy is arbitrage-free there exists an equivalent martingale measure $\tilde{\mathbb{N}}$ associated with $FX\tilde{N}$. Then the fundamental pricing formula (2.9) can be expressed in terms of $(FX\tilde{N}, \mathbb{N})$ as

$$\frac{V_t}{FX_t} = \tilde{N}_t \mathbb{E}_{\tilde{\mathbb{N}}}\left[\frac{V_T}{FX_T\tilde{N}_T}\middle|\mathcal{F}_t^A\right] \tag{5.2}$$

and the fundamental pricing formula for foreign claims (5.1) can be expressed in terms of $(FX\tilde{N}, \mathbb{N})$ as

$$\tilde{V}_t = \tilde{N}_t \mathbb{E}_{\tilde{\mathbb{N}}}\left[\frac{\tilde{V}_T}{\tilde{N}_T}\middle|\mathcal{F}_t^A\right]. \tag{5.3}$$

The importance of equations (5.2) and (5.3) is in showing that by changing the measure to the one associated with a foreign numeraire process, we effectively obtain the pricing formulae that one would get if we have chosen view the two-currency economy in the single currency economy denominated in foreign currency.

**Remark 5.1.** *A careful reader will notice that there is a slight difference between the two choices for the currency which is used to define the single-currency 'embedding' of the two-currency economy. In particular, the asset generated filtration $\mathcal{F}^A$ as defined in equation* (2.2) *will in general depend on the choice of the domestic currency. However, this is a minor difference which is not of practical importance. In particular, in the rest of the chapter we will simply condition on $\mathcal{F}_t$ in the fundamental pricing formula (see also Remark 2.10).*

Having defined the economy of our interest we can define the domestic and foreign deposits and forward rate agreements similarly as in Section 2.2. Which in turn allow us to define forward and spot LIBOR rates. For $i = 1, \ldots, n$, and $t \leq T_i$ we will denote the time $t$ value of the domestic forward/spot LIBOR by $L_t^i$ and the value of the foreign forward/spot LIBOR by $\tilde{L}_t^i$. One can than show using the same arguments as in Section 2.2 that

$$L_t^i = \frac{D_{t,T_i} - D_{t,T_{i+1}}}{\alpha_i D_{t,T_{i+1}}} \tag{5.4}$$

$$\tilde{L}_t^i = \frac{\tilde{D}t, T_i - \tilde{D}_{t,T_{i+1}}}{\alpha_i \tilde{D}_{t,T_{i+1}}}, \tag{5.5}$$

note that we implicitly assumed that the accrual factors $\alpha_i, i = 0, \ldots, n$, are the same in both economies, this can be easily relaxed and allow the accrual factors to be different.[3]

In similar fashion we can then define domestic and foreign swaptions and caplets. As the reader will probably guess we will add a tilde to the to the notation for the domestic instrument prices to denote the prices of foreign counterparts, in particular we will denote the time $t \leq T_i, i \in \{1, \ldots, n\}$ price of the digital caplet in-arrears with strike $K$ written on $L_{T_i}^i$ and $\tilde{L}_{T_i}^i$ by $V_t^{\text{dca},i}(K)$ and $\tilde{V}_t^{\text{dca},i}(K)$ respectively.

[3]For example it might be the case that two economies have different day-count conventions.

## 5.2 Two-currency Markov-functional Models

Recall that in a (single-currency) MFM the prices of (domestic) ZCBs can be expressed as functions of some driving process $x$ which is a Markov process under some equivalent martingale measure $\mathbb{N}$ corresponding to a numeraire process $N$. We now generalise this to a two-currency economy.

In a two-currency economy we will consider a model to be Markov-functional if there exist a triplet of processes $(x, y, z)$ which are Markov under the measure $\mathbb{N}$ and with respect to the augmented natural filtration generated by them. Moreover, we will additionally require that

1. for $i \in \{1, \ldots, n\}$ and $t \leq T_i$ the time $t$ price of the domestic $T_i$-maturity ZCB $D_{t,T_i}$ is a function of $x_t$;
2. for $t \leq T_{n+1}$ the time $t$ value of the foreign exchange rate $FX_t$ is a function of $y_t$;
3. for $i \in \{1, \ldots, n\}$ and $t \leq T_i$ the time $t$ price of the foreign $T_i$-maturity ZCB $\tilde{D}_{t,T_i}$ is a function of $z_t$.

Of our interest will be Markov-functional models driven by one-dimensional processes $x$, $y$ and $z$. In particular, we wish to construct a Markov-functional model by calibrating it to prices of domestic and foreign caplets and foreign exchange call options in the spirit of the MFMs from Sections 4.2 and 4.3.

As Fries and Rott (2004) observed this turns out to be a difficult problem. One of the reasons for this comes form the fact that when the numeraire process $N$ is a function of process $x$ only (for example if we take domestic $T_{n+1}$-maturity ZCB or a domestic discretely compounded rolling bank account as the numeraire) it is easy to see that processes $x$, $y$ and $z$ cannot be independent.

Fries and Rott (2004) address this issue by discretising the processes $x$, $y$ and $z$ in time and assumes that they are of the form

$$x_{T_i} = x_{T_{i-1}} + \sigma^x_{i-1}(W^x_{T_i} - W^x_{T_{i-1}}), \tag{5.6}$$

$$y_{T_i} = y_{T_{i-1}} + \sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}), \tag{5.7}$$

$$z_{T_i} = z_{T_{i-1}} + \sigma^z_{i-1}(W^z_{T_i} - W^z_{T_{i-1}}), \tag{5.8}$$

where $W^x, W^y$ and $W^z$ are independent Brownian motions, $\sigma^x_{i-1}, \sigma^y_{i-1}, \sigma^z_{i-1} > 0$, and $\mu_{i-1}$ is a drift term that is determined during the calibration process.

Under the above assumptions, it turns out that for any foreign claim $\tilde{V}_{T_i}$ expiring at time $T_i$ its time $T_j < T_i$ price is given by

$$\tilde{V}_{T_j} = \tilde{B}_{T_j} \mathbb{E}_{\mathbb{N}}\left[ \frac{\tilde{V}_{T_i}}{\tilde{B}_{T_i}} \middle| \mathcal{F}_{T_j} \right] \tag{5.9}$$

as long as the numeraire process $N$ is dependent on the process $x$ only (see Fries and Rott (2004)). Note that in equation (5.9) the foreign claim is discounted using the foreign rolling bank account $\tilde{B}$ but the expectation is taken with respect to the domestic EMM $\mathbb{N}$. This turns out to be the consequence of the fact that the one time-step increments of the discretised processes $x$, $y$ and $z$ are conditionally independent.

To retain the 'symmetry' – in the sense that the domestic and foreign currency can be interchanged – in their model, Fries and Rott (2004) choose to set up the model under the domestic spot measure. They calibrate the domestic and foreign ZCB markets to the domestic and foreign digital-caplets in-arrears – note that equation (5.9) allows this to be done using the approach from Section 4.3 independently for each of the currencies. Then they choose to calibrate the foreign exchange rate market iteratively forwards in time to prices of call options by using a parametric functional form to model the dependence of the foreign exchange rate on process $y$.

In the next section we will propose a construction that will allow for the Brownian motions $W^x, W^y$ and $W^z$ to be dependent and that will also use the Markov-functional sweep to calibrate the model to the prices of foreign exchange options.

## 5.3 The Model

In this section we propose a new algorithm that for constructing a two-currency MFM under the domestic spot measure. As was proposed by Fries and Rott (2004) we will discretise the driving process $(x, y, z)$ in time. In particular, we will assume that the driving process is of the form

$$x_{T_i} = x_{T_{i-1}} + \sigma^x_{i-1}(W^x_{T_i} - W^x_{T_{i-1}}), \tag{5.10}$$

$$y_{T_i} = \sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}), \tag{5.11}$$

$$z_{T_i} = z_{T_{i-1}} + \sigma^z_{i-1}(W^z_{T_i} - W^z_{T_{i-1}}), \tag{5.12}$$

where $x_0 = y_0 = z_0 = 0$, $W^x, W^y$ and $W^z$ are possibly correlated Brownian motions, $\sigma^x_{i-1}, \sigma^y_{i-1}, \sigma^z_{i-1} > 0$, and $\mu_{i-1}$ is a drift term that will be determined during the construction.

Next we make the following assumptions

1. In our model:

   1.1. $L^i_{T_i}, i = 1, \ldots, n$, can be written as an increasing càdlàg function of $x_{T_i}$;

   1.2. $FX_{T_i}, i = 1, \ldots, n+1$, can be written as an increasing càdlàg function of $y_{T_i}$;

   1.3. $\tilde{L}^i_{T_i}, i = 1, \ldots, n$, can be written as an increasing càdlàg function of $z_{T_i}$;

2. We are given:

   2.1. the initial value of the $T_1$-maturity domestic ZCB $D_{0,T_1}$ and prices of the *digital caplets in-arrears* written on $L^i_{T_i}, i = 1, \ldots, n$, for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$V_0^{\mathrm{dca},i}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{L^i_{T_i}>K\}}\right]; \tag{5.13}$$

2.2. the prices of digital call options written on the time $T_i, i = 1, \ldots, n+1$ value of the foreign exchange rate for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$V_0^{\mathrm{dFX,i}}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{FX_{T_i}>K\}}\right]; \tag{5.14}$$

2.3. the initial value of the $T_1$-maturity foreign ZCB $\tilde{D}_{0,T_1}$ and prices of the *digital caplets in-arrears* written on $\tilde{L}^i_{T_i}, i = 1, \ldots, n,$ for strikes $K \geq 0$ which are represented by a decreasing càdlàg function

$$\tilde{V}_0^{\mathrm{dca},i}(K) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\mathbf{1}_{\{\tilde{L}^i_{T_i}>K\}}\right]. \tag{5.15}$$

Note that assumptions 1.1 and 2.1 are exactly the same as assumptions 1 and 2 in the construction of the single-currency MFM under the spot measure in Section 4.3. Assumptions 1.1, 1.2 and 1.3 will ensure that the 'Markov-functional sweep' can be performed. On the other hand assumptions 2.1, 2.2 and 2.3 provide us with the market data needed to calibrate the model.

### 5.3.1 Main Idea

Ideally the one would like to perform the construction of an MFM iteratively forwards in time by performing the following steps at time $T_i, i \in \{1, \ldots, n\}$

1. Recover the functional form of $L^i_{T_i}$ from prices of domestic digital caplets in-arrears;
2. Recover the functional form of $FX_{T_i}$ from prices of digital foreign exchange call options;[4]
3. Recover the functional form of $\tilde{L}^i_{T_i}$ from prices of foreign digital caplets in-arrears.

Unfortunately, the procedure is not so straightforward. The reason for this comes from the fact that a model of a two-currency economy is determined by the dynamics of the numeraire – in our case the discretely compounded rolling bank account $B$ – and of the spot foreign exchange rate process $FX$. In particular, note that $FX_{T_{i-1}}\tilde{D}_{T_{i-1},T_i}$ is the time $T_{i-1}$ price of the claim paying $FX_{T_i}$ at time $T_i$ and therefore

$$FX_{T_{i-1}}\tilde{D}_{T_{i-1},T_i} = B_{T_{i-1}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\middle|\mathcal{F}_{T_{i-1}}\right]. \tag{5.16}$$

[4]Note that we need to perform this step also for time $T_{n+1}$.

Observing that $B_{T_i}$ is $\mathcal{F}_{T_{i-1}}$-measurable and that $\tilde{D}_{T_{i-1},T_i} = (1+\alpha_i \tilde{L}^{i-1}_{T_{i-1}})^{-1}$ then allows us to rewrite equation (5.16) as

$$FX_{T_{i-1}} \frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}} = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}\big|\mathcal{F}_{T_{i-1}}\big]. \tag{5.17}$$

In particular, observe that the functional form of $\tilde{L}^{i-1}_{T_{i-1}}$ which was determined in the previous step is also uniquely determined by the functional forms of $L^{i-1}_{T_{i-1}}$, $FX_{T_{i-1}}$ and $FX_{T_i}$. Or alternatively, the functional form of $FX_{T_i}$ has to be chosen so that equation (5.17) holds. This demonstrates the importance of the flexibility to choose the drift during the calibration. Had we chosen the drift term $\mu_{i-1}$ in advance, calibrating to the digital swap options would in general result in a functional form for $FX_{T_i}$ that would not satisfy equation (5.17).

However, the ability to freely choose the drift term does not solve problems of performing the step 2 in the above procedure entirely. In particular, it is not trivial to determine the drift term $\mu_{i-1}$ and the functional form of $FX_{T_i}$, because to determine one we need to know the other. From a theoretical perspective this is not a problem as we only need a solution – pair of functional forms $(FX_{T_i}, \mu_{i-1})$ – to exist. However, to apply the model in practice we need to be able to construct the solution, which is not a trivial task.

Here we propose to find a suitable pair $(FX_{T_i}, \mu_{i-1})$ using a predictor-corrector type of approach. In particular, we propose to use the following procedure instead of step 2:

2.1. Choose an initial functional form for $\mu_{i-1}$;

2.2. Determine the functional from of $FX_{T_i}$ by calibrating to digital call option prices;

2.3. Adjust the drift so that equation (5.17) is satisfied;

2.4. Re-evaluate prices of digital call options, if the fit to the market is acceptable proceed to step 3, otherwise go to step 2.1.

Our conjecture is that for a reasonable initial choice of $\mu_{i-1}$ the above algorithm converges.

In the next three subsections we describe in more detail how the $i$th time step of the proposed algorithm can be implemented. In particular, we assume that prior to $i$th step we have already recovered the functional forms of $L^j_{T_j}$, $\tilde{L}^j_{T_j}$, $FX_{T_j}$ and $\mu_{j-1}$ for $j \in \{1, \dots i-1\}$.

### 5.3.2 Calibration to the Domestic Digital Caplets In-arrears

We can determine the functional form of $L^i_{T_i}$ and consequently of $B_{T_{i+1}}$ from the prices of digital caplets in-arrears as was done in Section 4.3 for the single-currency MFM under the spot measure.

First, we define a function $J^{x,i}$ by

$$J^{x,i}(x^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{x_{T_i}>x^*\}}\right]. \tag{5.18}$$

Note that, we have already determined the functional form of $B_{T_i}$ in the previous step and therefore the function $J^{i,x}$ is well defined. We can now use assumptions 1.1 and 2.1 to recover the functional form of $L^i_{T_i}$ using the same argument as in Section 4.3, in particular

$$L^i_{T_i}(x^*) = \sup\big\{K \geq 0; V_0^{\mathrm{dca},i}(K) \geq J^{x,i}(x^*)\big\}. \tag{5.19}$$

Observe that determining the functional form of $L^i_{T_i}$ involves only integration over the joint distribution of process $x$. Therefore, one can perform it for all the time step independently of the calibration to the foreign exchange and the foreign interest rate markets.

### 5.3.3 Calibration to the Foreign Exchange Digital Call Options

To calibrate the model at time step $T_i$ to the prices of digital call options we have proposed a predictor-corrector type scheme. Before describing its two main steps, we need to choose an initial value for of the drift term. To do so, let us return back to equation (5.17)

$$FX_{T_{i-1}} \frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}} = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}\big|\mathcal{F}_{T_{i-1}}\big].$$

By assumption 1.2 $FX_{T_i}$ is an increasing function of

$$y_{T_i} = \sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}).$$

In particular, the Brownian increment $W^y_{T_i} - W^y_{T_{i-1}}$ is independent of the $\sigma$-algebra $\mathcal{F}_{T_{i-1}}$ while the drift term $\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})$ is $\mathcal{F}_{T_{i-1}}$-measurable. Therefore the conditional expectation on the right-hand side of equation (5.17) has to be $\sigma(\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}))$-measurable.

On the other hand, we know by observing the left-hand side of (5.17) the exact functional form of the conditional expectation on the right-hand side. Consequently, the drift term has to be measurable with respect to $\sigma$-algebra $\sigma\big(FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\big)$ where

$$FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}) := FX_{T_{i-1}}(y_{T_{i-1}}) \frac{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}{1+\alpha_{i-1}\tilde{L}^{i-1}_{T_{i-1}}(z_{T_{i-1}})} \tag{5.20}$$

is the time $T_{i-1}$ value of time $T_i$ forward foreign exchange rate and we can write

$$\mu_{i-1}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}}) = \mu_{i-1}\Big(FX_{T_{i-1},T_i}(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\Big). \tag{5.21}$$

Moreover, we can write equation (5.17) as

$$FX_{T_{i-1},T_i} = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}\big|\mathcal{F}_{T_{i-1}}\big]. \tag{5.22}$$

**Remark 5.2.** *Note that equation* (5.22) *is the reason for a difference between our setup of the process $y$ in* (5.11) *and the one in Fries and Rott (2004) in* (5.7)*. In particular, it allows* *us for the drift to be dependent on the* $(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})$ *only trough* $FX_{T_{i-1},T_i}$.

Having, fixed the initial drift, we can now describe the predictor step, that is recovering the functional form of $FX_{T_i}$ from foreign exchange digital call option prices. Similarly to the previous step we now define a function $J^{y,i}$ by

$$J^{y,i}(y^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{y_{T_i}>y^*\}}\right]. \tag{5.23}$$

Note, that the functional form of the numeraire $B_{T_i}$ is known and that we have already fixed the drift term $\mu_{i-1}$ and consequently the distribution of $y_{T_i}$. Therefore, the function $J^{y,i}$ is well defined. We can then use assumptions 1.2 and 2.2 to recover the functional form of $FX_{T_i}$ as

$$FX_{T_i}(y^*) = \sup\bigl\{K \geq 0; V_0^{\mathrm{dFX,i}}(K) \geq J^{y,i}(y^*)\bigr\}. \tag{5.24}$$

Determining the functional form of as above will in general result result in the foreign exchange rate $FX_{T_i}(y_{T_i})$ that no longer satisfies equation (5.17). Therefore, we need to perform the corrector step that adjusts the drift appropriately.

In particular, we need to determine a new value of drift $\mu_{i-1}(FX_{T_{i-1},T_i})$ such that

$$\begin{aligned}
FX_{T_{i-1},T_i} &= \mathbb{E}_{\mathbb{F}^0}\bigl[FX_{T_i}(y_{T_i})\big|(x_{T_{i-1}}, y_{T_{i-1}}, z_{T_{i-1}})\bigr] &\quad (5.25)\\
&= \mathbb{E}_{\mathbb{F}^0}\Bigl[FX_{T_i}\bigl(\sigma^y_{i-1}(W^y_{T_i} - W^y_{T_{i-1}}) + \mu_{i-1}(FX_{T_{i-1},T_i})\bigr)\Big|FX_{T_{i-1},T_i}\Bigr] &\quad (5.26)\\
&= \int_{-\infty}^{\infty} FX_{T_i}\bigl(\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}u + \mu_{i-1}(FX_{T_{i-1},T_i})\bigr)\phi(u)du, &\quad (5.27)
\end{aligned}$$

where $\phi$ is the density function of a standard normal random variable. By assumption 1.2 $FX_{T_i}$ is an increasing function of $y_{T_i}$ and therefore $\mu_{i-1}$ has to be an increasing function of $FX_{T_{i-1},T_i}$.

### 5.3.4 Calibration to the Foreign Digital Caplets In-arrears

Finally, we describe how to recover the functional form of the foreign LIBOR $\tilde{L}^i_{T_i}$. Unsurprisingly, we keep with the existing theme and define a function $J^{z,i}$ by

$$J^{z,i}(z^*) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}}{B_{T_i}}\mathbf{1}_{\{z_{T_i}>z^*\}}\right]. \tag{5.28}$$

Note, that we already know the functional form of $B_{T_i}$ and $FX_{T_i}$ and therefore the function $J^{z,i}$ is well defined. Again, we can use assumptions 1.3 and 2.3 to perform the Markov-functional sweep and determine the functional form of $\tilde{L}^i_{T_i}$ from

$$\tilde{L}^i_{T_i}(z^*) = \sup\bigl\{K \geq 0; \tilde{V}_0^{\mathrm{dca},i}(K) \geq J^{z,i}(z^*)\bigr\}. \tag{5.29}$$

**Remark 5.3.** *Recall that Fries and Rott (2004) used could calibrate to the foreign digital caplets in-arrears independently of calibrating to domestic ones and the foreign exchange* *rate. In our model this is in general not the case as we allow for the Brownian motions $W^x$, $W^y$ and $W^z$ to be correlated. Nevertheless, if the increments of Brownian motions are independent, one can show that* (5.9) *also holds in our model and the calibration to the foreign caplets can be performed independently.*

## 5.4 Numerical Implementation

Let us now outline how to implement the model presented in the previous section on a grid. For each of the time steps $T_i, i \in \{1, \dots, n\}$, we choose grid-points

$$h^x_{i,1} < \dots < h^x_{i,m} \tag{5.30}$$

$$h^y_{i,1} < \dots < h^y_{i,m} \tag{5.31}$$

$$h^z_{i,1} < \dots < h^z_{i,m} \tag{5.32}$$

corresponding to states of $x_{T_i}$, $y_{T_i}$ and $z_{T_i}$ respectively. Our aim is to recover the functional forms of $L^i(h^x_{i,j})$, $FX_{T_i}(h^y_{i,j})$, $\tilde{L}^i(h^z_{i,j})$ for $j = 1, \dots, m$, and determine the functional form of the drift $\mu_i(h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}) = \mu_{i-1}(FX_{T_{i-1},T_i}(h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}))$ for $j_x, j_y, j_z = 1, \dots, m$.

To ease the burden of notation we will adopt the following convention, by $s$ we will denote the vector valued process defined by

$$s_i := (x_{T_i}, y_{T_i}, z_{T_i}), \quad i = 0, \dots, n, \tag{5.33}$$

and by $h_{i,j}, i \in \{1, \dots, n\}, j = (j_x, j_y, j_z) \in \{1, \dots, m\}^3$, we will denote the grid-point

$$h_{i,j} := (h^x_{i,j_x}, h^y_{i,j_y}, h^z_{i,j_z}). \tag{5.34}$$

First we recall that we can calibrate to the domestic digital caplet in-arrears prices and determine the functional form of $L^i$ independently of the calibration to the foreign digital caplets in-arrears and foreign exchange digital call options. To do so, we can use the algorithm presented in Section 4.3. In the rest of the section we will therefore assume we have already determined the functional forms of $L^i_{T_i}$ on the grid points and therefore also of the rolling bank account.

One of the problems we face when implementing the proposed model is that distribution of $s_{T_i}$ is not Gaussian for $i \geq 2$ (we later discuss the case $i = 1$ separately) and we can only build it iteratively by observing that the conditional distribution $s_{T_i}|s_{T_{i-1}}$ is Gaussian. In particular, we need to build the information about the joint dynamics of process $s$ in a way that will allow us to efficiently evaluate the functions $J^{y,i}$ and $J^{z,i}$ on the grid-points and later also allow us to price other derivatives.

Recall that we were faced with a similar, albeit only one-dimensional, problem in Sections 4.2 and 4.3. There, we introduced the piecewise polynomial basis functions and defined suitable expectations $E_{i,j}$'s that allowed us to build up the distribution of them model efficiently. Here, we will outline how to extend this idea to our setting.

Let $i \in \{1, \dots, n\}$ we will say that functions $b_{i,j} : \mathbb{R}^3 \to \mathbb{R}, j = (j_x, j_y, j_z) \in \{1, \dots, m\}^3$ are basis functions if they are piecewise polynomial[5] and satisfy the following condition for all $j = (j_x, j_y, j_z), k = (k_x, k_y, k_z) \in \{1, \dots, m\}^3$

$$b_{i,j}(h_{i,k}) = \delta_{j_x,k_x}\delta_{j_y,k_y}\delta_{j_z,k_z}. \tag{5.35}$$

Then we can define for any function $f : \mathbb{R}^3 \to \mathbb{R}$ a function $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{f}(x, y, z) := \sum_{\substack{j=(j_x,j_y,j_z) \\ j\in\{1,\dots,m\}^3}} f(h_{i,j})b_{i,j}(x, y, z), \quad x, y, z \in \mathbb{R}. \tag{5.36}$$

In particular, note that the two functions agree on gridpoints

$$\tilde{f}(h_{i,j}) = f(h_{i,j}), \quad j \in \{1, \dots, m\}^3. \tag{5.37}$$

Moreover, for a suitable choice of basis functions and any 'smooth enough'[6] function $f$ the approximation $\tilde{f}$ is a 'good' (piecewise polynomial) approximation of $f$ on the domain $[h^x_1, h^x_m] \times [h^y_1, h^y_m] \times [h^z_1, h^z_m]$.

In particular, at each time step $T_i, \in \{1, \dots, n\}$, we use the basis functions to define constants $E_{i,j}, j \in \{1, \dots, m\}^3$, by

$$E_{i,j} := \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{B_{T_i}}\right]. \tag{5.38}$$

We now sketch how $E_{i,j}$'s and the functional forms of $FX_{T_i}$ and $\tilde{L}^i_{T_{i-1}}$ can be recovered. First, we note that $i = 1$ is a special case since at time zero the drift term $\mu_0(x_0, y_0, z_0)$ is constant and therefore the joint distribution of $(x_{T_1}, y_{T_1}, z_{T_1})$ is Gaussian. In particular, without loss of generality we can set $\mu_0 = 0$ since any deterministic drift can be absorbed into the functional form of $FX_{T_1}$. Moreover, observe that by fixing $\mu_0 = 0$ we have also made $E_{1,j}$'s well defined.

We can therefore evaluate $J^{y,1}$ on the grid-points directly

$$J^{y,1}(h^y_{1,j_y}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_1}}\mathbf{1}_{\{y_{T_1} > h^y_{1,j_y}\}}\right] \tag{5.39}$$

$$= (1 + \alpha_0 L^0_0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}_{\{\sigma^y_0 W^y_{T_1} > h^y_{1,j_y}\}}\right] \tag{5.40}$$

$$= (1 + \alpha_0 L^0_0)^{-1}\Phi\left(-\frac{h^y_{1,j_y}}{\sigma^y_0\sqrt{T_1}}\right), \tag{5.41}$$

where $\Phi$ is the cumulative distribution of a standard normal random variable. Consequently we can recover, the functional form of $FX_{T_1}$.

[5]Note that one will need to sensibly divide the $\mathbb{R}^3$ into partitions on which the coefficients are constants, two natural choices are cuboids or triangular pyramids with vertices corresponding to grid-points.

[6]The smoothness condition depends on the choice of basis functions.

Finally, we can determine the functional form of $\tilde{L}^1_{T_1}$ by first evaluating $J^{z,1}$ on the grid-points

$$J^{z,1}(h^z_{1,j_z}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_1}(y_{T_1})}{B_{T_1}}\mathbf{1}_{\{z_{T_1}>h^z_{1,j_z}\}}\right] \tag{5.42}$$

$$= (1+\alpha_0 L^0_0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_1}(y_{T_1})\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}_{\{z_{T_1}>h^z_{1,j_z}\}}\middle|y_{T_1}\right]\right] \tag{5.43}$$

$$= (1+\alpha_0 L^0_0)^{-1}\mathbb{E}_{\mathbb{F}^0}\left[FX_{T_1}(y_{T_1})\Phi\left(\frac{\frac{\sigma^z_0}{\sigma^y_0}\rho^{y,z}_{T_1}y_{T_1} - h^z_{1,j_z}}{\sigma^z_0\sqrt{T_1\big(1-(\rho^{y,z}_{T_1})^2\big)}}\right)\right], \tag{5.44}$$

where $\rho^{y,z}_{T_1} = \mathrm{Corr}(W^y_{T_1}, W^z_{T_1})$. We have manipulated $J^{1,z}(h^z_{1,j_z})$ to equation (5.44) which only involves integrating over a one-dimensional Gaussian distribution and can be performed by many existing numerical integration techniques.

Now we show how to perform a general time step. In particular, we assume that we have recovered the functional forms of $L^{i-1}_{T_{i-1}}$ $FX_{T_{i-1}}$, $\tilde{L}^{i-1}_{T_{i-1}}$ and the values of $E_{i-1,j}$'s for some $i \in \{2,\ldots,n\}$.

We first show how to recover the functional form of $FX_{T_i}$ by evaluating $J^{y,i}$ on the grid-points. We assume that we have chosen an initial guess for the drift $\mu_{i-1}$ or we have obtained it from the corrector step.

$$J^{y,i}(h^y_{i,j_y}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbf{1}_{\{y_{T_i}>h^y_{i,j_y}\}}\right] \tag{5.45}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}_{\{y_{T_i}>h^y_{i,j_y}\}}\middle|\mathcal{F}_{T_{i-1}}\right]\right] \tag{5.46}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\left[\mathbf{1}_{\{y_{T_i}>h^y_{i,j_y}\}}\middle|s_{T_{i-1}}\right]\right] \tag{5.47}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{(1+\alpha^{i-1}L^{i-1}_{T_{i-1}})B_{T_{i-1}}}\Phi\left(\frac{\mu_{i-1}(s_{T_{i-1}}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right)\right]. \tag{5.48}$$

Note that $L^{i-1}_{T_{i-1}}$ is a known function of $x_{T_{i-1}}$. We can now use the approximation using the basis functions $b_{i-1,j}$ and express $J^{y,i}(h^y_{i,j_y})$ in terms of $E_{i,j}$'s which are already known.

$$J^{y,i}(h^y_{i,j_y}) \approx \sum_k D_{T_{i-1},T_i}(h^x_{i-1,k_x})\Phi\left(\frac{\mu_{i-1}(h_{i,k}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right)\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i-1,k}(s_{T_{i-1}})}{B_{T_{i-1}}}\right] \tag{5.49}$$

$$= \sum_k \frac{E_{i-1,k}}{1+\alpha^{i-1}L^{i-1}_{T_{i-1}}(h^x_{i-1,k_x})}\Phi\left(\frac{\mu_{i-1}(h_{i-1,k}) - h^y_{i,j_y}}{\sigma^y_{i-1}\sqrt{T_i - T_{i-1}}}\right). \tag{5.50}$$

After we have recovered the functional form of $FX_{T_i}$ we need to use the corrector step to make sure our model remains arbitrage-free. We then repeat the two step until we achieve sufficient level of convergence at which point we can freeze the drift term $\mu_{i-1}$ and determine the $E_{i,j}$'s, by conditioning on $\mathcal{F}_{T_{i-1}}$

$$E_{i,j} = \mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{B_{T_i}}\right] \tag{5.51}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\Big|\mathcal{F}_{T_{i-1}}\right]\right] \tag{5.52}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_{i-1}}}\mathbb{E}_{\mathbb{F}^0}\left[\frac{b_{i,j}(s_{T_i})}{1+\alpha_{i-1}L^{i-1}_{T_{i-1}}(x_{T_{i-1}})}\Big|s_{T_{i-1}}\right]\right] \tag{5.53}$$

$$\approx \sum_k \frac{E_{i-1,k}}{1+\alpha^{i-1}L^{i-1}_{T_{i-1}}(h^x_{i-1,k_x})}\mathbb{E}_{\mathbb{F}^0}\big[b_{i,j}(s_{T_i})\big|s_{T_{i-1}} = h_{i-1,k}\big] \tag{5.54}$$

Finally we can recover the functional form of $\tilde{L}^i_{T_i}$ by evaluating function $J^{z,i}$ on the gridpoints

$$J^{z,i}(h^z_{i,j_z}) = \mathbb{E}_{\mathbb{F}^0}\left[\frac{FX_{T_i}(y_{T_i})}{B_{T_i}}\mathbf{1}_{\{z_{T_i}>h^z_{i,j_z}\}}\right] \tag{5.55}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}(y_{T_i})\mathbf{1}_{\{z_{T_i}>h^z_{i,j_z}\}}\big|\mathcal{F}_{T_{i-1}}\big]\right] \tag{5.56}$$

$$= \mathbb{E}_{\mathbb{F}^0}\left[\frac{1}{B_{T_i}}\mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}(y_{T_i})\mathbf{1}_{\{z_{T_i}>h^z_{i,j_z}\}}\big|s_{T_{i-1}}, y_{T_i}\big]\right] \tag{5.57}$$

Now observe that, the random vector $(y_{T_i}, z_{T_i})|s_{T_{i-1}}$ has a known bivariate normal distribution and therefore $z_{T_i}|y_{T_i}, s_{T_{i-1}}$ has a known normal distribution. In particular, using the tower property of conditional expectation we can show that

$$\mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}(y_{T_i})\mathbf{1}_{\{z_{T_i}>h^z_{i,j_z}\}}\big|s_{T_{i-1}}\big] = \mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}(y_{T_i})\Phi\big(g_{i,j_z}(s_{T_{i-1}}, y_{T_i})\big)\big|s_{T_{i-1}}\big], \tag{5.58}$$

for a known function $g_{i,j_z} : \mathbb{R}^4 \to \mathbb{R}$. This, then allows us to evaluate conditional expectation $\mathbb{E}_{\mathbb{F}^0}\big[FX_{T_i}(y_{T_i})\mathbf{1}_{\{z_{T_i}>h^z_{i,j_z}\}}\big|s_{T_{i-1}}\big]$ on the grid points $h_{i-1,k}, k \in \{1,\ldots,m\}^3$, by using only one-dimensional numerical integration. Then we can finally determine the value of $J^{i,z}$ on the grid-points as a linear combination of $E_{i-1,j}$'s.

## 5.5 Conclusion

In this chapter we have proposed a new two-currency Markov-functional model, which can be calibrated to the smile in the domestic and foreign caplet prices and foreign exchange options. We have used the idea of Fries and Rott (2004) and discretised the driving process first and then constructed the model. By doing so we avoided dealing with state-dependent drifts which would occur in the continuous time setting.

The model proposed in Section 5.3 relies on the predictor-corrector step to converge. We are yet to prove under what conditions this is the case and test it numerically. It is our conjuncture that predictor-corrector step indeed converges (possibly under mild technical conditions).

# Chapter 6

# Conclusion

In this thesis we have been interested in the low-factor market models of interest rates. We have contributed to the existing literature on the LIBOR market model, Markov-functional model and two-currency Markov-functional model.

In Chapter 3 we were interested in the time-homogeneous separable LMMs. First, we extended the concept of separability to the LMMs driven by $d$-dimensional Brownian motions with correlated components and proved that the single time-step approximation remains a function of a $d$-dimensional Markov process under the extended definition of separability. Next we focused on the classification of the time-homogeneous separable LMMs. We have shown that the problem of finding the time-homogeneous parameterisations of separable LMM can be reduced to finding solutions of a Levi-Civitá equation subject to additional conditions. This allowed us to characterise the two- and three-factor separable time-homogeneous LMMs. In the two-factor case we found all the parameterisations and in the three-factor case we found all parameterisations that are of practical interest (see Remark 3.10). We have then analysed the parameterisations obtained and showed that they are indeed of practical interest. In particular, in the three-factor case we can obtain the popular Rebonato's *abcd* instantaneous volatility function.

In Chapter 4 we focused on the Markov-functional models. First we discussed the numeraire approach and provided sufficient conditions under which a term structure model defined by a numeraire pair is Markov-functional. Next we restricted ourselves to the one-dimensional Markov-functional models under the terminal and the spot measure. Unlike the existing literature that is focused on the Gaussian driving processes, we were interested in the diffusion driving processes with continuous marginal distributions. We introduced two new algorithms that can be used to implement a one-dimensional MFM under the terminal and the spot measure for such a driving process. The algorithms relied on the use of piecewise-polynomial 'basis functions' that allowed us to efficiently approximate smooth functions and build the distribution of the driving process iteratively forwards in time. We then shifted our attention to the problem of choosing an appropriate driving process. First, we used the copula theory to prove that it is the dependence structure of the driving process and not the marginal distributions that influence the dynamics of the one-dimensional MFM under the spot and the terminal measure. Next, we approached problem of choosing an appropriate driving process by relating the one-factor separable local-volatility LMMs to the one-dimensional MFMs under the terminal measure. To do so we first showed that the concept of separability and can be extended to the one-factor local-volatility LMMs. Based on this we proposed and gave heuristic justification for an approach that can be used to define one-factor MFMs under the terminal measure with dynamics similar to the dynamics of a one-factor separable local-volatility LMM. Finally, we demonstrated our approach on an example.

In Chapter 5 we proposed a new two-currency Markov-functional model that can calibrate to the domestic and foreign interest rate markets and the foreign exchange market. The novelty of our approach is the use of 'Markov-functional' sweep to determine the functional form of the foreign exchange rate and the predictor-corrector step that we conjectured will converge and ensure that the model is arbitrage-free. We outlined the numerical implementation of such an approach using 'basis functions' that ensure that the model can be implemented efficiently.

To conclude the thesis let us suggest some possibilities for further research and outline how the topics discussed are connected and can be used beyond the context in which they were presented.

Firstly let us discuss the concept of separability and the single time-step approximation. We have presented two extensions of the separability condition. In Chapter 3 we extended it to the (log-normal) LMM driven by a Brownian motion with correlated components. In Chapter 4 we have extended it to the one-factor local-volatility LMM. It is not difficult to see that we can combine both generalisations and extend the separability to the $d$-factor local-volatility LMM driven by a Brownian motion with correlated components. We can then show that the single time-step approximation of such a model is $d$-dimensional when the separability condition is fulfilled (a slightly different approach has been outlined in Section 12.8 in Joshi (2011), while the full details are not given there we believe that it is equivalent to our proposal). In particular, we can then use the two- and three-factor parameterisations obtained in Theorems 3.8 and 3.9 in the context of the two- and three-factor local-volatility LMM.

The two- and three-factor separable parameterisations are not only interesting in the context of LMMs but have practical implications for the MFMs as well. Recall that the one-factor separable LMM under the terminal measure has similar dynamics to the one-dimensional MFM under the terminal measure (see Section 4.5). Bennett and Kennedy (2005) believe that this is also the case for the multi-factor separable log-normal LMMs and the multi-dimensional MFMs under the terminal measure (see Section 19.5 in Hunt and Kennedy (2004) for construction of the multi-dimensional MFM). It is our view that this is also the case for the multi-factor separable local-volatility LMMs. We can then use the two- and three-factor parameterisations to define driving processes for the two- and three-dimensional MFMs under the terminal measure that have similar dynamics to the two-and three-factor separable (local-volatility) LMMs. Such MFMs are an arbitrage-free and computationally efficient alternative as the two- and three-factor separable LMMs.

One can also apply the concepts from the MFMs to the LMM setting. Recall that we made a comment that Pietersz et al. (2004) claimed that the single-time step approximation of a separable LMM can be used only under the terminal measure to avoid the path-dependence of the numeraire. In particular, when using the single time-step approximation of a separable LMM one wishes to implement the model on a grid. Therefore, it appears that using the rolling bank account as the numeraire would be problematic as the numeraire discounted payoffs can no longer be represented by a single grid-point but depend on the way we got to the grid-point. Note that this is exactly the problem one faces in the one-factor MFM under the spot measure. One can then implement the one-factor separable LMM under the spot measure by using the construction of the one-factor MFM under the spot measure. The only difference is that in the LMM the functional forms of the LIBORs at their setting date is determined by the drift approximation used and not by the 'Markov-functional sweep'. Note that this idea can also be generalised to a multi-dimensional setting.

The implementation of the single time-step approximation to the separable LMM under the spot measure suggested above gives rise to a natural question: Is there a connection between the separable LMM under the spot measure and the MFM under the spot measure? While it is reasonable to believe that the answer is positive this remains an open question. In particular, we can then ask ourselves what is the relationship between the MFMs under the terminal and the spot measure. Since, the prices of derivatives in the LMM do not depend on the EMM chosen and the MFMs under the terminal and the spot measure are have similar dynamics as the separable LMM under the terminal and respectively spot measure, how different or similar are the two MFMs?

The application of copula theory to MFMs offers a new way to view and study them. From theoretical perspective it would be interesting to extend Theorem 4.12 to the multi-dimensional MFMs. However, the practical applications of this approach are the most exciting part. Firstly, copula theory offers a simple approach that can be used to define grids with equal probability mass between the consecutive grid-points, which seems to be a desirable property. Secondly, there are several known parametric families of copulae that can describe the time dependence of a Markov process. This is interesting from a practical perspective as it allows to move away from the Gaussian dependence while retaining the numerical tractability as the transition densities are known explicitly.

Last but not least, let us mention the two-currency MFM. Finding the conditions under which the predictor-corrector step converges remains an open question and it is the author's plan at the time of writing to address in the near future. In the spirit of the earlier chapters it would be interesting to examine if the separability condition can be used to approximate the two currency LMM by a function of some low-dimensional Markov process and if this is the case can we relate it to a suitably defined two-currency MFM.

# Appendix A

# Basis Functions

The numerical algorithms presented in Sections 4.2 and 4.3 crucially depend on the ability to approximate well any 'smooth enough' function on a compact interval as a linear combination of piecewise polynomial basis functions. We now formalise this idea and present a construction of basis functions that will ensure that the resulting approximation is piecewise polynomial.

**Definition A.1.** *Let $[a, b] \subset \mathbb{R}$ be an interval and let $a = h_1 < h_2 < \ldots < h_m = b$ be a partition of $[a, b]$. We say that continuous functions $b_i : [a, b] \to \mathbb{R}, i = 0, \ldots, n,$ are basis functions for the interval $[a, b]$ (with respect to the above partition) if*

$$b_i(h_j) = \delta_{i,j}, \quad i, j = 1, \ldots, m. \tag{A.1}$$

Due to the nature of our problem we only consider basis functions that are continuous piecewise polynomials of order $q$ and have constant coefficients on the intervals $(h_i, h_{i+1}], i = 1, \ldots, m-1$, of the partition.[1] In particular, we assume that each basis function is of the following form

$$b_i(x) = \sum_{j=1}^{m-1} \sum_{k=0}^{q} b^i_{j,k} x^k \mathbf{1}_{\{h_j < x \leq h_{j+1}\}}. \tag{A.2}$$

We can now define an interpolation functional $I : \mathcal{C}([a, b]) \to \mathcal{C}([a, b])$ by

$$I(f) = \sum_{i=1}^{m} f(h_i) b_i. \tag{A.3}$$

Observe that by the definition of basis functions $I(f)(h_i) = f(h_i), i = 1, \ldots, n$. Moreover, $I(f)$ is piecewise polynomial of order $q$.

We now show how to choose a set of basis functions such that $I(f)$ will be a piecewise polynomial approximation of function $f$. Consider the $i$th basis function $b_i$ on the $j$th interval $[h_j, h_{j+1}]$. We determine the coefficients $\{b^i_{j,k}\}_{k=0}^{q}$ by fitting a polynomial through

[1] Actually we can drop the assumption of continuity as constant coefficients on each interval make the basis functions continuous on the interior of each interval of the partition and the definition of basis functions then ensures the continuity on the dividing points $h_i, i = 1, \ldots, m$.

the points $\{(h_{i,j+k}, \delta_{i,j+k})\}_{k=-\frac{q-1}{2}}^{\frac{q+1}{2}}$. Note that this can only be done when

$$\frac{q-1}{2} < j < m - \frac{q+1}{2} \tag{A.4}$$

When $j$ does not satisfy inequality (A.4), we reduce $q$ to the highest odd integer such that the inequality holds. Figure A.1 shows the construction of the basis function $b_1$ on the intervals $[h_1, h_2]$, $[h_2, h_3]$ and $[h_3, h_4]$ for $q = 5$. Note that on other intervals the function $b_1$ equals zero.

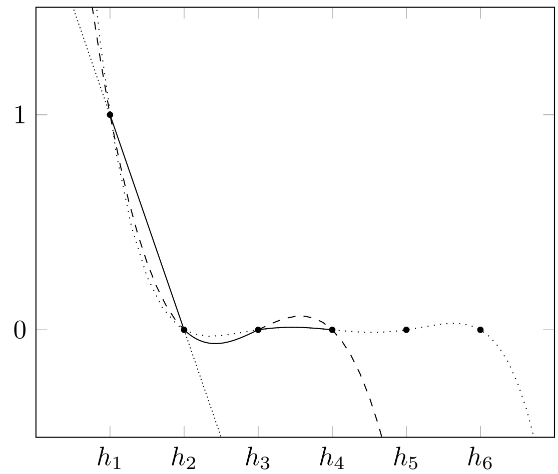



Figure A.1: Construction of basis function $b_1$ when $q = 5$.

The simplest example of the above construction occurs when $q = 1$. In this case inequality (A.4) hold for every $j = 1, \ldots, m-1$, and the $i$th basis function on the $j$th interval is a linear function through points $(h_j, \delta_{i,j})$ and $(h_{j+1}, \delta_{i,j+1})$. Observe that the support of $b_i$ is the interval $[h_{i-1} \vee 0, h_{i+1} \wedge h_m]$. Moreover $I(f)$ is a linear interpolation of a function $f$.

For a general any odd integer $q \leq \frac{m}{2}$ the support of the $i$th basis function will be the interval $[h_{i-\frac{q+1}{2}} \vee 0, h_{i+\frac{q+1}{2}} \wedge h_m]$. On the other hand on $j$th interval $[h_j, h_{j+1}]$, there will be only $q+1$ non-zero basis function,[2] $b_i, i = j - \frac{q-1}{2}, \ldots, j + \frac{q+1}{2}$. Note that polynomials $b_i|_{[h_j,h_{j+1}]}, i = j - \frac{q-1}{2}, \ldots, j + \frac{q+1}{2}$ are of degree $q$ and are linearly independent by construction, therefore they form a basis for the vector space of polynomials up to degree $q$. Furthermore, if we slightly abuse the notation,

$$\sum_{i=j-\frac{q-1}{2}}^{j+\frac{q+1}{2}} f(h_i) b^i\big|_{[h_j,h_{j+1}]}(h_i) = f(h_i). \tag{A.5}$$

Therefore $I(f)|_{[h_i,h_{i+1}]}$ is the polynomial approximation of degree $q$ of the function $f$ passing through points $(h_i, f(h_i)), i = j - \frac{q-1}{2}, \ldots, j + \frac{q+1}{2}$. Consequently, $I(f)$ is a piecewise

[2]Provided that the Inequality (A.4) is satisfied, otherwise we reduce the $q$ to the highest odd integer such that the inequality holds.)

polynomial approximation of function $f$.

# Appendix B

# Proofs

## B.1 Proof of Theorem 3.8

Let $v = (v_1, v_2), \sigma = (\sigma_1, \sigma_2) : \mathbb{R}_+ \to \mathbb{R}^2$, $\rho_{1,2} : \mathbb{R}_+ \to [-1, 1]$ and $\sigma^{inst} : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions that satisfy equation (3.31). Next define functions $g_i, h_i, i = 1, 2, 3$, and $f$ as in (3.32)–(3.36) and note that they are a solution to the functional equation

$$f(T - t) = \sum_{i=1}^{3} g_i(t) h_i(T). \tag{B.1}$$

Note that $g_i, h_i, i = 1, 2, 3$, and $f$ are continuous functions and we know from the discussion in the Section 3.3 that $f$ has to be of the form

$$f(y) = \sum_i P_i(y) \exp(-\lambda_i y), \tag{B.2}$$

where $\lambda_i \in \mathbb{C}$, $P_i$ is a polynomial (possibly with complex coefficients) and $\sum_i (1 + \deg P_i) = 3$. Similarly, functions $g_i, h_i, i = 1, 2, 3$, can be expressed as sum of exponential polynomials, i.e. for $i = 1, 2, 3$,

$$g_i(t) = \sum_i P_i^g(t) \exp(\lambda_i t), \tag{B.3}$$

$$h_i(T) = \sum_i P_i^h(T) \exp(-\lambda_i T). \tag{B.4}$$

$$\tag{B.5}$$

In particular $f$ has to be of one of the following forms

1. $x_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3,$

$$f(y) = \sum_{i=1}^{3} x_i \exp(-\lambda_i y); \tag{B.6}$$

2. $x_1 \neq 0$, $x_2, x_3, \lambda_1, \lambda_2 \in \mathbb{R}$

$$f(y) = (x_1 y + x_2)\exp(-\lambda_1 y) + x_3 \exp(-\lambda_2 y); \tag{B.7}$$

3. $x_1 > 0$, $x_2, x_3, \lambda \in \mathbb{R}$

$$f(y) = x_1(y^2 + x_2 y + x_3)\exp(-\lambda_1 y); \tag{B.8}$$

4. $x_1, \lambda_1 \in \mathbb{C} \setminus \mathbb{R}$, $x_2, \lambda_2 \in \mathbb{R}$

$$f(y) = x_1 \exp(-\lambda_1 y) + \overline{x}_1 \exp(-\overline{\lambda}_1 y) + x_2 \exp(-\lambda_2 y), \tag{B.9}$$

where $\overline{x}$ denotes the complex conjugate of $x$.

Before analysing the possible solutions note that equations (3.35) and (3.36) imply

$$h_3(T)^2 = h_1(T)h_2(T), \tag{B.10}$$

an observation which will be used throughout the proof.

**Case 1** We first analyse the case when $f$ is of the form (B.6). We can assume without loss of generality that $\lambda_1 < \lambda_2 < \lambda_3$. Clearly $f$ will be of desired form if the functions $g_i, h_i, i = 1, 2, 3$ are of the form,

$$g_i(t) = \alpha_i^2 \exp(2\beta_i t), \quad i = 1, 2 \tag{B.11}$$

$$g_3(t) = 2\gamma\alpha_1\alpha_2 \exp\big((\beta_1 + \beta_2)t\big), \tag{B.12}$$

$$h_i(T) = \exp(-2\beta_i T), \quad i = 1, 2, \tag{B.13}$$

$$h_3(T) = \exp\big(-(\beta_1 + \beta_2)T\big), \tag{B.14}$$

where $\alpha_1, \alpha_2 \geq 0$, $\beta_1, \beta_2 \in \mathbb{R}$ and $\gamma \in [-1, 1]$. Then $f$ is given by

$$f(x) = \alpha_1^2 \exp(-2\beta_1 x) + \alpha_2^2 \exp(-2\beta_2 x) + 2\gamma\alpha_1\alpha_2 \exp\big(-(\beta_1 + \beta_2)x\big) \tag{B.15}$$

and functions $v$, $\sigma$ and $\rho_{1,2}$ are given by

$$v(T) = \begin{bmatrix} \exp(-\beta_1 T) \\ \exp(-\beta_2 T) \end{bmatrix}, \tag{B.16}$$

$$\sigma(t) = \begin{bmatrix} \alpha_1 \exp(\beta_1 t) \\ \alpha_1 \exp(\beta_2 t) \end{bmatrix}, \tag{B.17}$$

$$\rho_{1,2}(t) = \gamma. \tag{B.18}$$

Note that this is indeed a valid parameterisation since $\rho_{1,2}(t) = \gamma \in [-1, 1]$, and corresponds to parameterisation 2.1 in the statement of the theorem.

Next we show that one cannot get a more general parameterisation when $f$ is of the form (B.6). We will refer to the parameterisation given by (B.16)–(B.18) as the 'original parameterisation'. The most general form we can expect of functions $g_i, h_i, i = 1, 2, 3$, is

$$g_i(t) = \alpha_i \exp(\lambda_1 t) + \beta_i \exp(\lambda_2 t) + \gamma_i \exp(\lambda_3 t), \tag{B.19}$$

$$h_i(T) = a_i \exp(-\lambda_1 T) + b_i \exp(-\lambda_2 T) + c_i \exp(-\lambda_3 T), \tag{B.20}$$

for some constants $\alpha_i, \beta_i, \gamma_i, a_i, b_i, c_i \in \mathbb{R}, i = 1, 2, 3$. Then $f$ has to be of the form

$$f(y) = \sum_{i=1}^{3} \big( a_i \alpha_i \exp(-\lambda_1 y) + b_i \beta_i \exp(-\lambda_2 y) + c_i \gamma_i \exp(-\lambda_3 y) \big). \tag{B.21}$$

In particular, observe that $f$ can be represented by the original parameterisation if

$$\sum_{i=1}^{n} a_i \alpha_i > 0, \tag{B.22}$$

$$\sum_{i=1}^{n} c_i \gamma_i > 0, \tag{B.23}$$

$$\left( \sum_{i=1}^{n} b_i \beta_i \right)^2 \leq 4 \left( \sum_{i=1}^{n} a_i \alpha_i \right) \left( \sum_{i=1}^{n} c_i \gamma_i \right). \tag{B.24}$$

If we can show that any parameterisation of the form (B.19) and (B.20) satisfies (B.22)–(B.24), then the original parameterisation covers all the parameterisations when $f$ is of the form (B.6). Note that (B.22) is true for any parameterisation of our interest since $f$ needs to be a positive function.

Another simple way of showing that a parameterisation offers no generality over the original one is by showing that $h_1, h_2$ and $h_3$ are linearly dependent. In such a case there exist constants $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ such that $\max\{\xi_1^2, \xi_2^2, \xi_3^2\} > 0$ and

$$\sum_{i=1}^{3} \xi_i h_i(T) = 0, \qquad T \geq 0. \tag{B.25}$$

Then at least one of the constants $\xi_1$ and $\xi_2$ is non-zero. Without loss of generality we assume $\xi_1 = 1$. Then

$$f(T - t) = \sum_{i=1}^{3} g_i(t) h_i(T) = \sum_{i=2}^{3} (g_i(t) - \xi_i g_1(t)) h_i(T) \tag{B.26}$$

and $f$ can only be a sum of two exponential functions, thus less general than the original parameterisation.

Now we use the restriction in equation (B.10). We will split the analysis into two possibilities: $\lambda_2 = \frac{1}{2}(\lambda_1 + \lambda_2)$ and $\lambda_2 \neq \frac{1}{2}(\lambda_1 + \lambda_2)$.

First assume that $\lambda_2 \neq \frac{1}{2}(\lambda_1 + \lambda_3)$. Then equation (B.10) implies

$$a_1 a_2 = a_3^2, \tag{B.27}$$

$$b_1 b_2 = b_3^2, \tag{B.28}$$

$$c_1 c_2 = c_3^2, \tag{B.29}$$

$$a_1 b_2 + a_2 b_1 = 2a_3 b_3, \tag{B.30}$$

$$a_1 c_2 + a_2 c_1 = 2a_3 c_3, \tag{B.31}$$

$$b_1 c_2 + b_2 c_1 = 2b_3 c_3. \tag{B.32}$$

Suppose that $a_1 = 0$, then $a_3 = 0$ and we are only interested in this parameterisation if $a_2 \neq 0$. Which in turn implies $b_1 = c_1 = 0$ and therefore $b_3 = c_3 = 0$. Thus $h_1 = h_3 \equiv 0$ and $f(T-t) = g_2(t)h_2(T)$ which leads to $f$ being an exponential, thus offering no generality over the original parameterisation. Note that assuming that any of the constants $a_i, b_i, c_i, i = 1, 2, 3$ equals zero would lead to the conclusion that $f$ is an exponential function (for example we simply relabel constants appropriately).

Next assume that $a_i b_i c_i \neq 0, i = 1, 2, 3$

Then it is easy to observe that

$$(a_1 b_2 - a_2 b_1)^2 = 0, \tag{B.33}$$

$$(a_1 c_2 - a_2 c_1)^2 = 0, \tag{B.34}$$

$$(b_1 c_2 - b_2 c_1)^2 = 0 \tag{B.35}$$

and therefore

$$a_1 b_2 = a_2 b_1, \qquad a_1 c_2 = a_2 c_1, \qquad b_1 c_2 = b_2 c_1. \tag{B.36}$$

Since $a_i b_i c_i \neq 0, i = 1, 2, 3$, then

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = x \tag{B.37}$$

and therefore $h_2 = x h_1$. In particular, functions $h_1, h_2$ and $h_3$ are linearly dependent, thus offering no generality over the original parameterisation.

Therefore we need to assume that $\lambda_2 = \frac{1}{2}(\lambda_1 + \lambda_3)$. In this case equation (B.10) implies

$$a_1 a_2 = a_3^2, \tag{B.38}$$

$$c_1 c_2 = c_3^2, \tag{B.39}$$

$$a_1 b_2 + a_2 b_1 = 2a_3 b_3, \tag{B.40}$$

$$b_1 c_2 + b_2 c_1 = 2b_3 c_3, \tag{B.41}$$

$$b_1 b_2 + a_1 c_2 + a_2 c_1 = b_3^2 + 2a_3 c_3. \tag{B.42}$$

Without loss of generality we can assume that $a_1 = 1$ and split the analysis into two cases $a_2 \neq 0$ and $a_2 = 0$.

First let us consider the case when $a_2 = 0$. Then equation (B.38) implies $a_3 = 0$ and consequently equation (B.40) implies $b_2 = 0$. Then (B.41) and (B.42) can be rewritten as

$$b_1 c_2 = 2 b_3 c_3 \tag{B.43}$$

$$c_2 = b_3^2 \tag{B.44}$$

Note that if $b_3 = 0$ then $c_3 = 0$ and therefore $h_3 \equiv 0$, this would clearly offer no generality over the original parameterisation. Therefore we can assume $b_3 \neq 0$ and we can express $b_1, c_1, c_2$ in terms of $b_3$ and $c_3$ as

$$b_1 = 2\frac{c_3}{b_3}, \qquad c_1 = \frac{c_3^2}{b_3^2}, \qquad c_2 = b_3^2. \tag{B.45}$$

We can now turn back to equation (B.1), since we know that $f$ is of the from as in equation (B.6), we know that the sum of coefficients in front of term of the form $\exp(-\lambda_i T + \lambda_j t)$ is 0 when $i \neq j$. Therefore $\beta_1 = \gamma_1 = 0$ (take $i = 1$ and $j = 2, 3$) and setting $i = 1, 2$ and $j = 1$ yields

$$\frac{c_3}{b_3}\alpha_1 + b_3\alpha_3 = 0 \tag{B.46}$$

$$\frac{c_3^2}{b_3^2}\alpha_1 + b_3^2\alpha_2 + c_3\alpha_3 = 0 \tag{B.47}$$

then $\frac{c_3^2}{b_3^2}\alpha_1 + c_3\alpha_3 = 0$ and therefore $\alpha + 2 = 0$.

On the other hand equation (3.34) implies $4g_1(t)g_2(t) \geq g_3(t)^2$. In particular, the coefficients $\gamma_1, \gamma_2, \gamma_3$ in front of the exponential $\exp(2\lambda_3 t)$ have to satisfy the inequality

$$0 = 4\gamma_1\gamma_2 \geq \gamma_3^2 \tag{B.48}$$

and therefore $\gamma_3 = 0$. Then we have to compare the coefficients in front of $\exp((\lambda_2 + \lambda_3)t)$,

$$4\alpha_1\gamma_2 = 4(\alpha_1\gamma_2 + \alpha_2\gamma_1 + \beta_1\beta_2) \geq \beta_3^2 + 2\alpha_3\gamma_3 = \beta_3^2. \tag{B.49}$$

Now we can show that the parameterisation satisfies conditions (B.23) and (B.24). First observe that $\gamma_2 \geq 0$ since $g_2$ is non-negative function. Moreover, since $\gamma_1 = \gamma_3 = 0$ it has to be the case that $\gamma_3 > 0$. Then

$$\sum_{i=1}^{3} c_i\gamma_i = c_3\gamma_3 = b_3^2\gamma_3 > 0 \tag{B.50}$$

and condition (B.23) holds. Next observe that

$$4\left(\sum_{i=1}^{3} a_i\alpha_i\right)\left(\sum_{i=1}^{3} c_i\gamma_i\right) = 4\alpha_1\gamma_2 c_2 \tag{B.51}$$

$$= 4\alpha_1\gamma_2 b_3^2 \tag{B.52}$$

$$\leq b_3^2\beta_3^2 = \left(\sum_{i=1}^{3} b_i\beta_i\right)^2 \tag{B.53}$$

and condition (B.24) also holds. Therefore, this parameterisation offers no generality over the original one.

Finally we need to consider the case when $a_2 \neq 0$. In this case we can without loss of generality assume $a_1 = a_2 = a_3 = 1$.

First observe that we are only interested in the case when $b_1c_1 \neq 0$ or $b_2c_2 \neq 0$ as it is easy to see that in other case the parameterisation offers no generality over the original parameterisation. For example if $b_1 = c_2 = 0$ it follows that either $c_3 = 0$ and $c_1b_2 = 0$ which in turn implies that either $c_1 = c_2 = c_3 = 0$ or $b_1 = b_2 = b_3 = 0$.

Therefore, we can assume without loss of generality that $b_1c1 \neq 0$. Since that $c_1c_2 = c_3^2 \geq 0$ we can rewrite (B.41) to

$$0 = b_1c_2 + b_2c_1 - 2\,\mathrm{sgn}(c_3)\sqrt{c_1c_2} \tag{B.54}$$

$$= \mathrm{sgn}(c_1)b_1|c_2| + \mathrm{sgn}(c_1)b_2|c_1| - 2\,\mathrm{sgn}(c_3)\sqrt{c_1c_2} \tag{B.55}$$

$$= b_1|c_2| + b_2|c_1| - 2\,\mathrm{sgn}(c_1c_3)\sqrt{c_1c_2} \tag{B.56}$$

$$= \left(b_1\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)b_2\sqrt{|c_1|}\right)\left(\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)\sqrt{|c_1|}\right) \tag{B.57}$$

Therefore either $\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)\sqrt{|c_1|} = 0$ or $b_1\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)b_2\sqrt{|c_1|} = 0$.

Note that $\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)\sqrt{|c_1|} = 0$ if and only if $c_1 = c_2$ and $\mathrm{sgn}(c_1c_3) = 1$, therefore $c_1 = c_2 = c_3$ and functions $h_1, h_2, h_3$ are linearly dependent and such parameterisation can offer no generality over the original parameterisation.

We can then assume that $b_1\sqrt{|c_2|} - \mathrm{sgn}(c_1c_3)b_2\sqrt{|c_1|} = 0$. In particular this implies

$$b_1\sqrt{|c_2|} = \mathrm{sgn}(c_1c_3)b_2\sqrt{|c_1|} \tag{B.58}$$

$$\frac{b_2}{b_1} = \mathrm{sgn}(c_1c_3)\frac{\sqrt{|c_2|}}{\sqrt{|c_1|}} =: x \in \mathbb{R}. \tag{B.59}$$

Then we can use (B.38)–(B.41) to express $b_2, b_3, c_1, c_3$ in terms of $b_1, c_1$ and $x$ as

$$b_2 = b_1x, \qquad b_3 = \frac{1}{2}b_1(1+x), \qquad c_2 = c_1x^2, \qquad c_3 = c_1x.$$

Next we use (B.42) to obtain a relationship between $b_1, c_1$ and $x$.

$$b_1b_2 + c_1 + c_2 = b_3^2 + 2c_3 \tag{B.60}$$

$$4c_1(1+x^2) + 4b_1^2 x = b_1^x(1+x)^2 + 8c_1 x \tag{B.61}$$
$$4c_1(1-x)^2 = b_1^2(1-x)^2 \tag{B.62}$$

Note that $x = 1$ implies $h_1 = h_2 = h_3$ which is not an interesting case. We then assume $x \neq 1$ and thus

$$c_1 = \frac{1}{4} b_1^2. \tag{B.63}$$

Recall that it is enough for us to show that (B.23) and (B.24) hold. First observe that the coefficient in front of the term $\exp(-\lambda_i T + \lambda_j t)$ on the right-hand side of equation (B.1) has to be 0 when $i \neq j$. In particular, for $i = 1, 2$ and $j = 3$

$$0 = \gamma_1 + \gamma_2 + \gamma_3 \tag{B.64}$$
$$0 = b_1\gamma_1 + b_1 x\gamma_2 + \frac{1}{2} b_1\gamma_3 \tag{B.65}$$

and therefore $\gamma_2 = \gamma_1$ and $\gamma_3 = -2\gamma_1$. Furthermore, $g_1$ is non-negative and therefore $\gamma_1 \geq 0$. In particular,

$$\sum_{i=1}^{3} c_i\gamma_i = \frac{1}{4} c_1\gamma_1(1-x)^2 \geq 0 \tag{B.66}$$

and (B.23) holds.

Next we prove that (B.24) also holds. Recall that $4g_1(t)g_2(t) \geq g_3(t)^2, t \geq 0$, then

$$
\begin{aligned}
0 &\leq 4g_1(t)g_2(t) - g_3(t)^2 && \text{(B.67)}\\
&= (4\gamma_1\gamma_2 - \gamma_3^2)\exp(2\lambda_3 t)\\
&\quad + (4\beta_1\gamma_2 + 4\beta_2\gamma_1 - 2\beta_3\gamma_3)\exp\big((\lambda_2+\lambda_3)t\big) && \text{(B.68)}\\
&\quad + (4\beta_1\beta_2 + 4\alpha_1\gamma_2 + 4\alpha_2\gamma_1 - \beta_3^2 - 2\alpha_3\gamma_3)\exp(2\lambda_2 t) + \dots\\
&= 4\gamma_1(\beta_1+\beta_2+\beta_3)\exp\big((\lambda_2+\lambda_3)t\big)\\
&\quad + \big(4\beta_1\beta_2 - \beta_3^2 + 4\gamma_1(\alpha_1 + 4\alpha_2 + 2\alpha_3)\big)\exp(2\lambda_2 t) + \dots, && \text{(B.69)}
\end{aligned}
$$

where we omitted smaller exponents. Next observe that $\sum_{i=1}^{3}\beta_i = 0$ and therefore

$$4\beta_1\beta_2 - \beta_3^2 + 4\gamma_1(\alpha_1 + 4\alpha_2 + 2\alpha_3) \geq 0 \tag{B.70}$$
$$4\gamma_1(\alpha_1 + 4\alpha_2 + 2\alpha_3) \geq (-\beta_1 - \beta_2)^2 - 4\beta_1\beta_2 \tag{B.71}$$
$$\gamma_1(\alpha_1 + 4\alpha_2 + 2\alpha_3) \geq \frac{1}{4}(\beta_1 - \beta_2)^2 \tag{B.72}$$

Then

$$\left(\sum_{i=1}^{3} b_i\beta_i\right)^2 = b_1^2(\beta_1 + \beta_2 x + \frac{1}{2}(-\beta_1 - \beta_2)(1+x))^2 \tag{B.73}$$

$$= \frac{1}{4}b_1^2(1-x)(\beta_1 - \beta_2)^2 \tag{B.74}$$

$$\leq b_1^2\gamma_1(\alpha_1 + \alpha_2 + \alpha_3)(1-x)^2. \tag{B.75}$$

On the other hand

$$4\left(\sum_{i=1}^{3} a_i\alpha_i\right)\left(\sum_{i=1}^{3} c_i\gamma_i\right) = 4(\alpha_1 + \alpha_2 + \alpha_3)\frac{1}{4}c_1\gamma_1(1-x)^2 \tag{B.76}$$

$$= b_1^2\gamma_1(\alpha_1 + \alpha_2 + \alpha_3)(1-x)^2 \tag{B.77}$$

and therefore equation (B.24) holds and this parameterisation offers no generality over the original one.

**Case 2** Next we analyse the case when $f$ is of the form as in equation (B.7). Then $h_1, h_2, h_3$ are of the form

$$h_i(T) = (a_iT + b_i)\exp(-\lambda_1 T) + c_i\exp(-\lambda_2 T), \tag{B.78}$$

where $\lambda_1 \neq \lambda_2$. Then equation (B.10) implies

$$(a_1T + b_1)(a_2T + b_2) = (a_3T + b_3)^2 \tag{B.79}$$

Note that $a_1 = 0$ implies $a_2 = a_3 = 0$ reducing the problem to Case 1. Similarly $a_2 = 0$ implies $a_1 = a_3 = 0$. Then we can without loss of generality assume $a_1a_2a_3 \neq 0$ and therefore

$$a_1a_2\Big(T + \frac{b_1}{a_1}\Big)\Big(T + \frac{b_2}{a_2}\Big) = a_3^2\Big(T + \frac{b_3}{a_3}\Big)^2. \tag{B.80}$$

In particular, there exists $x \in \mathbb{R}$ such that

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = x \tag{B.81}$$

and

$$h_i(T) = a_i(T + x)\exp(-\lambda_1 T) + c_i\exp(-\lambda_2 T). \tag{B.82}$$

Now we can use (B.10) again and obtain

$$a_1c_2(T + x) + a2c_1(T + x) = 2a_3c_3(T + x) \tag{B.83}$$

$$a_1a_2 = a_3^2 \tag{B.84}$$

$$c_1c_2 = c_3^2 \tag{B.85}$$

And it is straightforward to deduce

$$(a_1c_2 - a_2c_1)^2 = 0. \tag{B.86}$$

It then follows that

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = x' \tag{B.87}$$

for some $x' \in \mathbb{R}$ and

$$a_1a_2 = a_3c_3x'. \tag{B.88}$$

Therefore $\frac{c_3}{a_3} = x'$, $h_2 = \frac{a_2}{a_1}h_1$ and $h_3 = \frac{a_3}{a_1}h_1$. Then $f$ solves

$$f(T-t) = g(t)h(T), \tag{B.89}$$

where $g = g_1 + \frac{a_2}{a_1}g_2 + \frac{a_3}{a_1}g_3$ and $h = h_1$. In particular, $f$ is an exponential thus contradicting the assumption that $f$ is of the form as in equation (B.7) where $x_1 \neq 0$.

**Case 3** Next we analyse the case when $f$ is of the form as in equation (B.8). Then $g_i, h_i, i = 1, 2, 3$, are of the form

$$g_i(t) = p_i(t)\exp(\lambda_1 t), \tag{B.90}$$

$$h_i(T) = q_i(T)\exp(-\lambda_1 t), \tag{B.91}$$

where $p_i$ and $q_i$ are polynomials of degree two or less. Without loss of generality we can assume that the leading coefficient in polynomials $q_i, i = 1, 2, 3$, is equal to 1. Denote by $P$ the quadratic polynomial $P(y) := f(y)\exp(\lambda_1 y)$ and note that

$$P(T-t) = \sum_{i=1}^{3} p_i(t)q_i(T). \tag{B.92}$$

Note hat equation (B.10) implies that polynomials $q_1$ and $q_2$ have to be of the same parity and at least one of the polynomials $q_1$ and $q_2$ has to be quadratic. We can therefore without loss of generality assume that $q_1$ is a quadratic polynomial. We then have two possibilities: $\deg q_2 = 0$ and $\deg q_2 = 2$.

We first explore the former, i.e. $\deg q_2 = 0$, then $\deg q_1 = 1$. In particular, the polynomials $q_1, q_2, q_3$ are then of the form

$$q_1(T) = (T+c)^2, \quad q_2(T) = 1, \quad q_3(T) = T + c, \tag{B.93}$$

for some $c \in \mathbb{R}$. Differentiating (B.92) twice with respect to $T$ then yields

$$P''(T-t) = \sum_{i=1}^{3} p_i(t)q_i''(T) = 2p_1(t) \tag{B.94}$$

and therefore $p_1(t) = a$. Since $g_1$ is a non-negative function $a > 0$ (note that $a = 0$ leads to Case 1). Next we differentiate (B.92) once with respect to $T$ and obtain

$$P'(T-t) = \sum_{i=1}^{3} p_i(t) q_i'(T) = 2a(T+c) + p_3(t). \tag{B.95}$$

Since the left-hand side in (B.95) is a function of $T - t$ only $p_3$ has to be of the form $p_3(t) = -2a(t+b), b \in \mathbb{R}$. Finally,

$$P(T-t) = a(T+c)^2 + p_2(t) - 2a(t+b)(T+c) \tag{B.96}$$

$$= a(T-t+c-b)^2 - a(t+b)^2 + p_2(t) \tag{B.97}$$

and $p_2(t) = a(t+b)^2 + d, d \in \mathbb{R}$. Then $f$, $v$ and $\sigma$ are of the form

$$f(T-t) = \big(a(T-t+c-b) + d\big) \exp\big(-\lambda_1(T-t)\big) \tag{B.98}$$

$$v(T) = \begin{bmatrix} (T+c)\exp(-\frac{1}{2}\lambda_1 T) \\ \exp(-\frac{1}{2}\lambda_1 T) \end{bmatrix}, \tag{B.99}$$

$$\sigma(t) = \begin{bmatrix} \sqrt{a}\exp(\frac{1}{2}\lambda_1 t) \\ \sqrt{a(t+b)^2 + d}\exp(\frac{1}{2}\lambda_1 t) \end{bmatrix} \tag{B.100}$$

and we can determine $\rho_{1,2}$ from equation (3.34)

$$\rho_{1,2} = -\frac{(t+b)}{\sqrt{(t+b)^2 + d}}. \tag{B.101}$$

Recall that $\rho_{1,2}(t) \in [-1, 1], t \geq 0$ which implies $d \geq 0$, moreover since $f$ depends only on the parameters difference $b, c$ only through their difference we can set $c = 0$. We then obtain Parameterisation 2.2 as in the statement of the theorem by introducing $\alpha = \sqrt{a}, \beta = b, \gamma = d$ and $\lambda = \lambda_1$.

Now let us analyse the case when $\deg q_1 = \deg q_2 = 2$. Note that equation (B.10) implies that when $q_1$ has two distinct roots then $q_2$ and $q_3$ have the same roots as $q_1$ and therefore functions $h_1, h_2$ and $h_3$ are linearly dependent which in turn implies that $f$ cannot be a quadratic polynomial multiplied by an exponential. Then $q_1, q_2$ and $q_3$ have to be of the form

$$q_1 = (T+c)^2, \qquad q_2 = (T+d)^2, \qquad q_3 = (T+c)(T-d), \tag{B.102}$$

where $c \neq d$. We can now follow the same argument as for the case $\deg q_2 = 0$ and observe that

$$P''(T-t) = 2(p_1(t) + p_2(t) + p_3(t)) = 2a > 0 \tag{B.103}$$

and

$$\begin{aligned} P'(T-t) &= 2(T+c)p_1(t) + 2(T+d)p_2(t) + (2T+c+d)p_3(t) && \text{(B.104)} \\ &= 2aT + 2cp_2(t)2dp_2(t) + (c+d)p_3(t) && \text{(B.105)} \end{aligned}$$

and therefore

$$2cp_2(t)2dp_2(t) + (c+d)p_3(t) = -2a(t+b). \tag{B.106}$$

Then

$$\begin{aligned} P(T-t) &= (T+c)^2p_1(t) + (T+d)^2p_2(t) + (T+c)(T+d)p_3(t) && \text{(B.107)} \\ &= aT^2 + 2aT(t+b) + c^2p_1(t) + d^2p_2(t) + cdp_3(t) && \text{(B.108)} \end{aligned}$$

and therefore

$$c^2p_1(t) + d^2p_2(t) + cdp_3(t) = a((t+b)^2+e). \tag{B.109}$$

Note that $p_1, p_2, p_3$ then solve the following system of linear equations

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2c & 2d & c+d \\ c^2 & d^2 & cd \end{bmatrix}}_{:=A} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \begin{bmatrix} a \\ -2a(t+b) \\ a\big((t+b)^2+e\big) \end{bmatrix} \tag{B.110}$$

Recall that $c \neq d$, then the determinant $\det A = (c-d)^3 \neq 0$ and therefore $A$ is an invertible matrix. Consequently, functions $p_1, p_2, p_3$ are well defined by (B.110). In particular,

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix} = \frac{1}{(c-d)^2} \begin{bmatrix} a\big((t+b+d)^2+e\big) \\ a\big((t+c+d)^2+e\big) \\ -2a\big((t+b+c)(t+b+d)+e\big) \end{bmatrix}. \tag{B.111}$$

Finally, we need to find the conditions under which $\rho_{1,2}(t) \in [-1,1], t \geq 0$. Note that this is the case if and only if $4p_1(t)p_2(t) \geq p_3(t)^2$ (see equation (3.34)). In particular,

$$\begin{aligned} 0 &\leq 4p_1(t)p_2(t) - p_3(t)^2 && \text{(B.112)} \\ &= \frac{4a^2e}{(c-d)^2} && \text{(B.113)} \end{aligned}$$

and therefore $e > 0$. Then $f$ is of the form

$$f(y) = a\big((y+b)^2+e\big)\exp(-\lambda y), \tag{B.114}$$

where $a > 0$, $b, \lambda \in \mathbb{R}$ and $e \geq 0$. Therefore this parameterisation offers no generality over the previous case.

**Case 4** Finally we analyse the case when $f$ is of the form as in equation (B.9). We will denote the imaginary unit by $\iota$ to avoid confusion with index $i$. Without loss of generality we can take $\lambda_1 = \lambda - \iota\theta, \lambda, \theta \in \mathbb{R}$ and $x_1 = \frac{1}{2}(u - \iota w), u, w \in \mathbb{R}$. We can then express $f$ as

$$f(y) = \big(u\cos(\theta y) + w\sin(\theta y)\big)\exp(-\lambda y) + x_3\exp(-\lambda_2 y), \tag{B.115}$$

and $g_i, h_i, i = 1, 2, 3$, have to be of the form

$$g_i(t) = \big(\alpha_i\cos(\theta t) + \beta_i\sin(\theta t)\big)\exp(\lambda t) + \gamma_i\exp(\lambda_2 t), \tag{B.116}$$

$$h_i(T) = \big(a_i\cos(\theta T) + b_i\sin(\theta T)\big)\exp(-\lambda T) + c_i\exp(-\lambda_2 T), \tag{B.117}$$

for some $\alpha_i, \beta_i, \gamma_i, a_i, b_i, c_i \in \mathbb{R}$.

If $\lambda \neq \lambda_2$ equation (B.10) implies

$$a_1a_2 = a_3^2, \tag{B.118}$$

$$b_1b_2 = b_3^2, \tag{B.119}$$

$$c_1c_2 = c_3^2, \tag{B.120}$$

$$a_1b_2 + a_2b_1 = 2a_3b_3, \tag{B.121}$$

$$a_1c_2 + a_2c_1 = 2a_3c_3, \tag{B.122}$$

$$b_1c_2 + b_2c_1 = 2b_3c_3. \tag{B.123}$$

Recall that we got the same set of equations in Case 1 ($\lambda_2 \neq \frac{1}{2}(\lambda_1 + \lambda_3)$) in particular it is easy to observe that vectors $(a_i, b_i, c_i), i = 1, 2, 3$, are co-linear, thus implying that $f$ is a solution to a Pexider equation, in particular $u = w = 0$. Thus it offers no generality over Case 1.

We can therefore assume $\lambda_2 = \lambda$, note that since $h_1, h_2 \geq 0$ this implies $c_1, c_2 > 0$.[1] We can without loss of generality assume $c_1 = c_2 = 1$. Then equation (B.10) implies

$$a_1a_2 + 1 = a_3^2 + c_3^2, \tag{B.124}$$

$$b_1b_2 + 1 = b_3^2 + c_3^2, \tag{B.125}$$

$$a_1b_2 + a_2b_1 = 2a_3b_3, \tag{B.126}$$

$$a_1 + a_2 = 2a_3c_3, \tag{B.127}$$

$$b_1 + b_2 = 2b_3c_3. \tag{B.128}$$

Let us first consider the case when $a_1 + a_2 = 0$ and $c_3 = 0$. It will turn out that we will be able to capture all the parameterisations where $f$ is of the form as in (B.9) and is a non-negative function.

Equation (B.128) then implies $b_1 + b_2 = 0$. Define $a := a_1 = -a_2$ and $b := b_1 = -b_2$,

[1] If $c_1 = 0$ it follows $h_1 = h_3 \equiv 0$ and similarly if $c_2 = 0$ then $h_2 = h_3 \equiv 0$ and in both cases it is trivial to observe that $f$ is an exponential function.

then it follows from equations (B.124)–(B.126)

$$a^2 + b^2 = 1, \qquad a_3^2 = b^2, \qquad b_3^2 = a^2 \tag{B.129}$$

moreover $a_3 b_3 = -ab$. We can then assume without loss of generality that $b = \sqrt{1-a^2}$ and

$$h_1(T) = \big(a\cos(\theta T) + b\sin(\theta T) + 1\big)\exp(-\lambda T), \tag{B.130}$$

$$h_2(T) = \big(-a\cos(\theta T) - b\sin(\theta T) + 1\big)\exp(-\lambda T), \tag{B.131}$$

$$h_3(T) = \big(b\cos(\theta T) - a\sin(\theta T)\big)\exp(-\lambda T). \tag{B.132}$$

Next we would like to use (B.1) to find the constraints on the parameters $\alpha_i, \beta_i, \gamma_i, i = 1, 2, 3$. First, recall that $\cos(T-t)$ and $\sin(T-t)$ can be expanded as

$$\cos(T-t) = \cos T\cos t + \sin T\sin t, \tag{B.133}$$

$$\sin(T-t) = -\cos T\sin t + \sin T\cos t. \tag{B.134}$$

Therefore equation (B.1) implies

$$a\gamma_1 - a\gamma_2 + b\gamma_3 = 0 \tag{B.135}$$

$$b\gamma_1 - b\gamma_2 - a\gamma_3 = 0 \tag{B.136}$$

$$\alpha_1 + \alpha_2 = 0 \tag{B.137}$$

$$\beta_1 + \beta_2 = 0 \tag{B.138}$$

$$a(\alpha_1 - \alpha_2) + b\alpha_3 = b(\beta_1 - \beta_2) - a\beta_3 \tag{B.139}$$

$$-a(\beta_1 - \beta_2) - b\beta_3 = b(\alpha_1 - \alpha_2) - a\alpha_3 \tag{B.140}$$

In particular, $\alpha := \alpha_1 = -\alpha_2$, $\beta := \beta_1 = -\beta_2$, $\gamma := \gamma_1 = \gamma_2$ and $\gamma_3 = 0$. Then we can use (B.139) and (B.140) to find $\alpha_3 = 2\beta$ and $\beta_3 = -2\alpha$, hence

$$g_1(t) = \big(\alpha\cos(\theta t) + \beta\sin(\theta t) + \gamma\big)\exp(\lambda t), \tag{B.141}$$

$$g_2(t) = \big(-\alpha\cos(\theta t) - \beta\sin(\theta t) + \gamma\big)\exp(\lambda t), \tag{B.142}$$

$$g_3(t) = \big(2\beta\cos(\theta t) - 2\alpha\sin(\theta t)\big)\exp(\lambda t). \tag{B.143}$$

Recall that $g_2$ and $g_2$ have to be non-negative, which implies that $\gamma > 0$ and $\alpha^2 + \beta^2 \leq \gamma^2$ and it is easy to verify that inequality $4g_1(t)g_2(t) \geq g_3(t)^2$, which ensures that $\rho_{1,2}(t) \in [-1, 1], t \geq 0$.

Then, $f$ is of the form

$$f(y) = 2\big((a\alpha + b\beta)\cos(\theta y) + (b\alpha - a\beta)\sin(\theta y) + \gamma\big)\exp(-\lambda y). \tag{B.144}$$

Moreover, $f$ is a non-negative function if and only if $\gamma > 0$ and $\alpha^2 + \beta^2 \leq \gamma^2$ which is our assumption. Next observe that fixing $a = 0$ implies $b = 1$ and

$$f(y) = 2\big(\beta \cos(\theta y) + \alpha \sin(\theta y) + \gamma\big) \exp(-\lambda y). \tag{B.145}$$

Which clearly losses no generality. Finally, we need to determine functions $v, \sigma$ and $\rho_{1,2}$. Note, that defining $v_i(T) := \sqrt{h_i(T)}, T \geq 0, i = 1, 2$, will not work since $h_3(T) = v_1(T)v_2(T)$ assumes values between $[-1, 1]$. We then have account the sign of the function $h_3$ while maintaining continuity of functions $v_1$ and $v_2$.

Note that

$$1 + \sin(\theta T) = 0 \Leftrightarrow \cos\frac{\theta T}{2} + \sin\frac{\theta T}{2} = 0, \tag{B.146}$$
$$1 - \sin(\theta T) = 0 \Leftrightarrow \cos\frac{\theta T}{2} - \sin\frac{\theta T}{2} = 0 \tag{B.147}$$

and

$$\cos(\theta x) = \Big(\cos\frac{\theta x}{2} + \sin\frac{\theta x}{2}\Big)\Big(\cos\frac{\theta x}{2} - \sin\frac{\theta x}{2}\Big). \tag{B.148}$$

The $v$ defined as

$$v(T) = \begin{bmatrix} \operatorname{sgn}(\cos\frac{\theta T}{2} + \sin\frac{\theta T}{2})\sqrt{1 + \sin(\theta T)}\exp(-\frac{1}{2}\lambda T) \\ \operatorname{sgn}(\cos\frac{\theta T}{2} - \sin\frac{\theta T}{2})\sqrt{1 - \sin(\theta T)}\exp(-\frac{1}{2}\lambda T) \end{bmatrix} \tag{B.149}$$

is a continuous function and $v_1(T)v_2(T) = h_3(T)$.

Now let us turn our attention to $\sigma$ and $\rho_{1,2}$. When $\alpha^2 + \beta^2 < \gamma^2$ $g_1$ and $g_2$ are strictly positive functions and $4g_1(t)g_2(t) > g_3^2(t)$ for all $t$. Then $\sigma$ and $\rho_{1,2}$ can be parametrised as

$$\sigma(t) = \begin{bmatrix} \sqrt{\alpha \cos(\theta t) + \beta \sin(\theta t) + \gamma}\exp(\frac{1}{2}\lambda t) \\ \sqrt{-\alpha \cos(\theta t) - \beta \sin(\theta t) + \gamma}\exp(\frac{1}{2}\lambda t) \end{bmatrix}, \tag{B.150}$$
$$\rho_{1,2} = \frac{\beta \cos(\theta t) - \alpha \sin(\theta t)}{\sqrt{\gamma^2 - (\alpha \cos(\theta t) + \beta \sin(\theta t))^2}} \tag{B.151}$$

Note that $\rho_{1,2}$ is well defined and continuous since $\alpha^2 + \beta^2 < \gamma^2$.

On the other hand when $\alpha^2 + \beta^2 = \gamma^2$ it follows that $4g_1(t)g_2(t) = g_3(t)^2$ for all $t$ which implies $\rho_{1,2}(t)^2 = 1$, in particular since $\rho_{1,2}$ has to be a continuous function it has to be constant. We can then parameterise $\sigma$ by first observing that a constant $\phi$ defined by

$$\phi = \begin{cases} \arccos\frac{\alpha}{\gamma^2}; & \beta \geq 0 \\ -\arccos\frac{\alpha}{\gamma^2}; & \beta < 0 \end{cases} \tag{B.152}$$

satisfies $\cos\phi = \frac{\alpha}{\gamma^2}$ and $\sin\phi = \frac{\beta}{\gamma^2}$. Then

$$\gamma + \alpha\cos(\theta t) + \beta\sin(\theta x) = 0 \Leftrightarrow \cos\frac{\theta t - \phi}{2} = 0, \tag{B.153}$$

$$\gamma - \alpha\cos(\theta t) - \beta\sin(\theta x) = 0 \Leftrightarrow \sin\frac{\theta t - \phi}{2} = 0 \tag{B.154}$$

and

$$\beta\cos(\theta t) - \alpha\sin(\theta t) = -2\cos\frac{\theta t - \phi}{2}\sin\frac{\theta t - \phi}{2}. \tag{B.155}$$

Then we can parametrise $\sigma$ and $\rho_{1,2}$ as

$$\sigma(t) = \begin{bmatrix} \operatorname{sgn}(\cos\frac{\theta t-\phi}{2})\sqrt{\alpha\cos(\theta t) + \beta\sin(\theta t) + \gamma}\exp(\frac{1}{2}\lambda t) \\ -\operatorname{sgn}(\sin\frac{\theta t-\phi}{2})\sqrt{-\alpha\cos(\theta t) - \beta\sin(\theta t) + \gamma}\exp(\frac{1}{2}\lambda t) \end{bmatrix}, \tag{B.156}$$

$$\rho_{1,2} = 1 \tag{B.157}$$

in particular note that $\sigma$ is a continuous function.

And we can get the Parameterisation 2.3 from the statement theorem by rescaling parameter $\lambda \mapsto 2\lambda$.

## B.2 Proof of Lemma 4.2

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{N})$ be a filtered probability space and let $x$ be a Markov process under the measure $\mathbb{N}$. Suppose that $0 \leq t < T$. Let $\mathcal{V} \subset \mathcal{L}^1(\mathbb{N})$ be a collection of $\sigma(x_s, s \in [t,T])$-measurable such that for $V \in \mathcal{V}$

$$\mathbb{E}[V|\mathcal{F}_t] = \mathbb{E}[V|x_t]. \tag{B.158}$$

Note that $\mathcal{V}$ is a linear subspace of $\mathcal{L}^1(\mathbb{N})$ since the conditional expectation is a linear functional.

Define a collection of sets $\mathcal{A}$ by

$$\mathcal{A} := \{A = A_{t_1} \cap \cdots A_{t_n}; \forall n \in \mathbb{N}, \forall i \in \{1,\ldots,n\} : A_{t_i} \in \sigma(x_{t_i}), t_i \in [t,T]\}. \tag{B.159}$$

Then for any set $A \in \mathcal{A}$ its indicator function can be written as

$$\mathbf{1}_A = \mathbf{1}_{A_{t_1}} \ldots \mathbf{1}_{A_{t_n}}. \tag{B.160}$$

Since $x$ is a Markov process and $A_{t_i} \in \sigma(x_{t_i})$

$$\mathbb{E}[\mathbf{1}_A|\mathcal{F}_t] = \mathbb{E}[\mathbf{1}_A|x_t] \tag{B.161}$$

and therefore $\mathbf{1}_A \in \mathcal{V}$. Moreover, observe that the collection $\mathcal{A}$ is closed under finite intersections and contains the whole probability space $\Omega$. Therefore $\mathcal{A}$ is a $\pi$-system.

Moreover $\sigma(\mathcal{A}) = \sigma(x_s, s \in [t,T])$.

Now define a collection $\mathcal{B}$ by

$$\mathcal{B} := \{B \subset \Omega; \mathbf{1}_B \in \mathcal{V}\}. \tag{B.162}$$

We will show that $\mathcal{B}$ is a $\lambda$-system containing collection $\mathcal{A}$, thus allowing us to apply Dynkin's $\pi$-$\lambda$ Theorem. Trivially $\mathbf{1}_\Omega \equiv 1 \in \mathcal{V}$, and since $\mathcal{V}$ is a vector space $\mathbf{1}_{B^c} = 1 - \mathbf{1}_B \in \mathcal{V}$. We therefore only need to show that for an increasing sequence of sets $(B_i)_{i\in\mathbb{N}}$ the union

$$B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{B}. \tag{B.163}$$

Since the sequence of the sets $(B_i)_{i\in\mathbb{N}}$ is increasing the sequence of indicator functions $(\mathbf{1}_{B_i})_{i=1}^{\infty}$ is also increasing and

$$\lim_{i\to\infty} \mathbf{1}_{B_i}(\omega) = \mathbf{1}_B(\omega), \quad \omega \in \Omega. \tag{B.164}$$

We can therefore use the conditional monotone convergence theorem to show

$$\mathbb{E}[\mathbf{1}_B|\mathcal{F}_t] = \lim_{i\to\infty} \mathbb{E}[\mathbf{1}_{B_i}|\mathcal{F}_t] = \lim_{i\to\infty} \mathbb{E}[\mathbf{1}_{B_i}|x_t] = \mathbb{E}[\mathbf{1}_B|x_t]. \tag{B.165}$$

Therefore $B \in \mathcal{B}$ and $\mathcal{B}$ is a $\lambda$-system. Furthermore, $\mathcal{A} \subset \mathcal{B}$ since $\mathbf{1}_A \in \mathcal{V}, A \in \mathcal{A}$. Therefore we can apply Dynkin's $\pi$-$\lambda$ Theorem to show

$$\sigma(x_s, s \in [t,T]) = \sigma(\mathcal{A}) \subset \mathcal{B}. \tag{B.166}$$

Therefore, $\mathcal{V}$ contains all the indicator functions of $\sigma(x_s, s \in [t,T])$-measurable sets and since $\mathcal{V}$ is a vector space over $\mathbb{R}$ it contains all the simple functions (linear combinations of indicator functions). Furthermore, the collection $\mathcal{V}$ is closed under limits of increasing sequences by the conditional monotone convergence theorem. Therefore, we can use the standard construction to prove that $\mathcal{V}$ contains all integrable positive $\sigma(x_s; s \in [t,T])$-measurable functions. Finally, we can prove that $\mathcal{V}$ contains all integrable $\sigma(x_s; s \in [t,T])$ functions by viewing them as a difference of positive and negative parts and using the fact that $\mathcal{V}$ is a vector space.

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