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FOUR DIMENSIONAL GEOMETRIES

by

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CONTENTS

INTRODUCTION

The idea that geometric methods can play an important role in the resolution of topological questions, dormant since the turn of the century, has been revived recently by the work of Thurston (see [14] for a summary). Such methods are based on the notion of a "geometric" manifold. This is a manifold equipped with a Riemannian metric for which the pseudo-group of local isometries is transitive. The universal cover of a geometric manifold is also geometric and, in fact, has a transitive group of isometries since any local isometry of a simply connected Riemannian manifold is the restriction of a global isometry. To determine which manifolds admit a geometric structure it is, therefore, first necessary to determine which simply connected manifolds admit such structures. Since several metrics on a given manifold may have the same group of isometries - or at least the same connected component of the identity - we are led to the following problem: Determine all simply connected manifolds admitting a transitive Lie group action such the stabilizer subgroup at each point is compact. Such a structure is called a geometry in keeping with Klein's Erlanger Programme. The problem can be reduced somewhat by noting that any group action with compact stabilizer is contained in a largest such action so we need only classify maximal geometries.

The most important subclass of geometries consists of the ones covering compact manifolds or, more generally, manifolds of finite volume with respect to the volume form coming from some locally homogeneous Riemannian metric. In dimension 1 this problem is trivial and the classical Uniformization Theorem gives the 2-dimensional geometries as the Euclidean plane, the

Hyperbolic plane or the round 2-sphere. In dimension 3 the maximal geometries covering manifolds of finite volume have been determined by Thurston (see Scott [13] for a good survey of results in the field of geometric 3-manifolds). It turns out that there are only 8 maximal 3-dimensional geometries covering manifolds of finite volume and they all cover compact manifolds. These geometries are either the 3-dimensional spaces of constant curvature, have trivial stabilizer or fibre over one of the two dimensional geometries.

In this thesis we extend the idea of fibering a geomtry over a lower dimensional one to determine all the maximal 4-dimensional geometries. We find that there is a countable infinity of inequivalent such geomtries two of which have quotients of finite volume but no compact quotients. The classification is done on a case by case analysis of the possible stabilizer subgroups which can be regarded as subgroups of SO(4). In Chapter 1 we give a formal definition and collect together various results that will be needed for the remainder of this work. In particular the possibilities for the stabilizer are determined. In the last section of Chapter 1 we fix some notation and assumptions. In Chapter 2 we show how the existence of invariant distributions on a geometry gives rise to equivariant fiberings over a lower dimenionsal homogeneous space. The geometries with stabilizer not trivial or isomorphic to SO(2) are shown, in Chapter 3, to be Riemannian globally symmetric spaces. In Chapters 4 and 5 the case where the stabilizer is isomorphic to SO(2) is analysed. Finally the maximal geometries with trivial stabilizer are determined in Chapter 6 where it is shown that there is a countable infinity of inequivalent such geometries. A summary of the maximal 4-dimensional goemetries determined in Chapters 3-6 is then provided.

CHAPTER 1 : PRELIMINARIES.

1.0 Introduction

In this chapter we will define the objects to be studied and collect together a few preliminary results concerning them. Since the purpose is to classify four dimensional geometries we will need to know the compact connected subgroups of $SO(4)$. These will be determined in Section 2. Finally, in Section 3, a certain amount of notation will be fixed for the rest of this thesis.

1.1 Geometries

The objects that will be of interest to us are defined as follows:

Definition

A geometry is a triple (M,G,α) with M a connected, simply connected, smooth manifold M, G a Lie group and $\alpha:G\times M \to M$ a smooth transitive effective action of G on M such that the stabilizer subgroup G_x is compact for each $x \in M$.

Two geometries (M_1,G,α_1) and (M_2,G,α_2) will be considered equivalent if there is a diffeomorphism $\phi:M_1 \to M_2$ such that $\alpha_2(g,\phi(x)) = \phi(\alpha_1(g,x))$ for all $x \in M$, $g \in G$. Where the action of G on M is understood we will generally write $\alpha(g,x)$ as $g\cdot x$ or $g(x)$ and the geometry as (M,G).

The coset space G/G_x for $x \in M$ has a natural smooth (even analytic) manifold structure. The mapping $\phi:G/G_x \to M$ defined by $\phi(gG_x) = g(x)$ is a smooth bijection. By the Rank Theorem ϕ is, in fact, a diffeomorphism and it is easy to see that it provides an equivalence between the left action of G on G/G_x and the action of G on M. Hence the geometry (M,G,α) can be identified with the coset space G/G_x equipped with the natural left action of G. Moreover in the definition of equivalence above we may suppose the diffeomorphism is analytic since an equivalence corresponds to left multiplication by an element of G.

By the previous paragraph the set $G_i \cdot x$ is open in M for any connected component G_i of G and any two such are disjoint or coincide. Since M is connected this implies that the identity component G_0 of G is transitive on M. Because of this we will henceforth assume that G is connected.

Remark

M simply connected implies that it is orientable so if G is connected it acts by orientation preserving diffeomorphisms.

Proposition 1.1.1

The stabilizer G_x of any point $x \in M$ is connected.

Proof

Let $(G_x)_0$ denote the connected component of e in G_x. Then we have a fibre bundle:

$$G_x/(G_x)_0 \to G/(G_x)_0 \to G/G_x \cong M .$$

Since G_x is compact $G_x/(G_x)_0$ is finite so this is a covering map and hence a diffeomorphism by the assumption $\pi_1(M) = 0$. Therefore $G_x = (G_x)_0$. □

It is possible that the action of G on M may extend to the action of some larger group G' containing G as a proper subgroup. For example if we equip M with a Riemannian metric λ it may happen that the largest connected group of isometries $\mathrm{Isom}_0(M,\lambda)$ is strictly larger than G . The most obvious example of this comes from taking $\mathbb{R}^2$ acting on itself by translations. The Euclidean metric on $\mathbb{R}^2$ is invariant under translations but also includes rotations in its group of isometries. However we can simplify matters with

Proposition 1.1.2

Any geometry (M,G,α) is contained in a maximal geometry (M,G',α') where G' is the connected component of e in $\mathrm{Isom}(M,\lambda)$ for some Riemannian metric λ on M .

Proof

If (M,G,α) is a geometry we can find a metric λ on M such that G acts as a group of isometries of (M,λ) . If G_x is the stabilizer of $x \in M$ then the derivative dg_x acts as an isometry on the inner product space $(T_x M,\lambda_x)$ for each $g \in G_x$. Hence we obtain a representation $d_x : G_x \to SO(m)$ where $m = \dim(M)$. This representation is faithful (see [6] Chap. I Sec. 11) so $\dim(G_x) \leq \dim(SO(m)) = m(m-1)/2$. Since $\dim(M) = \dim(G) - \dim(G_x)$ we have $\dim(G) \leq m(m+1)/2$. If (M,G^i,α_i) is a sequence of geometries with $G^i \subseteq G^{i+1}$ and α_{i+1} restricted to $G^i xM$

is equal to α_i , we must, therefore, have $\dim(G^i) = \dim(G^{i+1})$ for i large enough. Since each G^i is assumed connected it follows that $G^i = G^{i+1}$ for all i large enough.

Clearly if (M,G,α) is a maximal geometry and λ is any metric for which G acts as a group of isometries then we must have $G = \mathrm{Isom}_0(M,\lambda)$. $\square$

It is an interesting question whether a given geometry is contained in a unique maximal geometry. We will make a few remarks on this subject at the end of this section. From now on all geometries will be assumed to be maximal and G will be the connected component of e in the group of isometries of (M,λ) for some Riemannian metric λ on M .

To the G invariant metric λ on M we can associate the G-invariant smooth Riemannian volume form ω with its associated measure μ and any two such measures differ by a constant. If $\Gamma \subseteq G$ is a discrete subgroup such that the quotient $\Gamma\backslash M$ is a manifold then μ descends to give a measure μ_Γ on $\Gamma\backslash M$. It is the purpose of this paper to classify four dimensional geometries.such that $\mu_\Gamma(\Gamma\backslash M)$ is finite for some such $\Gamma \subseteq G$ i.e. geometries which cover manifolds of finite volume. We will, therefore, henceforth assume that all geometries considered possess such a quotient.

In the sequel we will need the following two propositions, the first of which is an easy generalization to homogeneous spaces of standard results on lattices in Lie groups.

Proposition 1.1.3

If there is a discrete subgroup $\Gamma \subseteq G$ such that $\Gamma\backslash G/G_x$ has finite volume then G is unimodular.

Proof

Let μ denote the left invariant smooth measure on $M = G/G_x$ associated to a left invariant Riemannian metric. If ν denotes a Haar measure on G and $\pi : G \to G/G_x = M$ the quotient map then the measure μ' defined by $\mu'(E) = \nu(\pi^{-1}(E))$ is obviously left invariant. Since left invariant measures on homogeneous spaces are unique up to a scalar multiple we can assume, without loss of generality, that $\mu = \mu'$.

For $\Gamma \subseteq G$ a discrete subgroup let $p : G/G_x \to \Gamma\backslash G/G_x$ and $q : G \to \Gamma\backslash G$ denote the quotient maps. Assume that $\mu_\Gamma(\Gamma\backslash G/G_x) < \infty$. We can choose a collection $\{V_i\}_{i\in I}$ of subsets $V_i \subseteq G/G_x$ such that $\Gamma\backslash G/G_x = \bigcup_{i\in I} p(V_i)$ and $\sum_{i\in I} \nu(V_i) < \infty$. Setting $W_i = \pi^{-1}(V_i)$ we have $\Gamma\backslash G = \bigcup_{i\in I} q(W_i)$ and $\sum_{i\in I} \nu(W_i) < \infty$. Let $\hat{\nu}$ denote the measure on $\Gamma\backslash G$ induced by ν . Then, if $E \subseteq \Gamma\backslash G$, we have $\hat{\nu}(E) = \inf(\sum_{\alpha\in A} \nu(F_\alpha))$ where the infimum is taken over all covers of E . i.e. collections of sets $\{F_\alpha\}_{\alpha\in A}$ with $F_\alpha \subseteq G$ and $E \subseteq \bigcup_{\alpha\in A} q(F_\alpha)$. We have $\hat{\nu}(\Gamma\backslash G) \leq \sum_{i\in I} \nu(W_i) < \infty$. Let Δ denote the modular function of G . If $g \in G$ then the collection $\{W_i \cdot g\}$ is a cover of $\Gamma\backslash G$ so $\hat{\nu}(\Gamma\backslash G) \leq \sum_{i\in I} \nu(W_i \cdot g) = \Delta(g) \sum_{i\in I} \nu(W_i) = c\Delta(g)$ for some $c \in \mathbb{R}$. If Δ is non-trivial we can find a $g \in G$ such that $\Delta(g) < 1$. Then $\Delta(g^n) \to 0$ as $n \to \infty$. This implies that $\hat{\nu}(\Gamma\backslash G) = 0$ contradicting the discreteness of Γ . Hence Δ is trivial and G is unimodular.

□

Proposition 1.1.4

Let G be a unimodular Lie group, N a closed normal subgroup and $\pi: G \to G/N$ the quotient homomorphism. Then N is unimodular and, if $W \subseteq N$, we have $\nu_N(W) = \Delta_{G/N}(\pi(g)) \cdot \nu_N(gWg^{-1})$ where ν and Δ denote Haar measure and the modular function.

Proof

We first show how Haar measure on G may be constructed from Haar measures on N and G/N. Let ν_N and $\nu_{G/N}$ denote the Haar measures on N and G/N. If $E \subseteq G$ is a Borel subset we set

$$f_E(\pi(x)) = \nu_N(x^{-1}E \cap N) \quad \text{for} \quad x \in G .$$

If $y = xn$ for some $n \in N$ then

$$\begin{aligned} f_E(\pi(y)) &= \nu_N(n^{-1}x^{-1}E \cap N) \\ &= \nu_N(n^{-1}(x^{-1}E \cap N)) \\ &= \nu_N(x^{-1}E \cap N) \quad \text{since } \nu_N \text{ is left invariant.} \end{aligned}$$

Hence f_E is a well defined map on G/N. In fact f_E is a Borel map (see [4] Section 63) so we can define

$$\nu_G(E) = \int_{G/N} f_E \, d\nu_{G/N} .$$

If $g \in G$ then $f_{gE}(\pi(x)) = f_E(\pi(g)^{-1}\pi(x))$ so

$$\nu_G(gE) = \int_{G/N} f_E(\pi(g)^{-1}\pi(x)) \, d\nu_{G/N}$$

$$= \int_{G/N} f(\pi(x))d\nu_{G/N} \quad \text{since } \nu_{G/N} \text{ is left invariant.}$$

Therefore ν_G is left invariant. It is easy to see that ν_G is finite on compact sets and so is a Haar measure for G.

Since N is closed we can find a subset $U \subseteq G/N$ such that $\nu_{G/N}(U) \neq 0,\infty$ and there is a section $\phi:U \to G$. To show that N is unimodular let $W \subseteq N$ be a subset and set $A = \phi(U)\cdot W$. By the above construction we have $\nu_G(A) = \nu_{G/N}(U)\nu_N(W)$. If $n \in N$ then $An = \phi(U)\cdot Wn$ so $\nu_G(An) = \nu_{G/N}(U)\cdot\nu_N(Wn)$. But G is unimodular and so $\nu_G(An) = \nu_G(A)$. Hence, since $\nu_{G/N}(U) \neq 0,\infty$ we must have $\nu_N(W) = \nu_N(Wn)$ and N is unimodular.

To prove the second part let W,U,A be as above. If $g \in G$ then $Ag^{-1} = (\phi(U)\cdot g^{-1})\cdot(gWg^{-1})$ and $\nu_G(Ag^{-1}) = \nu_{G/N}(\pi(\phi(U)\cdot g^{-1}))\cdot\nu_N(gWg^{-1})$. But $\pi(\phi(U)) = U$ and $\nu_G(Ag^{-1}) = \nu_G(A)$. So we have $\nu_{G/N}(U)\cdot\nu_N(W) =$ $= \nu_{G/N}(U\pi(g)^{-1})\cdot\nu_N(gWg^{-1})$. But $\nu_{G/N}(U\pi(g)^{-1}) = \Delta_{G/N}(\pi(g))$ and $\nu_{G/N}(U) \neq 0$ and therefore $\nu_N(W) = \Delta_{G/N}(\pi(g))\cdot\nu_N(gWg^{-1})$.

$\square$

<u>Remarks on unique maximality</u>

Consider the simplest case of a simply connected Lie group G acting on itself by left translations. Let $\mathrm{Isom}_0(G,\lambda)$ denote the connected group of isometries for some left invariant metric λ on G. If there is a normal subgroup $G' \subseteq \mathrm{Isom}_0(G,\lambda)$ isomorphic to G then $\mathrm{Isom}_0(G,\lambda) \cong G' \rtimes H$ where H is the stabilizer of e. This holds for all left invariant metrics on G in the following cases:

(1) G nilpotent : Wilson [12]

(2) G compact : Ozeki (see [9] Chapter 3).

(3) G semisimple : Gordon [3].

In this case the unique maximal geometry containing G is clearly $(G, G \rtimes H)$ where H is a maximal compact group of automorphisms of G (any two such are equivalent since maximal compact subgroups are conjugate).

1.2 Subgroups of SO(4)

As the first step in classifying four dimensional geometries we need to determine what groups can occur as the stabilizers G_x. From Propositions 1.1.1 and 1.1.2 G_x must be isomorphic to a compact, connected subgroup of $SO(4)$. This section will be devoted to classifying, up to conjugacy in $SO(4)$, all connected compact subgroups of $SO(4)$. If we let $\pi : SU(2) \times SU(2) \to SO(4)$ denote the universal covering map the task is simplified by:

Lemma 1.2.1

Let H be a compact connected subgroup of $SO(4)$ and denote by $\bar{H}$ the connected component of e in $\pi^{-1}(H)$. Then $\bar{H}$ is compact, connected and $\pi(\bar{H}) = H$.

Proof

Obvious. □

We will first determine, up to conjugacy, all compact connected subgroups of $SU(2) \times SU(2)$.

Lemma 1.2.2

If K is a subgroup of $SU(2)\times SU(2)$ isomorphic to $SU(2)/\Gamma$ with Γ a discrete central subgroup then $\Gamma = \{e\}$ and K is conjugate to one of the following subgroups:

(1) $I_1 = \{(x,e) \mid x \in SU(2)\}$;

(2) $I_2 = \{(e,x) \mid x \in SU(2)\}$;

(3) $D_1 = \{(x,x) \mid x \in SU(2)\}$;

(4) $D_2 = \{(x,\phi(x)) \mid x \in SU(2)\}$;

where ϕ is the unique outer automorphism of $SU(2)$. (See footnote.)*

Proof

Let $i:K \to SU(2)\times SU(2)$ denote the inclusion and $\pi_1,\pi_2:SU(2)\times SU(2)\to SU(2)$ the projection onto the first and second factors. Setting $p_1 = \pi_1 o i$ $p_2 = \pi_2 o i$ it is easy to see that p_j is trivial or an isomorphism for $j = 1,2$. Since $K \neq \{e\}$ at least one of the p_j is non-trivial. So $\Gamma = \{e\}$ and $K \cong SU(2)$. If we represent $SU(2)$ by matrices:

$$SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in GL(2,\mathbb{C}) \mid |a|^2 + |b|^2 = 1\}$$

then the unique outer automorphism $\phi:SU(2) \to SU(2)^*$ is given by $\phi(a,b) = (\bar{a},\bar{b})^*$. Hence there are $g_1,g_2 \in SU(2)$ such that $Ad(g_j)op_j$ is either trivial, the identity or ϕ^* for $j = 1,2$. Thus taking the conjugate of K by (g_1,g_2) we can assume that p_j is trivial, the identity or ϕ^*. The result follows immediately.

□

* The outer automorphism group of $SU(2)$ is trivial so D_2 is redundant. Whenever this redundancy occurs in this section an '*' will be placed.

Let T_1 and T_2 denote the maximal tori $(\cong SO(2))$ of the subgroups I_1 and I_2 .

Proposition 1.2.3

Any compact connected subgroup of $SU(2)\times SU(2)$ is conjugate to one of the following:

(1) $SU(2) \times SU(2)$

(2) $I_1 \times T_2$ or $T_1 \times I_2$

(3) I_1, I_2, D_1, D_2*

(4) A torus contained in $T_1 \times T_2$.

Proof

If H is a compact, connected subgroup of $SU(2)\times SU(2)$ then $H \cong K\times T^n/\Gamma$ where K is compact semisimple, T^n is a torus and Γ is a discrete central subgroup.

If the torus factor is trivial then $K \cong SU(2)\times SU(2)$ or $K \cong SU(2)$. Hence $H = SU(2)\times SU(2)$ or, by Lemma 1.2.2, H is conjugate to I_1, I_2, D_1, D_2*.

If $K = \{e\}$ then H is a torus and, up to conjugacy, we can assume that it is contained in the maximal torus $T_1 \times T_2$ of $SU(2)\times SU(2)$.

If both K and T^n are non-trivial then $K \cong SU(2)$. Let $p: K\times T^n \to H$ be the projection map. By Lemma 1.2.2 we can assume, up to conjugacy, that $p(K)$ is one of I_1, I_2, D_1, D_2*. Now $p(K)$ has a non-trivial connected centralizer and the centralizers of D_1 and D_2* are

* See note pg. 9

discrete. So $p(K)$ is conjugate to I_1 or I_2. Since $p(T^n)$ is connected we must then have $n = 1$ and $p(T) \subseteq I_2$ or I_1. Since all maximal tori in $SU(2)$ are conjugate we have H conjugate to $I_1 \times T_2$ or $T_1 \times I_2$. □

If the quaternions $\mathbb{H}$ are provided with the usual Euclidean metric such that $\{1,i,j,k\}$ is an orthonormal basis, $SU(2)$ is isomorphic to the subgroup of unit quaternions and the universal covering map $\pi: SU(2)\times SU(2) \to SO(4)$ is given by $(y_1,y_2) \to T(y_1,y_2) \in SO(4)$ where $T(y_1,y_2)(x) = y_1 x y_2^{-1}$ for $x \in \mathbb{R}^4$.

<u>Proposition 1.2.4</u>

With respect to the basis $\{1,i,j,k\}$ of $\mathbb{H}$ any compact connected subgroup of $SO(4)$ is represented, up to conjugacy, by one of the following groups of matrices:

(1) $SO(4)$.

(2) The subgroup isomorphic to $SO(3)$ which fixes 1.

(3) The subgroup isomorphic to $SU(2)$ of the form:

$$\left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in GL(4,\mathbb{R}) \;\middle|\; A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix},\quad B = \begin{pmatrix} a_3 & a_4 \\ a_4 & -a_3 \end{pmatrix},\quad \sum_{i=1}^{4} a_i^2 = 1\right\}.$$

(4) The subgroup isomorphic to $SU(2)$ of the form:

$$\{\begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix} \in GL(4,\mathbb{R}) \mid A = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_3 & a_4 \\ -a_4 & a_3 \end{pmatrix}$$

$$\sum_{i=1}^{4} a_i^2 = 1\} .$$

(5) The subgroup isomorphic to $S(U(2)\times U(1)) \subseteq SU(3)$:

$$\{\begin{pmatrix} AC & BC^t \\ -BC & AC^t \end{pmatrix} \in GL(4,\mathbb{R}) \mid A,B \text{ as in (3) above,}$$

$$C = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}, \quad x_1^2 + x_2^2 = 1\} .$$

(6) The subgroup isomorphic to $S(U(2)\times U(1)) \subseteq SU(3)$:

$$\{\begin{pmatrix} AC & BC \\ -B^tC & A^tC \end{pmatrix} \in GL(4,\mathbb{R}) \mid A,B \text{ as in (4) above,}$$

$$C = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}, \quad x_1^2 + x_2^2 = 1\} .$$

(7) The maximal torus of the form:

$$\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(4,\mathbb{R}) \mid A,B \in SO(2)\} .$$

(8) A subgroup isomorphic to $SO(2)$ of the form:

$$\{\begin{pmatrix} A_{n\theta} & 0 \\ 0 & A_{m\theta} \end{pmatrix} \quad GL(4,\mathbb{R}) \mid A_k = \begin{pmatrix} \mathrm{Cos}k\theta & \mathrm{Sin}k\theta \\ -\mathrm{Sin}k\theta & \mathrm{Cos}k\theta \end{pmatrix}$$

$$n,m \in \mathbb{Z} \text{ coprime}\} .$$

Proof

If H is a compact connected subgroup of $SO(4)$ then by Lemma 1.2.1 $H = \pi(\tilde{H})$ where $\tilde{H}$ is a compact connected subgroup of $SU(2)\times SU(2)$. Since conjugacy in $SU(2)\times SU(2)$ implies conjugacy in $SO(4)$ we can assume that $\bar{H}$ is one of the groups listed in Proposition 1.2.3.

(i) $\bar{H} = SU(2) \times SU(2)$.

Then $H = SO(4)$.

(ii) $\bar{H} = I_1$.

Then $T(y,e)$ is the map $x \to yx$. If we choose $y = a_1+ia_2+ja_3+ka_4 \in \mathbb{H}$, $\Sigma\, a_1^2 = 1$ then:

$$T(y,e)(1) = a_1+ia_2+ja_3+ka_4$$

$$T(y,e)(i) = -a_2+ia_1+ja_4-ka_3$$

$$T(y,e)(j) = -a_3-ia_4+ja_1+ka_2$$

$$T(y,e)(k) = -a_4+ia_3-ja_2+ka_1$$

and H is the group (3).

(iii) $\bar{H} = I_2$.

A similar calculation to (ii) above show that H is the group (4) . Here $T(e,y)$ is the map $x \to xy^{-1}$

(iv) $\bar{H} = I_1\times T_2$ or $T_1\times I_2$.

Combining (ii) and (iii) above and using the fact that π is a homomorphism we obtain the groups (5) and (6). In both cases we have $Ker(\pi) \cap \bar{H} = \{(1,1),(-1,-1)\}$ and $SU(2)\times SO(2)/Ker(\pi) \cong S(U(2)\times U(1))$.

(v) Then $T(y,y)(x) = yxy^{-1}$ so $T(y,y)$ fixes ± 1 for each $y \in SU(2)$ and we have the group (2).

(vi) $\bar{H} = D_2^{*}$

In terms of the basis $\{1,i,j,k\}$ of $\mathbb{H}$ the outer automorphism $\phi:SU(2) \to SU(2)$ is given by $\phi(a_1+ia_2+ja_3+ka_4) - a_1-ia_2+ja_3-ka_4$. Hence $T(y,\phi(y))(x)$ is given by:

$$T(y,\phi(y))(x) = (a_1+ia_2+ja_3+ka_4)(x)(a_1+ia_2-ja_3+ka_4) \quad .$$

It is easy to see that $T(y,\phi(y))$ fixes $\pm j$ for each $y \in SU(2)$ and $Ker(\pi) \cap \bar{H} = \{(1,1),(-1,-1)\}$ so, again H is conjugate to the standard representation of $SO(3)$ in $SO(4)$.

(vii) $\bar{H}$ is a torus

Then H is a torus. The subgroup (7) is a maximal torus in $SO(4)$ so H must be conjugate to a subgroup of (7). Hence H is conjugate to the group (7) or a subgroup of the form (8). □

In the next Chapter we will need the following:

Corollary 1.2.5

Any nontrivial compact subgroup of $SO(4)$ not isomorphic to $SO(3)$ or $SO(2)$ contains the mapping of $\mathbb{R}^4$ which sends x to $-x$.

Proof

If $H \subseteq SO(4)$ contains this mapping then so does any subgroup conjugate to H in $SO(4)$. The Corollary now follows by inspecting the possibilities given by Proposition 1.2.4.

□

* see note pg. 9

1.3 Notation and Assumptions

1.3.1 For a manifold M we have:

(i) M is assumed finite dimensional, smooth, paracompact with countably many components.

(ii) The tangent bundle of M will be denoted TM with the tangent space at $x \in M$ denoted T_xM . If $D \subseteq TM$ is a smooth distribution on M then the corresponding subspace of T_xM will be denoted D_x .

(iii) Diff(M) will denote the group of smooth diffeomorphisms of M with $\mathrm{Diff}^+(M)$ denoting the orientation preserving ones.

(iv) The universal cover of M is written $\tilde{M}$.

1.3.2 For a Lie group G we use the following notation:

(i) The identity element of G will be denoted e, id, or I .

(ii) The subscript 0 will denote the connected component of the identity in a Lie group. e.g. $\mathrm{Aut}_0(G)$ denotes the connected component of e in the automorphism group of G .

(iii) The Lie algebra of a Lie group will be written in lower case script letters. e.g. the Lie algebra of G is $\mathcal{g}$, of $SL(2, \mathbb{R})$ is $\mathcal{sl}(2, \mathbb{R})$ etc.

(iv) The centre of G is denoted $Z(G)$.

(v) Ad (resp.ad) will denote the adjoint representation of a Lie group (resp. Lie algebra).

(vi) $\mathrm{Aut}(G)$, $\mathrm{Der}(\mathfrak{g})$ will denote the automorphism group of G and the derivation algebra of $\mathfrak{g}$ respectively. If $\phi: H \to \mathrm{Aut}(G)$ (resp. $\phi: \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$) is a homomorphism then $G \ltimes_{\alpha} H$ (resp. $\mathfrak{g} \ltimes_{\alpha} \mathfrak{h}$) will denote the semi-direct product of G by H (resp. $\mathfrak{g}$ by $\mathfrak{h}$) with action ϕ . If there is no possibility of misunderstanding then $G \ltimes_{\alpha} H$ will be written $G \ltimes H$.

(vii) $K \overset{\alpha}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} H$ will denote a short exact sequence of groups i.e. α injective, β surjective, $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

1.3.3 The triple (M,G,α) will denote a Lie group G acting smoothly on a manifold M via the map $\alpha: G \times M \to M$. If there is no possibility of confusion (M,G,α) will be written (M,G) and $\alpha(g,x)$ as $g \cdot x$ or $g(x)$. For $x \in M$ the stabilizer of x will be denoted G_x . If G acts transitively on M with compact stabilizer and M is simply connected then (M,G,α) is called a geometry. For a geometry we will make the following assumptions:

(i) G is connected.

(ii) (M,G,α) is maximal.

(iii) M is equipped with some G-invariant Riemannian metric usually denoted by $\langle -,- \rangle$.

(iv) M has the real analytic structure induced from the equivalence between (M,G,α) and $(G/G_x,G)$.

(v) There is a discrete subgroup $\Gamma \subseteq G$ such that M/Γ is a manifold of finite volume with respect to the measure inherited from a G-invariant volume form on M .

1.3.4 If $f:A \to B$ is a mapping and $C \subseteq A$ a subset then $f|C$ will denote the restriction of f to C .

1.3.5 The n dimensional sphere, Euclidean space and Hyperbolic space will be denoted S^n , E^n and H^n respectively.

1.3.6 As usual $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ will denote the natural numbers, integers, rationals, real numbers, complex numbers and quaternions. The orthogonal, unitary and special linear groups will have their usual notation $SO(n)$, $SU(n)$, $SL(n, \mathbb{R})$.

1.3.7 We assume some familiarity with the ideas associated with the theory of foliations. (See [7] Chapter 1.)

CHAPTER 2 : FIBERING GEOMETRIES.

2.0 Introduction

We will show that for a geometry (M,G) the existence of G-invariant distributions on M will enable us, in some cases, to fibre a geometry over a simply connected manifold of lower dimension. The fibering of M is G-invariant so the action of G descends to give an action of G on the quotient. However the action of G on the quotient space does not, in general, have a compact stabilizer and is therefore not always a geometry. In Section 1 we prove some general statements about G-invariant distributions and their integrability. In Section 2 these results are applied in two situations that will concern us in Chapters 3,4,5. We will assume that M is provided with the analytic structure given by the equivalence between (M,G) and $(G/G_x,G)$ together with a G-invariant Riemannian metric.

2.1 G-invariant Distributions

If D is a distribution on M then for $x \in M$ we denote by D_x the corresponding subspace of T_xM. We first have the following, standard, proposition.

Proposition 2.1.1

Let F be a G-invariant foliation on $M = G/G_x$. Then there is a connected Lie subgroup L of G containing G_x such that the leaves of F are the translates of L/G_x by elements of G.

Proof

First note that if L is a connected Lie subgroup of G containing G_x then the translates of L/G_x by elements of G obviously form an (analytic) foliation of G/G_x . For a G-invariant foliation F we denote by TF the corresponding sub-bundle of TM . If $p:G \to G/G_x$ denotes the quotient map then it is easy to see that $dp^{-1}(TF)$ is a left invariant distribution on G . If we identify the Lie algebra g of G with the space of left invariant vector fields then $dp^{-1}(TF)$ is a subspace of g containing g_x . The integrability of TF obviously implies the integrability of $dp^{-1}(TF)$. Hence $dp^{-1}(TF)$ is a subalgebra of g . If L is the corresponding connected subgroup then L contains G_x and $p(L) = L/G_x$ is a leaf of F . The result follows. □

It would seem that from the existence of a G-invariant foliation one can construct a fibre bundle $L/G_x \to G/G_x \to G/L$. For this to hold it is necessary that L be a closed subgroup of G .

Proposition 2.1.2

Let F be a G-invariant foliation and let L be the corresponding subgroup of G given by Proposition 2.1.1. Then L is closed if for each $x \in M$ TF_x contains the space of vectors left fixed by the action of G_x on T_xM . In this case the projection $\zeta:M \to M/F$ has the G-equivariant fibre-bundle structure $L/G_x \to G/G_x \to G/L$ with G/L simply connected. If $\hat{\zeta}:G \to \mathrm{Diff}(M/F)$ denotes the G action on M/F then $\hat{\zeta}(G)$ is orientation preserving and $\mathrm{Ker}(\hat{\zeta}) \cap G_x = \{g \in G_x \mid dg_x|(TF_x)^{\perp} = id\}$. Here $(TF_x)^{\perp}$ is the orthogonal complement of TF_x in T_xM .

Proof

To show that L is closed it is sufficient to show that the leaves of F are closed. By the transitivity of G on M we need only show that one leaf of F is closed. Let F_{x_0} denote the leaf of F through the point $x_0 \in M$ and $\exp_{x_0}: T_{x_0}M \to M$ the Riemannian exponential map at x_0. If $B(x_0,\varepsilon)$ is the ball of radius ε around 0 in $TF_x^{\perp}$ we set $N_{x_0}(\varepsilon) = \exp_{x_0}(B(x_0,\varepsilon))$. Choose ε_0 sufficiently small that

(i) $N_{x_0}(\varepsilon_0)$ is an embedded disc transverse to F.

(ii) Any point $x_1 \in N_{x_0}(\varepsilon_0)$ is joined to x_0 by a unique geodesic lying in $N_{x_0}(\varepsilon_0)$.

Assume that there is a point $x_1 \neq x_0$ in $F_{x_0} \cap N_{x_0}(\varepsilon_0)$. If $g \in G_{x_0}$ then $g(F_{x_0}) = F_{x_0}$ and $g(N_{x_0}(\varepsilon_0)) = N_{x_0}(\varepsilon_0)$. Thus $g(N_{x_0}(\varepsilon_0) \cap F_{x_0}) = {}$ $= N_{x_0}(\varepsilon_0) \cap F_{x_0}$. Since G_{x_0} is connected the subset $\{g(x_1) | g \in G_{x_0}\}$ is a connected subset of $F_{x_0} \cap N_{x_0}(\varepsilon_0)$. But $F_{x_0} \cap N_{x_0}(\varepsilon_0)$ is countable since $N_{x_0}(\varepsilon)$ is transverse to F. It follows that $g(x_1) = x_1$ for each $g \in G_{x_0}$. If $\gamma:[0,1] \to N_{x_0}(\varepsilon_0) \subseteq M$ is the unique shortest geodesic connecting x_0 and x_1 then each $g \in G_{x_0}$ must fix γ pointwise. Thus $dg_{x_0}(\gamma'(0)) = \gamma'(0)$ and so, by hypothesis, $\gamma'(0) \in TF_{x_0}$. This is a contradiction and we must therefore have $F_{x_0} \cap N_{x_0}(\varepsilon_0) = \{x_0\}$. This implies that L is locally closed in G i.e. there is a neighbourhood U of L in G such that $L = U \cap \bar{L}$. But a locally closed subgroup of a Lie group is closed. Hence L is closed.

Since L is closed the quotient $\zeta : M \to M/F$ is given by the fibre bundle projection $\zeta : G/G_x \to G/L$ with fibre L/G_x . Clearly ζ is G-equivariant for the actions of G on G/G_x and G/L . From the exact homotopy sequence of a fibre bundle we have

$$\to \pi_1(M) \to \pi_1(M/F) \to \pi_0(F) .$$

where F denotes a typical leaf of F . Now $\pi_1(M) = 0$ by hypothesis, and $\pi_0(F) = 0$ since F is connected. It follows that $\pi_1(M/F) = 0$ and M/F is orientable. Since $\zeta(G)$ is connected it must preserve any choice of orientation on M/F .

To demonstrate the final statement let $g \in \mathrm{Ker}(\hat{\zeta}) \cap G_{x_0}$ and let $N_{x_0}(\varepsilon_0)$ be the transverse neighbourhood defined above. If F_x is the leaf of F through $x \in M$ then $g(F_x) = F_x$ and so $g(F_x \cap N_{x_0}(\varepsilon_0)) = F_x \cap N_{x_0}(\varepsilon_0)$. It follows that $g|N_{x_0}(\varepsilon_0) = \mathrm{id}$. Hence $dg_x|(TF_{x_0})^{\perp} = \mathrm{id}$. Now assume that there is a $g \in G_{x_0}$ such that $dg_{x_0}|(TF_{x_0}) = \mathrm{id}$. Then clearly $g|N_{x_0}(\varepsilon_0) = \mathrm{id}$ and so $\hat{\zeta}(g)$ is the identity in a neighbourhood of $\zeta(x_0)$. But the action of $\hat{\zeta}(G)$ on M/F is analytic. Thus $\hat{\zeta}(g) = \mathrm{id}$ and $g \in \mathrm{Ker}(\hat{\zeta}) \cap G_{x_0}$.

□

We pause to point out a trap for the unwary. It is tempting at this stage to try and project the metric on $(TF)^{\perp}$ down to a metric on M/F. The idea is that the group $\hat{\zeta}(G)$ would act by isometries for this metric and that $(M/F,\hat{\zeta}(G))$ would be a geometry. That this is not necessarily the case will be demonstrated by counterexample in Theorem 4.2.2. In the proposition below we give some simple conditions for $(M/F,\hat{\zeta}(G))$ to have a compact stabilizer.

We denote by $\exp_x:T_xM \to M$ the Riemannian exponential map and by $B(x,\varepsilon) \subseteq (TF_x)^{\perp}$ the ball of radius ε around 0. We choose $\varepsilon_0 > 0$ such that $\exp_x:B(x,\varepsilon_0) \to N_x \subseteq M$ is an embedding transverse to F for each $x \in M$. If $x_1,x_0 \in F_{x_0}$ there is a holonomy map $\sigma(x_0,x_1)$: $U_{x_0} \subseteq N_{x_0} \to U_{x_1} \subseteq N_{x_1}$. Using this notation we have:

Proposition 2.1.3

With the notation of Proposition 2.1.2 the following statements are equivalent for F with closed leaves.

(a) The action of $\hat{\zeta}(G)$ on M/F has compact stabilizer.

(b) There is a G-invariant metric on M and a metric on M/F such that for each $x \in M$ $d\zeta_x:(TF_x)^{\perp} \to T_{\zeta(x)}M/F$ is an isometry.

(c) For some G-invariant metric on M we have $d\sigma(x,y):(TF_x)^{\perp} \to (TF_y)^{\perp}$ an isometry if $\zeta(x) = \zeta(y)$.

Proof

We first remark that for all $x \in M$ the map $d\zeta_x:(TF_x)^\perp \to T_{\zeta(x)}M/F$ is an isomorphism.

(a) <=> (b)

If the stabilizer in $\hat{\zeta}(G)$ of $y \in M/F$ is compact then there is a $\hat{\zeta}(G)$ invariant metric λ on M/F . If $<,>$ denotes the original G-invariant metric on M we define a new metric $<,>'$ on M by the conditions

(i) $<u,v>'_x = <u,v>_x$ if $u,v \in TF_x$.

(ii) $<u,v>'_x = 0$ if $u \in TF_x$, $v \in (TF_x)^\perp$.

(iii) $<u,v>'_x = \lambda_{\zeta(x)}(d\zeta_x(u),d\zeta_x(v))$ if $u,v \in (TF_x)^\perp$.

To show that $<,>'$ is G-invariant we need only show that $dg_x:(TF_x) \to (TF_{g(x)})$ is an isometry for $<,>'$. But this follows immediately from the fact that $g|N_x = (\zeta|N_{g(x)})^{-1} \circ (\hat{\zeta}(g)|\zeta(N_x) \circ (\zeta|N_x)$. Conversely if we have metrics satisfying (b) then the formula $\hat{\zeta}(g)|\zeta(N_x) =$
$= (\zeta|N_{g(x)}) \circ (g|N_x) \circ (\zeta|N_x)^{-1}$ shows that $\zeta(G)$ acts by isometries on $(M/F,\lambda)$.

(b) <=> (c)

We have, for $x,y \in F_x$,$(\zeta|N_y) = (\zeta|N_x) \circ \sigma(y,x)$.
Hence $(d\zeta_x|(TF_x)^\perp)^{-1} (d\zeta_y\ (TF_y)^\perp) = d\sigma_y(y,x)$. If (b) holds then $d\sigma_y$ is an isometry. Conversely if $y \in M/F$ and $\zeta(x_0) = y$ we can define an inner product on T_yM/F by:

$$\lambda_{x_0}(y)(u,v) = <(d\zeta_{x_0}|TF^{\perp}_{x_0})^{-1}(u),(d\zeta_{x_0}|TF^{\perp}_{x_0})^{-1}(v)>_{x_0} \quad .$$

For another point $x_1 \in \zeta^{-1}(y)$ we have $\zeta|N_{x_1} = \zeta|N_{x_0} \circ \sigma(x_1,x_0)$. So if (c) holds we have $\lambda_{x_0}(y) = \lambda_{x_1}(y)$. Hence we obtain a well defined metric λ satisfying (b).

□

Corollary 2.1.4

If Ker $(\hat{\zeta})$ is transitive on F_x for each $x \in M$ then $\hat{\zeta}(G)$ acts on M/F with compact stabilizer.

Proof

If $x,y \in F_x$ and $g(x) = y$ with $g \in \mathrm{Ker}(\hat{\zeta})$ then $g|N_x = \sigma(x,y)$. Since $dg_x:TF_x^{\perp} \to TF_y^{\perp}$ is an isometry the result follows from 2.1.3(c).

□

2.2 Applications

We present two simple applications of the propositions in 2.1.

Theorem 2.2.1

Let P denote the distribution defined by $P_x = \{v \in T_xM | dg_x(v) = v$ for all $g \in G_x\}$. Then P is G-invariant, parallelizable and integrable with G-invariant foliation F . The projection $\zeta:M \to M/F$ gives M the structure of a principal fibre bundle over M/F . In addition

(a) M/F is simply connected.

(b) There is an orientation preserving action of G on M/F such that $\zeta : M \to M/F$ is G equivariant.

(c) If $\hat{\zeta} : G \to \mathrm{Diff}^+(M/F)$ denotes the action of G on M/F then $\mathrm{Ker}(\hat{\zeta}) \cap G_x = \{e\}$ for each $x \in M$.

(d) If $g \in G_x$ then $g|F_x = \mathrm{id}$.

Proof

Let V denote the vector space of G-invariant vector fields on M . Clearly V is a finite dimensional Lie algebra. If $X \in V$ and $g \in G_x$ then $dg_x(X(x)) = X(x)$ so we have a mapping $\rho_x : V \to P_x$ for each $x \in M$. Clearly ρ_x is injective. If $X(x_0) \in P_{x_0}$ we can define a G-invariant vector field X by $X(x) = dg\, X(x_0)$ if $g(x_0) = x$. Hence ρ_x is surjective. It now follows immediately that P is G-invariant, parallelizable and integrable. From Propositions 2.1.1 and 2.1.2 there is a closed connected subgroup $L \subseteq G$ such that the quotient $\zeta : M \to M/F$ is given by the fibre bundle $L/G_{x_0} \to G/G_{x_0} \to G/L$. We wish to show that L is the connected component of e of the normalizer of G_{x_0} which we denote $N_0(G_{x_0})$. Clearly G_{x_0} acts trivially on L/G_{x_0} so $g[\ell] = [\ell]$ for $g \in G_x$, $\ell \in L$ where $[\ell]$ denotes the coset ℓG_{x_0} of ℓ in G . This means that $g\ell = \ell g'$ for some $g' \in G_{x_0}$ i.e. $\ell^{-1} g \ell \in G_{x_0}$. Hence $L \subseteq N_0(G_{x_0})$. It is also clear that G_{x_0} acts trivially on $N_0(G_{x_0})/G_{x_0}$. If $n(g_{x_0})$ denotes the subalgebra of g corresponding to $N_0(G_{x_0})$ and $p:G \to G/G_{x_0}$ is the projection then it follows that $dg_{x_0} = \text{id}$ on $dp_e(\mathfrak{n}(\mathfrak{g}_{x_0}))$ for $g \in G_{x_0}$. Hence $dp_e(\mathfrak{n}(\mathfrak{g}_{x_0})) \subseteq P_{x_0}$ and $\mathfrak{n}(\mathfrak{g}_{x_0})$ is contained in the subalgebra of $\mathfrak{g}$ corresponding to L . It follows that $N_0(G_{x_0}) \subseteq L$. Thus $L = N_0(G_{x_0})$ and the fibre of $\zeta:M \to M/F$ is the group $N_0(G_{x_0})/G_{x_0}$.

Both (a) and (b) are immediate from Proposition 2.1.2 and (d) follows since G_x acts by isometries on F_x in the metric induced from M . Finally (c) follows from Proposition 2.1.2 after observing that if $g \in G_x$ is such that $dg_x\, P_x^{\perp} = \text{id}$ then $dg_x = \text{id}$ and then g must be the identity.

□

We now consider the case when G_x is isomorphic to a torus.

<u>Theorem 2.2.2</u>

If G_x is isomorphic to a torus and $P_x = \{v \in T_xM \mid dg_x(v) = v \ \forall\ g \in G_x\}$ is trivial for each $x \in M$ then any irreducible G-invariant distribution Q is integrable. If F is the corresponding G-invariant foliation then the projection $\zeta:M \to M/F$ gives M the structure of a fibre bundle over M/F . In addition

(a) M/F is simply connected.

(b) There is an orientation preserving action of G on M/F with respect to which $\zeta:M \to M/F$ is G equivariant.

(c) In the metric induced from a G-invariant metric on M the leaves of F are isometric to E^2, H^2 or S^2.

Proof

If Q is a non-trivial G-invariant irreducible distribution on M then $\dim(Q) = 2$. If we form the exterior power $\Lambda^2 Q$ there is an induced action of G on $\Lambda^2 Q$. Since G_x acts on Q_x with determinant 1 the action of G_x on $(\Lambda^2 Q)_x$ is trivial. Hence, by Schur's Lemma, there are no non-trivial G_x equivariant homomorphisms from $(\Lambda^2 Q)_x$ to Q_x. Let $q:TM \to Q$ be the orthogonal projection. If X_1 and X_2 are two vector fields spanning Q over an open set $U \subseteq M$ then $(aX_1+bX_2)\wedge(cX_1+dX_2) = (ad-bc)X_1 \wedge X_2$ and $q[aX_1+bX_2, cX_1+dX_2] = (ad-bc)q([X_1,X_2])$ on U. Therefore there is a well defined G equivariant bundle homomorphism $\rho:\Lambda^2 Q \to Q^\perp$ given by $X_1 \wedge X_2 \to q[X_1,X_2]$ which is linear over the ring of C^∞ functions on M. Hence ρ must be trivial. It follows that $[X_1,X_2]$ is a vector field in Q if X_1, X_2 are vector fields in Q. Thus Q is integrable. The fibre bundle structure is given by Proposition 2.1.2.

The statements (a) and (b) follow from Proposition 2.1.2. To show (c) let $K_x = \{g \in G \mid g(F_x) = F_x\}$. K_x acts transitively by isometries on F_x in the induced metric. Hence the universal cover $\tilde{F}_x$ of F_x with the metric lifted from F_x is isometric to E^2, H^2 or S^2. If $\rho:K_x \to \mathrm{Isom}(F_x)$ is the obvious homomorphism then $\rho|G_x$ is non-trivial by the hypothesis $P_x = \{0\}$. The only Euclidean or Hyperbolic manifolds with a transitive group of isometries possessing a non-trivial connected stabilizer are E^2 or H^2. Hence $\tilde{F}_x$ isometric to E^2 or H^2 implies that F_x is isometric to E^2 or H^2. Assume that F_x is covered by S^2. Since G_x is connected for each $x \in M$ it preserves orientation on Q_x and so the leaves of F are orientable. Therefore F_x must be isometric to S^2.

□

Remarks

(1) If the foliation F has leaves isometric to S^2 it follows that $K_x = \{g \in G \mid g(F_x) = F_x\}$ is compact for each $x \in M$. The action of $\hat{\zeta}(G)$ on M/F has compact stabilizer.

(2) If the leaves of F are isometric to H^2 then $K_x = \{g \in G \mid g(F_x) = F_x\}$ is isomorphic to a quotient of $\widetilde{SL}(2, \mathbb{R})$ and is therefore simple and non-compact. $\hat{\zeta}(K_x)$ is the stabilizer of $\zeta(x)$. There is a homomorphism $\rho: \hat{\zeta}(K_x) \to GL(2(n-1), \mathbb{R})$ determined by the derivative. Then $\mathrm{Ker}(\rho \circ (\hat{\zeta}|K_x))$ is either K_x or a discrete central subgroup of K_x (K_x is connected). If $\mathrm{Ker}(\rho \circ (\hat{\zeta}(K_x))) \neq K_x$ then $\hat{\zeta}(K_x)$ contains a non-compact closed subgroup and $(M/F, \hat{\pi}(G))$ is *not* a geometry. If $\mathrm{Ker}(\rho \circ (\hat{\zeta}|K_x)) = K_x$ then it is easy to see that (M,G) is equivalent to $(H^2, PSL(2, \mathbb{R})) \times (M/F, G/\mathrm{Ker}(\hat{\zeta}))$.

CHAPTER 3 : SYMMETRIC SPACES.

3.0 Introduction

In this chapter we will consider the four dimensional geometries whose stabilizer subgroup is non-trivial and not isomorphic to $SO(2)$. From Proposition 1.2.4 the remaining possibilities for G_x are $SO(4)$, $SU(2)$ $SO(3)$, $S(U(2)\times U(1))$ and $SO(2)\times SO(2)$. It will be shown that the maximal geometries are the simply connected four dimensional Riemannian globally symmetric spaces. If M is such a space then the existence of a discrete group Γ of isometries of M with $\Gamma\backslash M$ a compact manifold is shown in Borel [2].

3.1 $G_x \cong SO(4), S(U(2)\times U(1)), SO(2)\times SO(2)$

For these cases the classification is particularly simple.

Theorem 3.1.1

If (M,G) is a maximal four dimensional geometry with $G_x \cong SO(4)$, $S(U(2)\times U(1))$ or $SO(2)\times SO(2)$ then for some G invariant metric M is isometric to one of the following spaces:

(a) Spaces of constant curvature : H^4, E^4, S^4.

(b) Complex Projective space $\mathbb{C}P^2 = SU(3)/S(U(2)\times U(1))$.
Complex Hyperbolic space $\mathbb{C}H^2 = SU(2,1)/S(U(2)\times U(1))$.

(c) Products of two dimensional geometries:
$E^2\times S^2$, $E^2\times H^2$, $S^2\times S^2$, $S^2\times H^2$, $H^2\times H^2$.

Proofs

(a) If $G_x \cong SO(4)$ then, since G_x is transitive on two-planes in T_xM, any G-invariant metric on M must have constant curvature. M is simply connected and therefore must be isometric to H^4, E^4 or S^4.

(b) If G_x is isomorphic to $S(U(2)\times U(1))$ then by Corollary 1.2.5 M is a Riemannian symmetric space with respect to any G-invariant metric. The only irreducible Riemannian symmetric spaces with this stabilizer are (see Helgason [6] pg.354):

$$\text{Complex Projective Space } \mathbb{C}P^2 = SU(3)/S(U(2)\times U(1))$$

$$\text{Complex Hyperbolic Space } \mathbb{C}H^2 = SU(2,1)/S(U(2)\times U(1)),$$

and these spaces are both simply connected. As they are written here G does not act faithfully on M since in both cases the centre $Z(G)$ of G is contained in G_x. However $Z(G) = \{\omega I \in GL(3,\mathbb{C}) \mid \omega^3 = 1\}$ and it is easy to see that $S(U(2)\times U(1))/Z(G) \cong S(U(2)\times U(1))$.

(c) If G_x is isomorphic to $SO(2)\times SO(2)$ then again, by Corollary 1.2.5, M is a Riemannian symmetric space for any G-invariant metric. If M is reducible we must have M isometric to one of $E^2\times S^2$, $E^2\times H^2$, $S^2\times S^2$, $S^2\times H^2$, $H^2\times H^2$. There are two further possibilities:

$$M = SO(4)/SO(2)\times SO(2)$$

$$M = SO_0(2,2)/SO(2)\times SO(2)$$

where $SO_0(2,2)$ is the connected component of the identity in the subgroup of $GL(4,\mathbb{R})$ preserving the quadratic form $q(x) = x_1^2+x_2^2-x_3^2-x_4^2$.

If $M = SO(4)/SO(2) \times SO(2)$ then $Z(SO(4)) \subseteq SO(2) \times SO(2)$ and $SO(4)$ does not act effectively on M. It is easy to see that $SO(4)/Z(SO(4)) \cong SO(3)\times SO(3)$. Hence M is isometric to $SO(3)\times SO(3)/SO(2)\times SO(2) = S^2\times S^2$. Finally $SO_0(2,2)$ is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/C$ where C is the subgroup $C = \{(I,I),(-I,-I)\}$. Thus the same argument shows that M is isometric to $PSL(2,\mathbb{R})\times PSL(2,\mathbb{R})/SO(2)\times SO(2) = H^2\times H^2$.

□

3.2 $G_x \cong SO(3)$

In this case we know from Proposition 1.2.4 that the subspace of vectors in T_xM left fixed by the action of G_x is one dimensional for each $x \in M$. If P denotes this distribution it is integrable with G-invariant foliation F. By Theorem 2.2.1 the quotient space M/F is a simply connected smooth manifold and we have a principal fibre bundle $F \to M \xrightarrow{\zeta} M/F$ where F denotes a typical leaf of F. Since $\dim(P) = 1$ F is diffeomorphic to $\mathbb{R}$ or S^1. There is also a homomorphism $\hat{\zeta}:G \to \mathrm{Diff}^+(M/F)$ such that $\hat{\zeta}(G)$ acts smoothly and transitively on M/F. In the situation we are considering of $\dim(M) = 4$ and $G_x \cong SO(3)$ we have the additional information.

Proposition 3.2.1

The pair $(M/F,\hat{\zeta}(G))$ is a geometry equivalent to one of the spaces E^3,H^3,S^3 equipped with its maximal connected group of isometries. $\mathrm{Ker}(\hat{\zeta})$ is connected, central in G and transitive on each leaf of F.

Proof

We first note that by Theorem 2.2.1(c) the subgroup of $\hat{\zeta}(G)$ fixing a point of M/F contains a group isomorphic to $SO(3)$. We know that $\dim(M/F) = 3$ and M/F is simply connected. Therefore if $(M/F,\hat{\zeta}(G))$ is a geometry it is a maximal geometry and M/F has constant curvature for any $\hat{\zeta}(G)$ invariant metric. Hence M/F with such a metric is isometric to E^3, H^3, S^3 and $\hat{\zeta}(G)$ is isomorphic to $\mathrm{Isom}_0(E^3)$, $SO(4)$, $SO_0(3,1)$.

To show that $(M/F,\hat{\zeta}(G))$ is a geometry it suffices, by Corollary 2.1.4, to show that $\mathrm{Ker}(\hat{\zeta})$ is transitive on each leaf of F. By Theorem 2.2.1 P is parallelizable and we have a G-invariant vector field X on M such that $X(x) \neq 0$, $||X(x)||_x = 1$ and $X(x) \in P_x$ for each $x \in M$. X is clearly globally integrable with a corresponding flow $\{\phi^t\}_{t\in\mathbb{R}}$ that commutes with the action of G on M. Now for each $t \in \mathbb{R}$ ϕ^t sends each leaf of F onto itself and the group $\{\phi^t\}_{t\in\mathbb{R}}$ acts transitively on any leaf of F. Thus if we can show that ϕ^t is an isometry for each $t \in \mathbb{R}$ then the maximality assumption on (M,G) will imply that $\phi^t \in \mathrm{Ker}(\hat{\zeta})$ for each $t \in \mathbb{R}$. Hence $\mathrm{Ker}(\hat{\zeta})$ will be transitive on each leaf of F and central on G. It remains, therefore, to show that ϕ^t is an isometry for each $t \in \mathbb{R}$. Let Q denote the orthogonal complement to P. For each $t \in \mathbb{R}$ ϕ^t commutes with the action of G on M so the distribution Q^t defined by $Q^t_{\phi_t(x)} = d\phi^t_x(Q_x)$ is G-invariant. It now follows, since the G_x invariant complement to P_x is unique, that we must have $Q^t_x = Q_x$ for each $x \in M$, $t \in \mathbb{R}$. Hence Q is invariant under the action of $\{\phi^t\}_{t\in\mathbb{R}}$. If $v_1 \in Q_{x_1}$, $v_2 \in Q_{x_2}$ and $||v_1||_{x_1} = ||v_2||_{x_2}$ then there is a $g \in G$ such that $g(x_1) = x_2$ and $dg_{x_1}(v_1) = v_2$, and we have:

$$\begin{aligned}
<d\phi^t_{x_2}(v_2), d\phi^t_{x_2}(v_2)>_{\phi^t(x_2)} &= <d\phi^t \circ dg_{x_1}(v_1), d\phi^t \circ dg_{x_1}(v_1)>_{\phi^t(x_2)} \\
&= <dg \circ d\phi^t(v_1), dg \circ d\phi^t(v_1)>_{\phi^t(x_2)} \\
&= <d\phi^t(v_1), d\phi^t(v_1)>_{\phi^t(x_1)} \quad .
\end{aligned}$$

It follows that there is a homomorphism $\alpha : \mathbb{R} \to \mathbb{R}^+_*$ such that $<d\phi^t_x(v), d\phi^t_x(v)> = \alpha(t)^2 <v,v>_x$ for all $x \in M$, $v \in Q_x$. Hence $d\phi^t$ preserves angles and expands volumes by $(\alpha(t))^3$. If $\Gamma \subseteq G$ is a discrete group and $\Gamma\backslash M$ is a manifold then the flow ϕ^t descends to give a flow $\tilde{\phi}^t$ on $\Gamma\backslash M$. Then $\tilde{\phi}^t$ still expands volume by $(\alpha(t))^3$. This contradicts the existence of a quotient of finite volume unless $\alpha(t) = 1$ for all $t \in \mathbb{R}$. Hence ϕ^t is an isometry for each $t \in \mathbb{R}$. $\mathrm{Ker}(\hat{\zeta})$ is connected since, for each $x \in M$, F_x is connected, $\mathrm{Ker}_0(\hat{\zeta})$ is transitive on F_x and $\mathrm{Ker}(\hat{\zeta}) \cap G_x = \{e\}$.

□

We can now show that in the current situation the complementary distribution to P is also integrable.

Lemma 3.2.2

Let D be a smooth n-dimensional distribution on M and let $\{X_1,X_2\}$, $\{Y_1,Y_2\}$ be two pairs of vector fields in D such that for some $x_0 \in M$ $X_1(x_0),X_2(x_0)$ are independent and the linear spans of $\{X_1(x_0),X_2(x_0)\}$ and $\{Y_1(x_0),Y_2(x_0)\}$ coincide. Then $[X_1,X_2](x_0) \in D_{x_0}$ iff $[Y_1,Y_2](x_0) \in D_{x_0}$.

Proof

Choose vector fields $X_3,\ldots,X_n$ such that $X_1,\ldots,X_n$ span D in a neighbourhood of x_0 . Then we can write Y_1 and Y_2 as:

$$Y_1(x) = \sum_{i=1}^{n} a_i(x)X_i(x) \quad , \quad Y_2(x) = \sum_{j=1}^{n} b_j(x)X_j(x)$$

with $a_k(x_0) = b_k(x_0) = 0$ for $3 \le k \le n$ and $a_1(x_0)b_2(x_0)-a_2(x_0)b_1(x_0) \neq 0$. Then we have

$$[Y_1,Y_2](x) = \sum_{i=1}^{n} f_i(x)X_i(x) + \sum_{i,j} (a_ib_j-a_jb_i)(x)[X_i,X_j](x)$$

for some smooth functions $f_1,\ldots,f_n$. If $x = x_0$ we obtain

$$[Y_1,Y_2](x_0) = \sum_{i=1}^{n} f_i(x_0)X_i(x_0) + (a_1b_2-a_2b_1)(x_0)[X_1,X_2](x_0) \quad .$$

The result now follows since $(a_1b_2-a_2b_1)(x_0) \neq 0$.

□

If Q_x denotes the orthogonal complement to P_x in T_xM then the Q_x form a smooth, G-invariant distribution on M. In the current situation with $G_x \cong SO(3)$ we have

Proposition 3.2.3

The distribution Q is integrable.

Proof

We will first show that if there is a pair of vector fields X_1, X_2 in Q with $X_1(x_0), X_2(x_0)$ independent and $[X_1,X_2](x_0) \in Q_{x_0}$ for some $x_0 \in M$ then Q is integrable. Secondly we will construct such a pair of vector fields.

Let X_1, X_2 be vector fields in Q with $[X_1,X_2](x_0) \in Q_{x_0}$ and $X_1(x_0), X_2(x_0)$ independent. Let Y_1, Y_2 be any two vector fields in Q and set

$$\Omega = \{x \in M \mid [Y_1,Y_2](x) \in Q_x\} .$$

Then Ω is closed and we wish to show that $\Omega = M$. Let A denote the set $\{x \in M \mid Y_1(x), Y_2(x) \text{ independent}\}$ and set $B = M-A$. Now G is transitive on M and G_x is transitive on two-planes in Q_x. Therefore if $x \in A$ we can choose $g \in G$ such that $g(x) = x_0$ and the span of $\{dg_x(Y_1(x)), dg_x(Y_2(x))\}$ is the same as the span of $\{X_1(x_0), X_2(x_0)\}$. By Lemma 3.2.2 this implies that $dg_x([Y_1,Y_2](x)) = [dg(Y_1), dg(Y_2)](x_0) \in Q_{x_0}$.

Since Q is G-invariant, $[Y_1,Y_2](x) \in Q_x$. Thus $A \subseteq \Omega$. Now let $x \in \mathrm{Int}(B)$. If $Y_1(x) \neq 0$ then $Y_1(y) \neq 0$ for all y in some neighbourhood N of x contained in B and we can write $Y_2(y) = \alpha(y)Y_1(y)$ for some function $\alpha:N \to \mathbb{R}$. Hence $[Y_1,Y_2] = Y_1(\alpha)Y_1$ on N and $[Y_1,Y_2](y) \in Q_y$ for all $y \in N$. So $x \in \Omega$. Similarly if $Y_2(x) \neq 0$ we have $x \in \Omega$. Now assume that $Y_1(x) = Y_2(x) = 0$. If either Y_1 or Y_2 is identically 0 in a neighbourhood of x then $[Y_1,Y_2] = 0$ in a neighbourhood of x and so $x \in \Omega$. If Y_1 is not identically zero in any neighbourhood of x there is a sequence of points $\{x_n\} \subseteq B$ converging to x and such that $Y_1(x_n) \neq 0$ for each $n \in \mathbb{N}$. Now $x_n \in \Omega$ as already demonstrated. Thus x is a limit of points in Ω and hence $x \in \Omega$ since Ω is closed. We now have $\bar{A} \subseteq \Omega$ and $\mathrm{Int}(B) \subseteq \Omega$ so $M = \bar{A} \cup \mathrm{Int}(B) \subseteq \Omega$.

To complete the proof we must find two vector fields X_1,X_2 in Q such that $X_1(x_0),X_2(x_0)$ are independent and $[X_1,X_2](x_0) \in Q_{x_0}$ for some $x_0 \in M$. Choose $x_0 \in M$ and let Y_1,Y_2,Y_3 be a local basis for Q in a neighbourhood of $x_0 \in M$. Let $X_1 = Y_3$ and $X_2 = aY_1+bY_2$ for some $a,b \in \mathbb{R}$. If $\phi:T_{x_0}M \to P_{x_0}$ denotes the orthogonal projection then the mapping $X_2(x_0) \to \phi([X_1,X_2](x_0))$ is a linear functional on the span of $Y_1(x_0),Y_2(x_0)$ in Q_{x_0} . It must have a non-trivial kernel so there exist $\alpha,\beta \in \mathbb{R}$ not both zero such that

$$[Y_3,\alpha Y_1+\beta Y_2](x_0) \in Q_{x_0} .$$

□

We can now determine the four dimensional geometries with stabilizer isomorphic to $SO(3)$.

Theorem 3.2.4

If (M,G) is a maximal four dimensional geometry with G_x isomorphic to $SO(3)$ then (M,G) is equivalent to $(E\times S^3, \mathbb{R}\times SO(4))$ or $(E\times H^3, \mathbb{R}\times SO_0(3,1))$.

Proof

By Proposition 3.2.3 the distribution Q is integrable and we denote by L the corresponding G-invariant foliation. F will, as usual, denote the foliation induced by P . We denote by F_x, L_x the leaves of F and L through $x \in M$. Combining Propositions 3.2.1 and 2.1.3 we see that there is a $\hat{\zeta}(G)$ invariant metric λ on M/F such that $d\zeta_x : Q_x \to T_{\zeta(x)}M/F$ is an isometry. We first wish to show that for any $x \in M$ the restriction of $\zeta : M \to M/F$ to L_x is an isometry onto M/F with respect to the induced metric on L_x . Since $\zeta : M \to M/F$ is a bundle projection by Theorem 2.2.1 and L_x is transverse to F the map $\zeta : L_x \to \zeta(L_x)$ is a covering map onto its image. Also, by the definition of the metric on M/F , $\zeta : L_x \to \zeta(L_x)$ is a local isometry. Since M/F is simply connected it only remains to show that $\zeta(L_x) = M/F$. Choose $x_0 \in M$ and denote by A the subset of M $A = \{y \in M | F_y \cap L_{x_0} \neq \emptyset\}$. Then $A = M$ iff $\zeta(L_{x_0}) = M/F$. Using holonomy map of F it is easy to see that A is open. Now assume that we have a sequence $\{y_n\} \subseteq A$ such that $y_n \to y_0$. Without loss of generality we can assume that $\{y_n\} \subseteq L_{y_0}$. Then for each $n \in \mathbb{N}$ there exists a point $x_n \in F_{y_n} \cap L_{x_0}$. By Proposition 3.2.1 $\mathrm{Ker}(\hat{\zeta})$ is transitive on F_{y_n} so there is a $g \in \mathrm{Ker}(\hat{\zeta})$ such that $g(y_n) = x_n$. Then $g(L_{y_0}) = g(L_{y_n}) = L_{x_n} = L_{x_0}$ so $g(y_0) \in L_{x_0}$. But $g(y_0) \in F_{y_0}$ so $F_{y_0} \cap L_{x_0} \neq 0$. Hence A is closed. It follows that $A = M$ since M is connected.

By Proposition 3.2.1 the pair $(M/F, \hat{\zeta}(G))$ is a geometry equivalent to E^3, H^3, S^3 so we have an exact sequence $\mathrm{Ker}(\hat{\zeta}) \rightarrowtail G \overset{\hat{\zeta}}{\twoheadrightarrow} \mathrm{Isom}_0^+(M/F,\lambda)$ where $\mathrm{Isom}_0^+(M/F,\lambda)$ is isomorphic to $SO(4)$, $SO(3,1)$, or $\mathrm{Isom}_0^+(E^3)$ and $\mathrm{Ker}(\hat{\zeta})$ is connected, central in G and isomorphic to $\mathbb{R}$ or S^1 , Let K_{x_0} denote the subgroup $K_{x_0} = \{g \in G \mid g(L_{x_0}) = L_{x_0}\}$. Then $G_{x_0} \subseteq K_{x_0}$ and K_{x_0} acts transitively on L_{x_0} in the induced metric. By the previous paragraph $\zeta|L_{x_0}$ is an isometry onto $(M/F,\lambda)$ so we must have $\hat{\zeta}(K_{x_0}) = \hat{\zeta}(G)$. If $g \in \mathrm{Ker}(\hat{\zeta}) \cap K_{x_0}$ then $g(x_0) \in F_{x_0} \cap L_{x_0}$. But from the previous paragraph we know that $F_{x_0} \cap L_{x_0} = \{x_0\}$. Therefore $g \in \mathrm{Ker}(\hat{\zeta}) \cap G_{x_0}$. Since $\mathrm{Ker}(\hat{\zeta}) \cap G_{x_0} = \{e\}$ by Theorem 2.2.1(c) we must have $g = e$. Hence $\hat{\zeta}|K_{x_0}$ is an isomorphism. The exact sequence $\mathrm{Ker}(\hat{\zeta}) \rightarrowtail G \twoheadrightarrow \hat{\zeta}(G)$ now splits and, $\mathrm{Ker}(\hat{\zeta})$ being connected and central by Proposition 3.2.1, we have $G \cong \mathrm{Ker}(\hat{\zeta}) \times K_{x_0}$. Hence $M \cong \mathrm{Ker}(\hat{\zeta}) \times K_{x_0}/G_{x_0}$.

Since M is assumed simply connected we must have $\mathrm{Ker}(\hat{\zeta}) \cong \mathbb{R}$. The result follows from Proposition 3.2.1 since $(E \times E^3, \mathbb{R} \times \mathrm{Isom}_0^+(E^3))$ is not maximal.

□

3.3 Maximality

The geometries of 3.1 and 3.2 are simply connected Riemannian globally symmetric spaces of the form $(E^n \times G'/K', \mathrm{Isom}_0^+(E^n) \times G')$ where

(i) G' is semi-simple and connected.

(ii) There is an involutory automorphism σ of G' such that $\sigma(g) = g$ iff $g \in K'$.

For any metric on $M = E^n \times G'/K'$ left invariant under the action of $\mathrm{Isom}_0^+(E^n) \times G'$ it is known that $\mathrm{Isom}_0(M) = \mathrm{Isom}_0^+(E^n) \times G'$ (see [6] Chap. II, 4.1). Hence these geometries are maximal. The existence of compact manifold quotients of a Riemannian globally symmetric space by discrete subgroups of isometries is shown in Borel [2]. Therefore the geometries of 3.1 and 3.2 are maximal.

CHAPTER 4 : STABILIZER ISOMORPHIC TO SO(2):I.

4.0 Introduction

In this chapter we will determine the maximal four dimensional geometries satisfying the conditions

A1 : G_x is isomorphic to $SO(2)$.

A2 : For each $x \in M$ there is a vector $v \in T_xM$ left fixed by the action of G_x .

In the notation of Chapter 2 condition A2 means that the distribution P of 2.2.1 is non-trivial. The case when $G_x \cong SO(2)$ and P is trivial will be dealt with in the next chapter.

For the purposes of this chapter we denote $P^{\perp}$ by Q . Since Q_x has even dimension $\dim(P) = \dim(Q) = 2$. From Theorem 2.2.1 we know that P is integrable with G invariant foliation F . There is also a transitive action of G on the quotient space M/F such that the quotient map $\zeta:M \to M/F$ becomes a G equivariant principal bundle. In Section 1 we show that there is a metric λ on M/F for which $\hat{\zeta}(G)$ acts conformally. With respect to this metric $(M/F,\lambda)$ is conformally equivalent to E^2 , H^2 or S^2 . These three possibilities are considered in sections 2-4. The geometries are, except for the one described in Theorem 4.2.2, products of lower dimensional geometries.

For the rest of this chapter the notations $P, Q, \zeta : M \to M/F$, $\hat{\zeta} : G \to \mathrm{Diff}^{+}(M/F)$ will have the meanings ascribed in Theorem 2.2.1 and assumptions A1 and A2 are in force.

4.1 Extensions of Theorem 2.2.1

We first show the existence of a metric on M/F with respect to which $\hat{\zeta}(G)$ acts conformally.

Theorem 4.1.1

There is a Riemannian metric λ on M/F with respect to which the group $\hat{\zeta}(G)$ acts conformally and transitively. The pair $(M/F, \lambda)$ is conformally equivalent to E^2 , H^2 , S^2 .

Proof

We first note that $d\hat{\zeta}_x : Q_x \to T_{\zeta(x)}M/F$ is an isomorphism for each $x \in M$. Let λ be a metric on M/F for which this map is conformal at each $x \in M$. Then the fact that $\hat{\zeta} : M \to M/F$ is G-equivariant immediately implies that $\hat{\zeta}(G)$ is a group of conformal diffeomorphisms with respect to this metric. It remains to construct such a metric. If $\zeta(x) = \zeta(y)$ we have a well defined map $h(x,y) = d\zeta_y^{-1} \circ d\zeta_x : Q_x \to Q_y$. Now $G_x = G_y$, $\dim(Q) = 2$ and ζ is G equivariant. Hence $h(x,y)$ is G_x equivariant. It follows that $h(x,y)$ is conformal since G_x preserves the metric on Q_x and Q_y . Let $\{N_i\}_{i \in I}$ be a collection of discs in M transverse to F and such that $\zeta|N_i$ is a diffeomorphism onto its image V_i and the collection $\{V_i\}_{i\in I}$ is a locally finite covering of M/F. On each V_i we can define a metric λ_i such that $d\zeta_x:Q_x \to T_{\zeta(x)}M/F$ is an isometry for $x \in N_i$. If $\{\phi_i\}_{i\in I}$ is a partition of unity subordinate to $\{V_i\}_{i\in I}$ then we define a metric λ on M/F by $\lambda(p)(u,v) = \Sigma\ \phi_i(p)\lambda_i(p)(u,v)$. Since $h(x,y)$ is conformal it follows that if $p \in M/F$ the metrics $\lambda_i(p)$ are conformally equivalent. Hence $d\zeta_x:Q_x \to T_{\zeta(x)}M/F$ is conformal with respect to λ.

From Theorem 2.1.3(a) we know that M/F is simply connected. Since $\dim(M/F) = 2$ we can use the theorem giving the existence of isothermal coordinates and the uniformization theorem to conclude that $(M/F,\lambda)$ is conformally equivalent to E^2, H^2 or S^2. We therefore choose the metric on M/F to be one of these three possibilities.

□

In the case being considered $\dim(G) = \dim(M) + \dim(G_x) = 5$. Since $\dim(M/F) = 2$, $\hat{\zeta}(G)$ is transitive on M/F and $\hat{\zeta}|G_x$ is an isomorphism, by Theorem 2.2.1(c), we have $\dim(\hat{\zeta}(G)) \geq 3$. Hence $\dim(\mathrm{Ker}(\hat{\zeta})) \leq 2$.

<u>Proposition 4.1.2</u>

Suppose $\dim(\mathrm{Ker}(\hat{\zeta})) = 2$. Then $\mathrm{Ker}(\hat{\zeta})$ is connected, abelian, central in G and the pair $(M/F,\hat{\zeta}(G))$ is a geometry.

Proof

Assume that $\dim(\mathrm{Ker}(\hat{\zeta})) = 2$. We have $\dim(F_x) = 2$ for each leaf F_x of F and so it follows from 2.2.1(c) that $\mathrm{Ker}(\hat{\zeta})$ is simply transitive on F_x . Since F_x is connected it follows that $\mathrm{Ker}(\hat{\zeta})$ is connected. Since $\dim(M/F) = 2$ $(M/F,\hat{\zeta}(G))$ is a geometry by Corollary 2.1.4. Now $\mathrm{Ker}(\hat{\zeta})$ is closed and normal in G and hence unimodular by Proposition 1.1.4. Therefore $\widetilde{\mathrm{Ker}}(\hat{\zeta}) \cong \mathbb{R}^2$. To show that $\mathrm{Ker}(\hat{\zeta})$ is central we will use a technique similar to that used to prove the analogous statement in Proposition 3.2.1. We recall that P is parallelizable by G invariant vector fields (Theorem 2.2.1). Let X_1 and X_2 be two such vector fields. The Lie bracket of two G-invariant vector fields is G invariant. Hence, P being integrable, we have $[X_1,X_2] = aX_1+bX_2$ for some $a,b \in \mathbb{R}$. Thus $\{X_1,X_2\}$ span a finite dimensional subalgebra k of vector fields on M . Both X_1 and X_2 are completely integrable and so by Palais [10] there is a connected Lie group K acting on M with the following property (K integrates k):

> If k_t is a one parameter subgroup of K and Y is the vector field defined by $Y(p) = \frac{d}{dt}(k_t(p))\Big|_{t=0}$ then $Y \in k$.

The action of K on M sends any leaf of F to itself and commutes with the action of G . In particular the actions of K and $\mathrm{Ker}(\hat{\zeta})$ restricted to any leaf F_x of F commute. Since $\mathrm{Ker}(\hat{\zeta})$ is abelian and simply transitive on F_x this implies that if $k \in K$ there is a $g \in \mathrm{Ker}(\hat{\zeta})$ such that $k|F_x = g|F_x$. To see this represent the action of $\mathrm{Ker}(\hat{\zeta})$ on F_x as a group acting on itself by left translations. Then K must act by right translations which coincide with left translations since $\mathrm{Ker}(\hat{\zeta})$ is abelian. It follows that K acts by isometries on F_x in the induced metric and therefore acts on P by isometries. If $k \in K$ the distribution Q^k defined by $Q^k_x = dk(Q_{k^{-1}(x)})$ is G invariant. Since the G_x-invariant complement to P_x is unique we must have $Q^k = Q$ for each $k \in K$. Therefore Q is invariant under K . Now K commutes with G so it is easy to show that if $k \in K$ there is an $\alpha(k) \in \mathbb{R}^+_x$ such that $\langle dk_x(v), dk_x(v)\rangle_{k(x)} = \alpha(k)\langle v,v\rangle_x$ for each $x \in M$, $v \in Q_x$. The action of K descends to give an action on any manifold of the form $\Gamma\backslash M$, Γ a discrete subgroup of G . The existence of such a quotient with finite volume implies that $\alpha(k) = 1$ for each $k \in K$ and so K acts by isometries. The action of G on M is assumed maximal so $K \subseteq G$. Therefore $K = \mathrm{Ker}_0(\hat{\zeta}) = \mathrm{Ker}(\hat{\zeta})$. Since K commutes with G we conclude that $\mathrm{Ker}(\hat{\zeta})$ is central.

$\square$

For the next three sections we assume that M/F is equipped with a metric λ satisfying 4.1.1 and such that $(M/F,\lambda)$ is isometric to E^2, H^2 or S^2 .

4.2 M/F has Euclidean metric

We are assuming in this section that (M,G) is a four dimensional geometry with M/F diffeomorphic to $\mathbb{R}^2$ and that $\hat{\zeta}(G)$ acts conformally on $\mathbb{R}^2$ with respect to the Euclidean metric. If we denote by $C^+(E^2)$ the group of orientation preserving conformal automorphisms of E^2 then $\hat{\zeta}(G)$ is a connected subgroup of $C^+(E^2)$ transitive on E^2 and containing a compact subgroup isomorphic to $SO(2)$.

Proposition 4.2.1

The group $\zeta(G)$ is isomorphic to either $C^+(E^2)$ or $Isom^+(E^2)$.

Proof

Since $\hat{\zeta}(G)$ is transitive on M/F and has non-trivial stabilizer at each point of M/F by 2.2.1(c) we have $\dim(\hat{\zeta}(G)) \geq 3$. If $\dim(\hat{\zeta}(G)) = 4$ then $\hat{\zeta}(G) = C^+(E^2)$ since both these groups are connected. If $\dim(\hat{\zeta}(G)) = 3$ then, since $\dim(G) = 5$, $\dim(Ker(\hat{\zeta})) = 2$. Therefore by 2.1.4 $(M/F,\hat{\zeta}(G))$ is a geometry. It is easy to see that in this case $(M/F,\hat{\zeta}(G))$ is equivalent to $(E^2,Isom^+(E^2))$ and so $\hat{\zeta}(G)$ is isomorphic to $Isom^+(E^2)$.

□

We first show the existence of a geometry satisfying the first possibility given by Proposition 4.2.1.

Theorem 4.2.2

Let G be the group defined by $\mathbb{R}^3 \rtimes_\alpha (\mathbb{R} \times SO(2))$ where $\alpha(t,\theta)$ is given in matrix form by

$$\alpha(t,\theta) = \begin{pmatrix} e^t\mathrm{Cos}\theta & e^t\mathrm{Sin}\theta & 0 \\ -e^t\mathrm{Sin}\theta & e^t\mathrm{Cos}\theta & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

and let K denote the subgroup of G defined by $K = \{(0,0,0,0,\theta) \mid \theta \in [0,2\pi)\}$. Then $(G/K,G)$ is a maximal four dimensional geometry satisfying the condition that $(M/F,\hat{\xi}(G))$ is equivalent to $(E^2,C^+(E^2))$.

Proof

The proof will be in three parts. First we will show that there is a discrete subgroup $\Gamma \subseteq G$ such that $\Gamma\backslash G/K$ is a compact manifold. Then $(G/K,G)$ will be shown to be maximal. Finally we show that $(M/F,\hat{\xi}(G))$ is equivalent to $(E^2,C^+(E^2))$.

Let G_1 be the group $\mathbb{R}^3 \rtimes_{\beta_1} \mathbb{R}$ where the action of $\mathbb{R}$ on $\mathbb{R}^3$ is given by

$$\beta_1(t) = \begin{pmatrix} e^{\lambda t}\mathrm{Cos}t & e^{\lambda t}\mathrm{Sin}t & 0 \\ -e^{\lambda t}\mathrm{Sin}t & e^{\lambda t}\mathrm{Cos}t & 0 \\ 0 & 0 & e^{-2\lambda t} \end{pmatrix}$$

This group has a compact group of automorphisms isomorphic to $SO(2)$ given by

$$\beta_2(\theta)\cdot(x_1,x_2,x_3,t) = (x_1 \operatorname{Cos}\theta+x_2 \operatorname{Sin}\theta,-x_1\operatorname{Sin}\theta+x_2\operatorname{Cos}\theta,x_3,t) \quad .$$

It is easy to see that there is an isomorphism $\phi:G \to G_1 \ltimes_{\beta_2} SO(2)$ which sends K onto the subgroup $K_1 = \{e\} \times SO(2)$. It follows that the geometries $(G/K,G)$ and $(G_1\ltimes_{\beta_2} SO(2)/K_1 \;,\; G_1\ltimes_{\beta_2} SO(2))$ are equivalent. It will be shown in 6.4.9 that for some λ there is a discrete subgroup $\Gamma \leq G_1$, such that $\Gamma\backslash G_1$ is compact. So $\phi^{-1}(\Gamma)\backslash G/K$ is a compact manifold. Therefore $(G/K,G)$ is a geometry.

Assume that $(G/K,G)$ is not maximal. Then we can find a left invariant metric λ on $M = G/K$ such that (M,λ) is isometric to one of the symmetric spaces of Chapter 3. Let $(M,\mathrm{Isom}_0(M,\lambda))$ be the corresponding maximal geometry. There is an embedding $\psi:G \to \mathrm{Isom}_0(M)$ where $\psi(G)$ is a closed subgroup of $\mathrm{Isom}_0 M$. Since G/K is diffeomorphic to $\mathbb{R}^4$ (M,λ) cannot be isometric to $E^2\times S^2$, $S^2\times K^2$, $S^2\times S^2$, $E\times S^3$, S^4 or $\mathbb{C}P^2$. The remaining possibilities are $E^2\times H^2$, $H^2\times H^2$, $E\times H^3$,H^4,E^4, $\mathbb{C}H^2$ all of which have non-positive curvature. Let G_2 be the group $\mathbb{R}^3\ltimes_{\beta_3}\mathbb{R}$ where $\mathbb{R}$ acts by

$$\beta_3(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

Clearly $G \cong G_2 \ltimes SO(2)$. A G invariant metric on G/K corresponds to a left invariant metric on G_2 which is also invariant under the action of $SO(2)$. The existence of such a metric of non-positive curvature was shown above. Now G_2 is unimodular and $\mathfrak{g}_2 \cong [\mathfrak{g}_2,\mathfrak{g}_2] \ltimes_{M_3} \mathbb{R}$ where

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

As noted in Theorem 1.6 of [8] one can use the results of sections 5.2, 4.4, 6.2 of [1] to show that $\mathbb{R}$ must be orthogonal to $[\mathfrak{g}_2,\mathfrak{g}_2] \cong \mathbb{R}^3$ and that M_3 is skew-adjoint with respect to an inner product on $\mathbb{R}^3$. If A is a positive definite matrix then the adjoint of M_3 with respect to the inner product $\langle Au,v \rangle$ is given by $M_3^* = AM_3^TA^{-1}$ where M_3^T denotes the transpose. But $M_3^T = M_3$ so $M_3^* = -M_3$ implies that M_3 and $-M_3$ have the same eigenvalues. This is clearly not the case. Hence (G/K,G) is maximal.

Representing G/K as $\mathbb{R}^4$ the action of $g = (u_1,u_2,u_3,u_4,\theta)$ on G/K is given by

$$g\cdot(x_1,x_2,x_3,x_4) = (u_1+e^{u_4}(x_1\operatorname{Cos}\theta+x_2\operatorname{Sin}\theta),u_2-e^{u_4}(x_1\operatorname{Sin}\theta-x_2\operatorname{Cos}\theta) \; ,$$

$$u_3 + e^{-2u_4}x_3,u_4 + x_4) \; .$$

The leaf of F through (x_1,x_2,x_3,x_4) is given by

$$F_{(x_1,x_2,x_3,x_4)} = \{(x_1,x_2,t,s) \mid t,s \in \mathbb{R}\}$$

and therefore the action of G on M/F is

$$g\cdot(x_1,x_2) = (u_1+e^{u_4}(x_1\mathrm{Cos}\theta+x_2\mathrm{Sin}\theta),u_2-e^{u_4}(x_1\mathrm{Sin}\theta-x_2\mathrm{Cos}\theta)) \quad .$$

Clearly, then, $\hat{\zeta}(G)$ is the conformal group of E^2 and $\mathrm{Ker}(\hat{\zeta}) = \{(0,0,u,0,0)(u \in \mathbb{R}\}$ which has dimension 1 .

□

In fact the geometry of Theorem 4.2.2 is the only four dimensional one satisfying the condition that $\hat{\zeta}(G) = C^+(E^2)$.

Theorem 4.2.3

If (M,G) is a four dimensional geometry with the pair $(M/F,\hat{\zeta}(G))$ equivalent to $(E^2,C^+(E^2))$ then (M,G) is equivalent to the geometry described in Theorem 4.2.2.

Proof

The following facts about $C^+(E^2)$ will be needed:

F1: The universal cover $\tilde{C}^+(E^2)$ of $C^+(E^2)$ is $\mathbb{C}\times\mathbb{C}$ with the multiplication $(z_1,z_2)\cdot(w_1,w_2) = (z_1+e^{z_2}w_1,z_2+w_2)$.

Topologically $\tilde{C}^{+}(E^2)$ is $\mathbb{R}^4$ with the multiplication

$$(x_1,x_2,x_3,x_4)\cdot(y_1,y_2,y_3,y_4) = (x_1 + e^{x_3}(y_1\mathrm{Cos}x_4 - y_2\mathrm{Sin}x_4)\ ,\ x_2 + e^{x_3}(y_2\mathrm{Cos}x_4 + y_1\mathrm{Sin}x_4)\ ,\ x_3 + y_3, x_4+y_4)\ .$$

F2: With respect to the multiplication on $\mathbb{R}^4$ given by F1 a basis for the Lie algebra $\mathfrak{c}(E^2)$ is given by the left invariant vector fields:

$$X_1 = e^{x_3}(\mathrm{Cos}x_4\partial/\partial x_1 + \mathrm{Sin}x_4\partial/\partial x_2)\ ,\ X_2 = e^{x_3}(-\mathrm{Sin}x_4\partial/\partial x_1 + \mathrm{Cos}x_4\partial/\partial x_2)$$

$$X_3 = \partial/\partial x_3,\ X_4 = \partial/\partial x_4$$

and so the Lie algebra $\mathfrak{c}(E^2)$ is given by:

$$\mathfrak{c}(E^2) = \{X_1,X_2,X_3,X_4 \mid [X_1,X_2] = 0\ ,\ [X_3,X_4] = 0\ ,\ [X_1,X_3] = -X_1\ ,\ [X_2,X_3] = -X_2,\ [X_1,X_4] = -X_2, [X_2,X_4] = X_1\}\ .$$

F3: Let $g = (x_1,x_2,x_3,x_4)$ then the adjoint map $\mathrm{Ad}:G \to \mathrm{Aut}(\mathfrak{g})$ is given by:

$$\mathrm{Ad}(g) = \begin{pmatrix} e^{x_3}\mathrm{Cos}x_4 & -e^{x_3}\mathrm{Sin}x_4 & -x_1 & x_2 \\ e^{x_3}\mathrm{Sin}x_4 & e^{x_3}\mathrm{Cos}x_4 & -x_2 & -x_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The modular homomorphism $\Delta_{\tilde{C}^{+}(E^2)}$ of $\tilde{C}^{+}(E^2)$ is given by

$\Delta_{\tilde{C}^+(E^2)}(g) = |\det(\mathrm{Ad}(g))|$ so we have:

$$\Delta_{\tilde{C}^+(E^2)}(x_1,x_2,x_3,x_4) = e^{2x_3} \quad .$$

(i) We have an extension $\mathrm{Ker}(\hat{\zeta}) \rightarrowtail G \twoheadrightarrow C^+(E^2)$ with $\dim(\mathrm{Ker}(\hat{\zeta})) = 1$. Let $\mathrm{Ker}_0(\hat{\zeta})$ denote the connected component of e in $\mathrm{Ker}(\hat{\zeta})$ and let $\phi : C^+(E^2) \to \mathrm{Aut}_0(\mathrm{Ker}_0(\hat{\zeta}))$ denote the homomorphism induced by conjugation. By Proposition 1.1.3 G must be unimodular since we are assuming the existence of a quotient of finite volume. Since $\mathrm{Ker}_0(\hat{\zeta})$ is closed in G it follows from Proposition 1.1.4 that ϕ is non-trivial. This, in turn, implies that $\mathrm{Ker}_0(\hat{\zeta})$ cannot be isomorphic to $SO(2)$ so we have $\mathrm{Ker}_0(\hat{\zeta}) \cong \mathbb{R}$, $\mathrm{Aut}_0(\mathrm{Ker}_0(\hat{\zeta})) \cong \mathbb{R}^+_*$ and $|\det d\phi(g)| = \phi(g)$ for each $g \in C^+(E^2)$. By Proposition 1.1.4 $|\det d\phi(g)| = \Delta^{-1}_{C^+(E^2)}(g)$ so, from F3 above, $\phi(x_1,x_2,x_3,x_4)\cdot t = e^{-2x_3}t$ for $t \in \mathrm{Ker}_0(\hat{\zeta})$. If $\phi_* : c(E^2) \to \mathbb{R}$ is the induced homomorphism of Lie algebras then, for the basis $\{X_1,X_2,X_3,X_4\}$ of F2 above we have $\phi_*(X_1) = \phi_*(X_2) = \phi_*(X_4) = 0$, $\phi_*(X_3)\cdot Y = -2Y$ where Y is a basis for $\mathbb{R}$ as a Lie algebra.

(ii) Fix a point $x_0 \in M$. Then, since $TF^\perp$ is irreducible and G-invariant there is an irreducible $\mathrm{ad}(g_{x_0})$ invariant two dimensional subspace $h \subseteq g$ such that $dq(h) = TF^\perp$ (here g is regarded as left invariant vector fields on G and $q : G \to G/G_{x_0}$ is the quotient map). Let $\hat{\zeta}_* : g \to c(E^2)$ denote the homomorphism induced from $\hat{\zeta}$ and $h \subseteq c(E^2)$

the ideal corresponding to translations. Then $\mathfrak{r} \cap \hat{\zeta}_*^{-1}(\mathfrak{h})$ is $\mathrm{ad}(\mathfrak{g}_{x_0})$ invariant since $\hat{\zeta}_*^{-1}(\mathfrak{h})$ is an ideal of $\mathfrak{g}$. Now $\dim(\hat{\zeta}_*(\mathfrak{h})) = 3$ and $\dim(\mathfrak{r}) = 2$ so $\mathfrak{r} \cap \hat{\zeta}_*^{-1}(\mathfrak{h}) \neq \{0\}$. Hence $\mathfrak{r} \subseteq \hat{\zeta}_*^{-1}(\mathfrak{h})$. Now since $\mathfrak{r}$ is irreducible and $[\mathfrak{g}_{x_0}, \mathrm{Ker}(\hat{\zeta}_*)] \subseteq \mathrm{Ker}(\hat{\zeta}_*)$ we also have $\hat{\zeta}_*|\mathfrak{r}$ injective.

(iii) The Lie algebra of G is spanned by $\{Y_1,Y_2,Y_3,Y_4,Y_5\}$ where $\hat{\zeta}_*(Y_1) = 0$, $\hat{\zeta}_*(Y_i) = X_{i-1}$ for $2 \le i \le 5$. By (ii) we can assume that $[Y_2,Y_5] = Y_3$, $[Y_3,Y_5] = -Y_2$, $[Y_2,Y_3] = aY_1$ since $\{X_1,X_2,X_3\}$ spans the subalgebra of $C(E^2)$ corresponding to $\mathrm{Isom}^+(E^2)$. From (i) we have $[Y_1,Y_2] = [Y_1,Y_3] = [Y_1,Y_5] = 0$, $[Y_4,Y_1] = -2Y_1$. Hence

$$\begin{aligned}\mathfrak{g} = \{Y_1,Y_2,Y_3,Y_4,Y_5 \mid\ & [Y_4,Y_1] = -2Y_1, [Y_2,Y_3] = aY_1, [Y_2,Y_5] = Y_3, \\ & [Y_3,Y_5] = -Y_2, [Y_1,Y_2] = [Y_1,Y_3] = [Y_1,Y_5] = 0, [Y_4,Y_2] = Y_2+bY_1 \\ & [Y_4,Y_3] = Y_3 + cY_1, [Y_4,Y_5] = dY_1\},\end{aligned}$$

is a presentation of $\mathfrak{g}$ for some $a,b,c,d \in \mathbb{R}$. Now we have:

$$[Y_2,[Y_3,Y_4]] + [Y_4,[Y_2,Y_3]] + [Y_3,[Y_4,Y_2]] = -4aY_1$$

$$[Y_4,[Y_3,Y_5]] + [Y_5,[Y_4,Y_3]] + [Y_3,[Y_5,Y_4]] = -bY_1$$

$$[Y_4,[Y_2,Y_5]] + [Y_5,[Y_4,Y_2]] + [Y_2,[Y_5,Y_4]] = cY_1 \quad .$$

Therefore, since g is a Lie algebra, we must have $a = b = c = 0$. Now $\{Y_1, Y_2, Y_3\}$ spans an abelian ideal with a complementary abelian subalgebra spanned by $\{Y_4, 1/2dY_1 + Y_5\}$. Hence $g \cong \mathbb{R}^3 \ltimes \mathbb{R}^2$ where $\mathbb{R}^2$ acts by the commuting matrices

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

An easy calculation shows that the universal cover $\tilde{G}$ of G is given by $\mathbb{R}^3 \ltimes_\alpha \mathbb{R}^2$ where

$$\alpha(s,t) = \begin{pmatrix} e^s \mathrm{Cos}\,t & e^s \mathrm{Sin}\,t & 0 \\ -e^s \mathrm{Sin}\,t & e^s \mathrm{Cos}\,t & 0 \\ 0 & 0 & e^{-2s} \end{pmatrix}$$

The centre of $\tilde{G}$ is $Z(\tilde{G}) = \{(0,0,0,0,2k\pi) \in \mathbb{R}^5 \mid k \in \mathbb{Z}\}$. Thus if G is to act effectively on M we must have $G = \tilde{G}/Z(\tilde{G})$ and the maximal compact subgroup of G is isomorphic to $SO(2)$. Clearly G is isomorphic to the group of Theorem 4.2.2 so (M,G) is equivalent to the geometry defined in 4.2.2.

$\square$

Moving on to the case $\hat{\zeta}(G) \cong \mathrm{Isom}^+(E^2)$ we again have a unique maximal geometry satisfying this condition. We denote by N the nilpotent group of matrices:

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\}$$

and by K the group of automorphisms of N isomorphic to $SO(2)$ given by:

$$K(\theta) \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\mathrm{Cos}\theta + z\mathrm{Sin}\theta & y - xz\mathrm{Sin}^2\theta + 1/4\,\mathrm{Sin}2\theta(z^2 - x^2) \\ 0 & 1 & -x\mathrm{Sin}\theta + z\mathrm{Cos}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Lemma 4.2.4

The group $K \times \{id\}$ is a maximal compact group of automorphisms of $N \times \mathbb{R}$.

Proof

Since $N \times \mathbb{R}$ is simply connected we need only show that the maximal compact group of automorphisms of the Lie algebra $n \times \mathbb{R}$ is isomorphic to $SO(2)$. Now $n \times \mathbb{R}$ is given by:

$$n \times \mathbb{R} = \{X_1, X_2, X_3, X_4 \mid [X_4, X_i] = 0 \text{ for } 1 \le i \le 3,\ [X_1, X_2] = 0,$$
$$[X_1, X_3] = 0,\ [X_2, X_3] = X_1\}.$$

With respect to this basis $\mathrm{Aut}(n \times \mathbb{R})$ is given by:

$$\mathrm{Aut}(n \times \mathbb{R}) = \left\{ \begin{pmatrix} a & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & 0 \\ 0 & b_3 & c_3 & 0 \\ 0 & b_4 & c_4 & d_2 \end{pmatrix} \mid b_2c_3 - b_3c_2 = a \right\}$$

It is easy to see that $\mathrm{Aut}(n \times \mathbb{R}) = R \cdot S$ where

$$R = \left\{ \begin{pmatrix} 1 & b_1 & b_2 & b_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c_1 & c_2 & c_3 \end{pmatrix} \middle| \; b_i, c_j \;\; \mathbb{R} \;\text{ for }\; 1 \le i,j \le 3 \right\}$$

$$S = \left\{ \begin{pmatrix} x_1y_2 - x_2y_1 & 0 & 0 & 0 \\ 0 & x_1 & y_1 & 0 \\ 0 & x_2 & y_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x_1y_2 - x_2y_1 \ne 0 \right\}$$

R is simply connected, closed, normal and solvable and $S \cong GL(2,\mathbb{R})$. It follows that a maximal compact subgroup of S is maximal compact in $\mathrm{Aut}(n \times \mathbb{R})$. The maximal compact subgroup of S is isomorphic to $SO(2)$ and so we conclude that K is maximal compact.

□

Theorem 4.2.5

Let G be the group $(N \times \mathbb{R}) \rtimes_\alpha SO(2)$ where $\alpha(SO(2))$ is the group of automorphisms of $N \times \mathbb{R}$ described in 4.2.4. If we denote by K the subgroup of G defined by $K = \{(e,0,A) | A \in SO(2)\}$ then $(G/K, G)$ is a maximal geometry such that $(M/F, \hat{\zeta}(G))$ is equivalent to $(E^2, \mathrm{Isom}^+(E^2))$.

Proof

If Γ denotes the subgroup of $N \times \mathbb{R}$ consisting of the direct product of $\mathbb{Z} \subseteq \mathbb{R}$ and the integral matrices in N then $\Gamma \backslash N \times \mathbb{R}$ is compact. Hence $\Gamma\backslash G/K$ is a compact manifold. Therefore $(G/K, G)$ is a geometry.

A G-invariant metric on G/K corresponds to a left invariant metric on $N \times \mathbb{R}$ also invariant under the adjoint action of K. Let λ be such a metric for which $\mathrm{Isom}_0(N \times \mathbb{R}, \lambda)$ is maximal. By [12] $N \times \mathbb{R}$ is normal in $\mathrm{Isom}_0(N \times \mathbb{R}, \lambda)$ so, denoting the stabilizer of e by K', we have a homomorphism $\phi : K' \to \mathrm{Aut}(N)$. Since K' is connected and $dk_e \neq \mathrm{id}$ for any $k \in K'$ the map ϕ is injective and $\phi(K')$ is a compact subgroup of $\mathrm{Aut}(N)$ containing $\mathrm{Ad}(K)$. Since $\mathrm{Ad}(K)$ is maximal by 4.2.4 we must have $\mathrm{Ad}(K) = \phi(K')$ and so $K = K'$. It follows that $(G/K, G)$ is maximal.

Let $g = (n, t, \theta)$ be an element of G with

$$n = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad \theta \in [0, 2\pi) .$$

If G/K is regarded as $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). then:

$$\begin{aligned} g \cdot (x_1, x_2, x_3, x_4) = (&u + x_1 \mathrm{Cos}\theta + x_3 \mathrm{Sin}\theta, x_2 + v - u(x_1 \mathrm{Sin}\theta - x_3 \mathrm{Cos}\theta) \\ &+ 1/4(x_3^2 - x_1^2)\mathrm{Sin}2\theta - x_1 x_3 \mathrm{Sin}^2\theta, w - x_1 \mathrm{Sin} + x_3 \mathrm{Cos}\theta , \\ &t + x_4) . \end{aligned}$$

The leaf of F through (x_1, x_2, x_3, x_4) is:

$$F_{(x_1,x_2,x_3,x_4)} = \{(x_1,t,x_3,s) \mid t,s \in \mathbb{R}\}$$

and therefore the action of G on M/F is:

$$g\cdot(x_1,x_2) = (u,v) + A_\theta(x_1,x_2) \ , \quad A_\theta = \begin{pmatrix} \mathrm{Cos}\theta & \mathrm{Sin}\theta \\ -\mathrm{Sin}\theta & \mathrm{Cos}\theta \end{pmatrix} \ .$$

So $\hat{\zeta}(G) = \mathrm{Isom}_0^+(E^2)$. We also have $\mathrm{Ker}(\hat{\zeta}) = \{(0,t,0,s,0) | t,s \in \mathbb{R}\}$, which is the centre of $N \times \mathbb{R}$, confirming Proposition 4.1.2.

□

Theorem 4.2.6

If (M,G) is a four dimensional geometry with $(M/F,\hat{\zeta}(G))$ equivalent to $(E^2,\mathrm{Isom}_0^+(E^2))$ then (M,G) is equivalent to the geometry described in Theorem 4.2.5.

Proof

We have an exact sequence $\mathrm{Ker}(\hat{\zeta}) \rightarrowtail G \xrightarrow{\hat{\zeta}} \!\!\!\!\to \mathrm{Isom}_0^+(E^2)$. By Proposition 4.1.2 we know that $\mathrm{Ker}(\hat{\zeta})$ is abelian and central in G . At the Lie algebra level, therefore, we have the exact sequence $\mathbb{R}^2 \rightarrowtail g \xrightarrow{\hat{\zeta}} \!\!\!\!\to \mathit{isom}(E^2)$ where $\mathit{isom}^+(E^2)$ acts trivially on $\mathbb{R}^2$. The Lie algebra of $\mathrm{Isom}^+(E^2)$ is given by:

$$\mathit{isom}(E^2) = \{Y_1,Y_2,Y_3 | [Y_1,Y_2] = 0 \ , \ [Y_1,Y_3] = -Y_2, [Y_2,Y_3] = Y_1\} \ .$$

Hence g is given by

$$g = \{X_1,X_2,Y_1,Y_2,Y_3 \mid [X_i,Y_j] = 0 \quad \text{for} \quad 1 \le i \le 2 \; , \; 1 \le j \le 3 \; ,$$
$$[Y_1,Y_2] = a_1X_1+a_2X_2, [Y_1,Y_3] = -Y_2+b_1X_1+b_2X_2 \; ,$$
$$[Y_2,Y_3] = Y_1+c_1X_1+c_2X_2, [X_1,X_2] = 0\}$$

for some choice of constants a_1,a_2,b_1,b_2,c_1,c_2 . Choosing a new basis $\bar{X}_1 = X_1$, $\bar{X}_2 = X_2$, $\bar{Y}_1 = Y_1+c_1X_1+c_2X_2$, $\bar{Y}_2 = Y_2-b_1X_1-b_2X_2$, $\bar{Y}_3 = Y_3$ we see that g can be written

$$g = \{X_1,X_2,Y_1,Y_2,Y_3 \mid [X_i,Y_j] = 0 \quad \text{for} \quad 1 \le i \le 2 \; , \; 1 \le j \le 3 \; ,$$
$$[X_1,X_2] = 0 \; , \; [Y_1,Y_2] = a_1X_1+a_2X_2, [Y_1,Y_3] = -Y_2 \; ,$$
$$[Y_2,Y_3] = Y_1\} \quad .$$

These are now two cases depending whether or not $[Y_1,Y_2]$ is trivial.

<u>Case (1)</u> : $a_1 = a_2 = 0$

In this case $g \cong \mathbb{R}^2 \times \mathit{isom}(E^3)$ so $\tilde{G} \cong \mathbb{R}^2 \times \widetilde{\mathrm{Isom}}_0^+(E^2)$. The group G is given by $\tilde{G}/D$ where D is a discrete central subgroup. Now $Z(\widetilde{\mathrm{Isom}}_0^+(E^2)) \cong \mathbb{Z}$ so D is a free abelian subgroup of $\mathbb{R}^2 \times \mathbb{Z}$. If $\mathrm{rank}(D) \ge 2$ then G contains a maximal torus T with $\dim(T) \ge 2$. Up to conjugacy in G we can assume that $G_x \subseteq T$ and we have the fibre bundle $T/G_x \to G/G_x \to G/T$. This gives an exact sequence

$\ldots \to \pi_2(G/T) \to \pi_1(T/G_x) \to \pi_1(G/G_x)$. Now, by assumption, $\pi_1(G/G_x) = 0$ and also G/T is contractible since G is solvable. Hence $\pi_1(T/G_x) = 0$, a contradiction. Thus $\operatorname{rank}(D) = 1$ ($\operatorname{rank}(D) \neq 0$ since $\tilde{G}$ contains no compact subgroups). We write $\tilde{G} = \mathbb{R} \times \tilde{G}'$ where $\tilde{G}' = \mathbb{R} \times \widetilde{\mathrm{Isom}}_0^+(E^2)$. Without loss of generality we can assume that $D \subseteq \tilde{G}'$. If $\tilde{G}'$ is written as $\mathbb{R} \times (\mathbb{C} \ltimes_\alpha \mathbb{R})$ where $\alpha(t)\cdot z = e^{it}z$ then $D = \{(k\alpha, 0, 2\pi km) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R} \mid k \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R}$, $0 \neq m \in \mathbb{Z}$. Then D is contained in $A = \{(k\alpha t, 0, 2\pi mt) \mid t \in \mathbb{R}\}$ so $\tilde{G}'/D \cong (\mathbb{R}\times\mathbb{C}) \ltimes A/D$. Since A/D preserves the Euclidean metric on $\mathbb{R} \times \mathbb{C}$ it follows that $(M,G) = (\mathbb{R} \times \mathbb{R} \times \mathbb{C}, \mathbb{R} \times (\mathbb{R} \times \mathbb{C}) \ltimes A/D)$ is not maximal.

<u>Case (2)</u> : $a_1 \neq 0$ or $a_2 \neq 0$

Choosing a new basis for g if necessary we can assume that $[Y_1, Y_2] = X_1$ and $g \cong \mathbb{R} \times h$ where h is the Lie algebra

$$h = \{Y, X_1, X_2, X_3 \mid [X_1, X_2] = Y \ , \ [Y, X_i] = 0 \ \text{ for } \ 1 \leq i \leq 3 \ ,$$
$$[X_1, X_3] = -X_2 \ , \ [X_2, X_3] = X_1\} \ .$$

The set $\{Y, X_1, X_2\}$ spans an ideal isomorphic to n and so $h \cong n \ltimes \mathbb{R}$. Thus $\tilde{G} \cong \mathbb{R} \times (N \ltimes \mathbb{R})$ where, if N is given by the upper triangular matrices:

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| \ x, y, z \in \mathbb{R} \right\}$$

then $\mathbb{R}$ acts by the compact group K of Lemma 4.2.4. Topologically $\tilde{G}$ is $\mathbb{R}\times\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^5$ and it is easy to see that the centre is $Z(\tilde{G}) = \{(t_1,0,t_2,0,2k\pi) \in \mathbb{R}^5 | t_1,t_2 \in \mathbb{R},\ k \in \mathbb{Z}\}$. We have $G \cong \tilde{G}/D$ where D is a discrete central subgroup. The argument used in Case (1) again shows that $\mathrm{rank}(D) = 1$. Hence $D = \{(kn_1,0,kn_2,0,2\pi kn_3)(k \in \mathbb{Z}\}$ for some $n_1,n_2,n_3 \in \mathbb{Z}$. D is contained in the one parameter subgroup $A = \{(tn_1,0,tn_2,0,2\pi n_3 t) \mid t \in \mathbb{R}\}$. If $n_3 = 0$ then $A/D \subseteq Z(G)$. Since A/D is isomorphic to $SO(2)$ and is the maximal compact subgroup of G this contradicts the fact that G_x acts faithfully on T_xM for each $x \in M$. Therefore $n_3 \neq 0$ and $A \cap (\mathbb{R} \times N \times \{0\})$ is trivial. Hence $\tilde{G} \cong (\mathbb{R} \times N) \ltimes A$ where A acts by conjugation and $G \cong (\mathbb{R} \times N) \ltimes A/D$ where A/D acts trivially on $\mathbb{R} \times \{e\}$ and by rotations in the x-z plane of N. The result follows immediately.

□

4.3 M/F has spherical metric

We are now assuming that (M,G) is a four dimensional geometry with M/F diffeomorphic to S^2 and that $\hat{\zeta}(G)$ acts conformally with respect to the standard metric on S^2. The group of conformal automorphisms of S^2 can be identified with the group $PSL(2,\mathbb{C})$ acting by fractional linear transformations on the extended complex plane. Hence $\hat{\zeta}(G)$ can be regarded as a transitive subgroup of $PSL(2,\mathbb{C})$ whose stabilizer subgroup at each point contains a compact subgroup isomorphic to $SO(2)$.

Proposition 4.3.1

The subgroup $\hat{\xi}(G)$ has dimension 3.

Proof

Since $\hat{\xi}(G)$ is transitive on S^2 with non-trivial stabilizer we have $\dim(\hat{\xi}(G)) \geq 3$. Also $\dim(\hat{\xi}(G)) \leq \dim(G) = 5$. Assume that $\dim(\hat{\xi}(G)) = 4$ or 5. We set $S^2 = \mathbb{C} \cup \{\infty\}$ and denote by A_∞ the stabilizer of ∞ in $\hat{\xi}(G)$. Then A_∞ is contained in the complex Affine group:

$$\mathrm{Aff}(\mathbb{C}) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL(2,\mathbb{C}) \mid ac = 1\}$$

whose Lie algebra is:

$$\mathfrak{aff}(\mathbb{C}) = \{\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \in M(2,\mathbb{C}) \mid x,y \in \mathbb{C} \} .$$

The adjoint action of $\mathrm{Aff}(\mathbb{C})$ on $\mathfrak{aff}(\mathbb{C})$ is given by

$$\mathrm{Ad}(a,b) \cdot \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} = \begin{pmatrix} x & a^2y - 2abx \\ 0 & -x \end{pmatrix}$$

Choosing $a = 1/\sqrt{y}$ if $x = 0$ and $b = ay/2x$ if $x \neq 0$ we see that the orbits of the adjoint action are represented by the matrices:

$$T_1(x) = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \quad \text{for } x \in \mathbb{C} \quad \text{or } T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If we denote by $\mathfrak{a}_\infty$ the Lie subalgebra corresponding to A_∞ we have, by assumption, $\dim(\mathfrak{a}_\infty) = 2$ or 3 . Hence $\mathfrak{a}_\infty$ contains two elements of $\mathfrak{aff}(\mathbb{C})$ independent over $\mathbb{R}$. Since A_∞ contains a compact subgroup isomorphic to $SO(2)$ we can assume, after conjugating by an element of $Aff(\mathbb{C})$, that $T_1(i) \in \mathfrak{a}_\infty$. There are now two possibilities.

Case 1 : $yT_2 \in \mathfrak{a}_\infty$ for $y \neq 0$

Conjugating by an element of $Aff(\mathbb{C})$ if necessary we can assume that that $y = 1$. Now $[T_1(i), T_2] = 2iT_2$ so for all $\alpha, \beta \in \mathbb{R}$ we have $(\alpha + i\beta)T_2 \in \mathfrak{a}_\infty$. Therefore $\mathfrak{a}_\infty$ contains the algebra:

$$\mathfrak{k}_1 = \{\begin{pmatrix} \alpha i & z \\ 0 & -\alpha i \end{pmatrix} | \alpha \in \mathbb{R} , z \in \mathbb{C} \}$$

and hence A_∞ contains the subgroup

$$K_1 = \{\begin{pmatrix} e^{i\theta} & z \\ 0 & e^{-i\theta} \end{pmatrix} | \theta \in \mathbb{R} , z \in \mathbb{C} \} .$$

Since $\hat{\xi}(G)$ is transitive on $\mathbb{C} \cup \{\infty\}$ there is a $g_0 \in \hat{\xi}(G)$, such that $g_0(\infty) = 0$. Such an g_0 must be of the form:

$$g_0 = \begin{pmatrix} 0 & a_0 \\ -1/a_0 & b_0 \end{pmatrix} .$$

Then $g_0 K_1 g_0^{-1} \subseteq \hat{\zeta}(G)$ and it is easy to see that

$$g_0 K_1 g_0^{-1} = \{ \begin{pmatrix} e^{-i\theta} & 0 \\ w & e^{i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} , w \in \mathbb{C} \} .$$

Hence $\hat{\zeta}_*(\mathcal{g})$ contains βT_3 for $\beta \in \mathbb{C}$ where $T_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Now $\hat{\zeta}_*(\mathcal{g})$ already contains αT_2 for $\alpha \in \mathbb{C}$ so, since $[\alpha T_2, \beta T_3] = T_1(\alpha\beta)$ we see that $\hat{\zeta}_*(\mathcal{g})$ contains all of $\mathfrak{sl}(2,\mathbb{C})$. This is a contradiction since $\dim(\hat{\zeta}(G)) \leq 5$.

<u>Case 2</u> : $T_1(x) + yT_2 \in \mathcal{a}_\infty$ for $x \neq 0$

If $\mathcal{a}_\infty$ contains an element $V = T_1(x) + yT_2$ for $x \neq 0$ then $[T_1(i), V] = 2iy\, T_2$ so we are again in Case (1) unless $y = 0$ in which case, since $T_1(i)$ and V are independent, $\mathcal{a}_\infty$ contains the subalgebra

$$\mathcal{k}_2 = \{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in \mathbb{C} \}$$

and A_∞ contains the subgroup:

$$K_2 = \{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mid z \in \mathbb{C} , z \neq 0 \}$$

Since $\hat{\zeta}(G)$ is assumed transitive on $\mathbb{C} \cup \{\infty\}$ there is $g_0 \in \hat{\zeta}(G)$ such that $g_0(\infty) = 1$. This g_0 is of the form

$$g_0 = \begin{pmatrix} a_0 & b_0 \\ a_0 & b_0 + 1/a_0 \end{pmatrix}$$

and so $g_0K_2g_0^{-1} \subseteq \hat{\zeta}(G)$ and it is easy to see that:

$$g_0K_2g_0^{-1} = \left\{ \begin{pmatrix} \lambda(z-1/z) + z & \lambda(1/z-z) \\ (\lambda+1)(z-1/z) & \lambda(1/z-z) + 1/z \end{pmatrix} \middle| \ z \in \mathbb{C} \right\}$$

where $\lambda = a_0b_0$. The Lie algebra of $g_0K_2g_0^{-1}$ is:

$$\mathrm{Ad}(g_0)\cdot k_2 = \left\{ \alpha \begin{pmatrix} 2\lambda+1 & -2\lambda \\ 2(\lambda+1) & -(2\lambda+1) \end{pmatrix} \middle| \ \alpha \in \mathbb{C} \right\} .$$

Since $\hat{\zeta}_*(g)$ contains $T_1(x)$ $\forall x \in \mathbb{C}$ we have $\begin{pmatrix} 0 & -\lambda \\ \lambda+1 & 0 \end{pmatrix} \in \hat{\zeta}_*(g)$.

If $\lambda = 0$ then $\hat{\zeta}(G)$ contains the subgroup:

$$K_3 = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \middle| \ z \in \mathbb{C} \right\}$$

but K_3 fixes $0 \in \mathbb{C}$ as does K_2 so the dimension of the stabilizer is ≥ 3 and we are back in Case (1). If $\lambda = -1$ then $\hat{\zeta}(G)$ contains the subgroup:

$$K_4 = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \middle| \ z \in \mathbb{C} \right\}$$

but K_4 fixes ∞ as does K_2 so the dimension of the stabilizer is ≥ 3 and we are back in Case (1). Now $\lambda \neq 0,1$ and setting

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad X_2 = \begin{pmatrix} 0 & -\lambda \\ \lambda+1 & 0 \end{pmatrix}$$

we have

$$[X_1,X_2] = \begin{pmatrix} 0 & -2\lambda \\ -2(\lambda+1) & 0 \end{pmatrix} \in \mathfrak{a}_\infty \quad .$$

Therefore $\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \in \hat{\zeta}_*(\mathfrak{g})$ for all $\alpha,\beta \in \mathbb{C}$ and, as in Case (1), this implies that $\hat{\zeta}_*(\mathfrak{g}) \subset \mathfrak{sl}(2,\mathbb{C})$. □

We now go on to classify the geometries with $(M/F,\hat{\zeta}(G)) \leq (S^2,PSL(2,\mathbb{C}))$.

Theorem 4.3.2

If (M,G) is a four dimensional geometry with $(M/F,\hat{\zeta}(G))$ equivalent to (S^2,L) where $L \leq PSL(2,\mathbb{C})$ acts transitively on S^2 then (M,G) is equivalent to either

(a) $(E^2{\times}S^2,\ \mathbb{R}^2{\times}SO(3))$

(b) $(E{\times}S^3,\ \mathbb{R}{\times}H)$ where H is a subgroup of $SO(4)$ preserving the Hopf fibration of S^3.

Neither of the geometries is maximal.

Proof

By Proposition 4.3.1 $\dim(\hat{\zeta}(G)) = 3$ so that $\dim(Ker(\hat{\zeta})) = 2$. Hence by Proposition 4.1.2 $(M/F,\hat{\zeta}(G))$ is a geometry which is easily seen to be equivalent to $(S^2,SO(3))$. Again by 4.1.2 we know that $Ker(\hat{\zeta})$ is abelian and central in G . Therefore at the Lie algebra level we have an extension $\mathbb{R}^2 \rightarrowtail g \twoheadrightarrow so(3)$ with $so(3)$ acting trivially on $\mathbb{R}^2$. Since $so(3)$ is simple g must be isomorphic to $\mathbb{R}^2 \times so(3)$ and hence the universal cover $\tilde{G}$ is isomorphic to $\mathbb{R}^2 \times SU(2)$. Thus $G = \mathbb{R}^2 \times SU(2)/D$ where D is a discrete central subgroup. Now $Z(\tilde{G}) \cong \mathbb{R}^2 \times \mathbb{Z}_2$ and the $\mathbb{Z}_2$ factor is contained in a maximal torus $A \leq SU(2)$. If T is a maximal torus in G then $T \leq (\mathbb{R}^2 \times A)/D$. Without loss of generality we can assume that $G_x \leq T$ and we have the fibre bundle $T/G_x \to G/G_x \to G/T$. If $\dim(T) = 3$ then $G/T \cong S^2$ and we have the exact sequence

$$\pi_2(T/G_x) \to \pi_2(G/G_x) \to \pi_2(G/T) \to \pi_1(T/G_x) \to \pi_1(G/G_x) \quad .$$

This is impossible since $\pi_2(T/G_x) = \pi_1(G/G_x) = 0$ and $\pi_2(G/T) \cong \mathbb{Z}$, $\pi_1(T/G_x) \cong \mathbb{Z} \times \mathbb{Z}$. Hence $\dim(T) \leq 2$ and we can assume that $G = \mathbb{R} \times ((\mathbb{R} \times SU(2))/D)$.

(i) <u>D is trivial</u>

Then $G = \mathbb{R}^2 \times SU(2)$. But any subgroup of $SU(2)$ isomorphic to $SO(2)$ contains the centre of $SU(2)$ so G does not act effectively on G/G_x.

(ii) <u>$D = \{(n,0) \in \mathbb{Z} \times \mathbb{Z}_2 \mid n \in \mathbb{Z}\}$</u>

Then $G \cong \mathbb{R} \times SO(2) \times SU(2)$. It is easy to see that any compact subgroups of G isomorphic to $SO(2)$ must intersect the centre of G non-trivially so G does not act effectively on G/G_x.

(iii) <u>$D = \mathbb{Z}_2$</u>

Then $G = \mathbb{R}^2 \times SO(3)$ and $(M,G) = (E^2 \times S^2, \mathbb{R} \times SO(3))$ which gives (a) of the theorem.

(iv) $\underline{D = \mathbb{Z} \times \mathbb{Z}_2}$

Then $G = \mathbb{R} \times SO(2) \times SO(3)$. If G_x is contained in the $SO(3)$ factor then $\pi_1(G/G_x) \neq \{0\}$ which contradicts the assumption on M. It is now easy to see that $G_x \cap Z(G) \neq \{e\}$, so, again, G does not get effectively on G/G_x.

(v) $\underline{D = \{(n,(-1)^n) \in \mathbb{Z} \times \mathbb{Z}_2 \mid n \in \mathbb{Z}\}}$

If we regard $SO(2)$ as the group of all complex numbers of modulus 1 and $SU(2)$ as the group of unit quaternions then there is a two-fold covering map $p: SO(2) \times SU(2) \to (\mathbb{R} \times SU(2))/D$ given by $p(e^{2\pi it},q) = [2t,q]$. Denoting the connected component of e in $p^{-1}(G_x)$ by K we have $p(K) = G_x$ and K is of the form

$$K = \{(e^{2\pi in\theta},e^{2\pi im\theta}) \mid \theta \in \mathbb{R}\}$$

for some n,m coprime and $m \geq 0$. Here we are writing a quaternion as z_1+jz_2 for $z_1,z_2 \in \mathbb{C}$.

If $m = 0$ then G_x is central in G so G does not act effectively on G/G_x. If we set $\theta = 1/2m$ then $(e^{2\pi in\theta},e^{2\pi im\theta}) = (e^{\ in/m},-1)$. This is not in $\mathrm{Ker}(p)$ if $m \neq 1$ or if $m = 1$ and n is even. In these cases $p(e^{\pi in/m},-1)$ is a non-trivial element of $G_x \cap Z(G)$ so G does not act effectively.

We now assume that $m = 1$ and n is odd then $\mathrm{Ker}(p)$ K so $K = p^{-1}(G_x)$ and the induced map $\bar{p}: SO(2) \times SU(2)/K \to D\backslash\mathbb{R}\times SU(2)/G_x$ is a diffeomorphism. Setting $H = SO(2)\times SU(2)$ we have the exact sequence:

$$\pi_2(H) \to \pi_2(H/K) \xrightarrow{\alpha} \pi_1(K) \xrightarrow{\beta} \pi_1(H) \to \pi_1(H/K) \to \pi_0(K) \quad .$$

Now $\pi_2(H) = \pi_0(K) = 0$, $\pi_1(H/G_x) \cong \pi_1(M) = 0$, and $\pi_1(K) \cong \pi_1(H) \cong \mathbb{Z}$. We know that β is injective and sends a generator of $\pi_1(K)$ onto n times a generator of $\pi_1(H)$. Hence α is trivial, $\pi_2(H/K) = \{0\}$ and $\pi_1(H/K) \cong \mathbb{Z}/n\mathbb{Z}$. Since $\pi_1(H/K) \cong \pi_1(M) = \{0\}$ we must have $n = \pm 1$. Using the automorphism of $SO(2) \times SU(2)$ given by $(e^{2\pi it},q) \to (e^{-2\pi it},q)$ we can assume that $n = 1$. We have $SO(2) \times SU(2)/K \cong SU(2)$ via the map $[e^{2\pi it},q] \to qe^{-2\pi it}$ and the action of $SO(2) \times SU(2)$ on $SU(2)$ is given by $(e^{2\pi it},q) .w = qwe^{-2\pi it}$. It is clear that this action preserves the Hopf fibration whose fibre through $w \in SU(2)$ is given by the set $\{we^{2\pi is} \mid s \in \mathbb{R}\}$. The action is not maximal since it is a restriction of the action of $SU(2) \times SU(2)$ on $SU(2)$ given by $(q_1,q_2)\cdot w = q_1wq_2^{-1}$. Hence we have (a) of the theorem.

Since (i)-(v) exhaust all the possible subgroups of $\mathbb{Z} \times \mathbb{Z}_2$ the proof is completed.

□

4.4 M/F has Hyperbolic metric

In this, final, section of the Chapter we will consider the case when $(M/F,\hat{\xi}(G))$ is equivalent to (H^2,L) where L acts by conformal transformations on the hyperbolic plane. In fact any conformal transformation of H^2 is an isometry so $\hat{\xi}(G)$ can be regarded as $PSL(2,\mathbb{R})$ acting on the upper half plane by fractional linear transformations. As in Section 4.2 we give first an existence result and then show uniqueness. We let $\widetilde{SL}(2,\mathbb{R})$ denote the universal cover of $PSL(2,\mathbb{R})$ and let K denote the maximal compact subgroup of the group of inner automorphisms of $\widetilde{SL}(2,\mathbb{R})$. Then $K \cong SO(2)$ and we have

Theorem 4.4.1

Let G denote the group $\mathbb{R} \times (\widetilde{SL}(2,\mathbb{R}) \ltimes K)$ and K' the subgroup $\{0\} \times \{e\} \times K$. Then $(G,G/K')$ is a maximal geometry with $(M/F,\hat{\zeta}(G))$ equivalent to $(H^2,PSL(2,\mathbb{R}))$.

Proof

Any semi-simple Lie group has a discrete subgroup with compact quotient (a result of Borel, see Ragunathan [11] Chapter XIV). Hence we can find a discrete group $\Gamma \leq \widetilde{SL}(2,\mathbb{R})$ such that $\mathbb{Z}\times\Gamma\backslash\mathbb{R}\times\widetilde{SL}(2,\mathbb{R})$ is compact. Hence $\mathbb{Z} \times \Gamma\backslash G/K'$ is a compact manifold and $(G,G/K')$ is a geometry.

Next we show that $(G/K',G)$ is maximal. If (M,G^1) is a maximal geometry extending $(G/K',G)$ then G^1_x must be isomorphic to one of the groups considered in Chapter 3. Hence (M,G^1) must be a symmetric space equipped with its maximal group of isometries. Now $\mathbb{R} \times \widetilde{SL}(2,\mathbb{R}) = G/K'$ is diffeomorphic to $\mathbb{R}^4$ so (M,G^1) must be either E^4, H^4 $E\times H^3$, $E^2\times H^2$, $H^2\times H^2$, or $\mathbb{C}H^2$ acted on by its group of isometries. This implies that there is a G-invariant metric of non-positive curvature on G/K' . This is the same as saying that there is a left invariant metric on $\mathbb{R} \times \tilde{SL}(2,\mathbb{R})$ of non-positive curvature also invariant under the maximal compact subgroup of the adjoint group. This contradicts the fact, proved in [\ ] Corollary 2.6, that a Lie group posessing a left invariant metric of non-positive curvature must necessarily be solvable. Thus $(G/K',G)$ is maximal.

To show that $(M/F,\hat{\zeta}(G))$ is equivalent to $(H^2,PSL(2,\mathbb{R}))$ it is clearly sufficient to show that $G/Ker(\hat{\zeta})$ is isomorphic to $PSL(2,\mathbb{R})$. This means, in turn, that if we denote the Lie algebra of $Ker(\hat{\zeta})$ by $\mathfrak{r}$ we must show that $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{sl}(2,\mathbb{R})$. By Theorem 2.1.3 the foliation F is given by the translates of $N_0(K')/K'$ where $N_0(K')$ is the normalizer of K' in G . The Lie algebra of G is:

$$\begin{aligned} \mathfrak{g} = \{Y_1,Y_2,X_1,X_2,X_3 \mid & [Y_1,X_i] = 0 \quad \text{for} \quad 1 \le i \le 3 \ , \ [Y_1,Y_2] = 0 \ , \\ & [X_1,X_2] = 2X_3 \ , \ [X_1,X_3] = 2X_2, \ [X_2,X_3] = 2X_1 \ , \\ & [Y_2,X_1] = -2X_3 \ [Y_2,X_2] = 0 \ , \ [Y_2,X_3] = 2X_1\} \ . \end{aligned}$$

The subalgebra corresponding to $N_0(K')$ is $\mathfrak{n}(Y_2)$ the normalizer of Y_2 in $\mathfrak{g}$. It is easy to see that $\mathfrak{n}(Y_2)$ is spanned by $\{Y_1,X_2,Y_2\}$.

Now $\mathrm{Ker}(\hat{\zeta})$ is the largest normal subgroup of $N_0(K')/K'$. Therefore $\mathfrak{n}$ is spanned by $\{Y_1, X_2\}$. If $p: g \to g/\mathfrak{n}$ denotes the quotient map we set $V_1 = p(-X_1)$, $V_2 = p(-\frac{1}{2}(X_2+Y_2))$, $V_3 = p(X_3)$. We have $[V_1,V_2] = 2V_3$, $[V_1,V_3] = 2V_2$, $[V_2,V_3] = 2V_1$ and so $g/\mathfrak{n} \cong \mathfrak{sl}(2,\mathbb{R})$. But $\dim(M/F) = \dim(G)-\dim(N_0(K')) = 2$ so $\hat{\zeta}(G)$ is a covering space of $PSL(2,\mathbb{R})$. If $\hat{\zeta}(G) \not\cong PSL(2,\mathbb{R})$ then any compact subgroup of $\hat{\zeta}(G)$ would contain the centre of $\hat{\zeta}(G)$ and $\hat{\zeta}(G)$ would not act effectively on M/F . Therefore $\hat{\zeta}(G) \cong PSL(2,\mathbb{R})$ as required.

□

The geometry described in 4.4.1 is the only one satisfying the hypothesis of this section.

Theorem 4.4.2

If (M,G) is a four dimensional geometry with $(M/F,\hat{\zeta}(G))$ equivalent to (H^2,L) with $L \subseteq PSL(2,\mathbb{R})$ then (M,G) is equivalent to the geometry described in Theorem 4.4.1.

Proof

Since the group $\mathbb{R} \times (\widetilde{SL}(2,\mathbb{R}) \ltimes K)$ has $\{0\} \times \{e\} \times K$ as a maximal compact subgroup it is sufficient to show that $G \cong \mathbb{R} \times (\widetilde{SL}(2,\mathbb{R}) \ltimes K)$. We know that $\hat{\zeta}(G) \cong PSL(2,\mathbb{R})$ since by Theorem 2.2.1(c) it is a transitive subgroup of isometries of H^2 of dimension ≥ 3 . Hence $\dim(\mathrm{Ker}(\hat{\zeta})) = 2$. By 4.1.2 $\mathrm{Ker}(\hat{\zeta})$ is abelian and central in G . So, at the Lie algebra level, there is an extension $\mathbb{R}^2 \rightarrowtail g \xrightarrow{\hat{\zeta}_*} \mathfrak{sl}(2,\mathbb{R})$

where $\mathfrak{sl}(2,\mathbb{R})$ acts trivially on $\mathbb{R}^2$. Since $\mathfrak{sl}(2,\mathbb{R})$ is simple this implies that $g \cong \mathbb{R}^2 \times \mathfrak{sl}(2,\mathbb{R})$ and $\tilde{G} \cong \mathbb{R}^2 \times \widetilde{SL}(2,\mathbb{R})$. Therefore $G \cong \tilde{G}/D$ with D a discrete central subgroup. Since $Z(\widetilde{SL}(2,\mathbb{R})) \cong \mathbb{Z}$ we can assume that D is a free abelian subgroup of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Now $\widetilde{SL}(2,\mathbb{R})$ can be written $J\cdot R$ where J covers $SO(2) \subseteq SL(2,\mathbb{R})$, $Z(\widetilde{SL}(2,\mathbb{R})) \subseteq J$, and R is isomorphic to the group:

$$\mathrm{Aff}^+(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \in GL(2,\mathbb{R}) \mid a > 0 \} .$$

Hence $\tilde{G}/D \cong D\backslash\tilde{G} = ((\mathbb{R}^2\times J)/D)\cdot R$. Therefore if $T \subseteq G$ is a maximal torus G/T is contractible. Without loss of generality we assume that $G_x \subseteq T$. From the fibration $T/G_x \to G/G_x \to G/T$ we have the exact sequence $\ldots \to \pi_2(G/T) \to \pi_1(T/G_x) \to \pi_1(G/G_x)$. Since $\pi_1(G/G_x) = \pi_2(G/T) = 0$ it follows that $\pi_1(T/G_x) = 0$, and $T = G_x$. Therefore $\mathrm{rank}(D) = 1$ and, without loss of generality, we can assume that $G \cong \mathbb{R}\times((\mathbb{R}\times\widetilde{SL}(2,\mathbb{R}))/D)$. There are two possibilities.

<u>Case (1)</u> : $D \subseteq J$

Then $G \cong \mathbb{R}^2 \times PSL^{(n)}(2,\mathbb{R})$ where $PSL^{(n)}(2,\mathbb{R})$ denotes the n-fold cover of $PSL(2,\mathbb{R})$. Clearly G extends to an action of $\mathrm{Isom}^+(E^2) \times PSL^{(n)}(2,\mathbb{R})$ on M and so (M,G) is not maximal.

<u>Case (2)</u> : $D \nsubseteq J$

D is contined in a one parameter subgroup A of $\mathbb{R} \times J$. The condition $D \nsubseteq J$ implies that the projection $p: \widetilde{SL}(2, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is injective when restricted to A. It follows that $(\mathbb{R} \times \widetilde{SL}(2, \mathbb{R}))/D \cong \widetilde{SL}(2, \mathbb{R}) \rtimes_{Ad} (A/D)$ where $Ad(A/D)$ is non-trivial since otherwise $D \subseteq \mathbb{R} \times \{e\}$ and G_x would be normal in G. Since $Ad(A/D)$ is maximal compact in $Aut(\widetilde{SL}(2, \mathbb{R}))$ it follows that G is isomorphic to the group defined in Theorem 4.4.1.

$\square$

CHAPTER 5 : STABILIZER ISOMORPHIC TO SO(2):II.

5.0 Introduction

In this Chapter we will determine the four dimensional geometries satisfying the conditions

B1: G_x isomorphic to $SO(2)$.

B2: For each $x \in M$ the action of G_x on T_xM leaves no vector fixed.

From Theorem 2.2.2 we know that there are two smooth, G-invariant, mutually orthogonal, integrable distributions $P^{(1)}, P^{(2)}$ on M with $\dim(P^{(1)}) = \dim(P^{(2)}) = 2$. The action of G_x restricted to $P_x^{(i)}$ is non-trivial for each $x \in M$. If $F^{(1)}, F^{(2)}$ are the corresponding G-invariant foliations then $M/F^{(i)}$ is a smooth simply connected manifold and $\zeta^{(i)} : M \to M/F^{(i)}$ is fibre bundle for $i = 1,2$. There is a smooth action of G on $M/F^{(i)}$ equivariant with respect to the projection $\zeta^{(i)} : M \to M/F^{(i)}$. Lastly we know that with respect to the metric induced from M the leaves of $F^{(i)}$ are isometric to E^2, H^2 or S^2 .

In Section 5.1 we show that the leaves of $F^{(i)}$ cannot be isometric to S^2 for $i = 1,2$. This leaves two possibilities: either one of the foliations $F^{(1)}, F^{(2)}$ has leaves isometric to H^2 or both of them have leaves isometric to E^2 . The first possibility is taken up in 5.2 where it is shown that there are two geometries satisfying the required condition. Both these geometries have quotients of finite volume but no compact quotients. In Section 5.3 we show that the second possibility implies that the geometry is not maximal and can be extended to the Euclidean geometry $(E^4, \mathrm{Isom}^+(E^4))$.

For the rest of this Chapter (M,G) will be a four dimensional geometry satisfying (1) and (2) above and P , F , ζ etc., will have the meanings assigned in Theorem 2.2.2.

5.1 Extensions of Theorem 2.2.2

We first determine $\mathrm{Ker}(\hat{\zeta}) \cap G_x$.

Proposition 5.1.1

If F is one of the two G-invariant foliations on M then, in the notation of Theorem 2.2.2, $\mathrm{Ker}(\hat{\zeta}) \cap G_x$ is finite.

Proof

From Proposition 2.1.2 we know that $\mathrm{Ker}(\hat{\zeta}) \cap G_x$ is given by $\mathrm{Ker}(\hat{\zeta}) \cap G_x = \{g \in G \mid dg_x|TF_x^{\perp} = id\}$. Now with respect to some orthonormal basis in $TF_x^{\perp}$ the action of G_x is given by the homomorphism $\rho : \mathbb{R}/2\pi\mathbb{Z} \to GL(2,\mathbb{R})$ where, for some non-zero $n \in \mathbb{Z}$

$$\rho(\theta) = \begin{pmatrix} \mathrm{Cos}\ n\theta & \mathrm{Sin}\ n\theta \\ -\mathrm{Sin}\ n\theta & \mathrm{Cos}\ n\theta \end{pmatrix} , \quad \theta \in [0,2\pi) \quad .$$

If $\rho(\theta) = I$ then $\theta = 2k\pi/n$ for $0 \le k < n$. Hence $\mathrm{Ker}(\hat{\zeta}) \cap G_x = \{\theta \in [0,2\pi) \mid \rho(\theta) = I\}$ is finite.

$\square$

From Theorem 2.2.2(c) we know that for a G-invariant metric on M the leaves of $F^{(1)}$ and $F^{(2)}$ are isometric to E^2, H^2 or S^2 . In the current situation we can eliminate the case S^2 .

Proposition 5.1.2

With respect to the metric induced from a G-invariant metric on M the leaves of $F^{(1)}$ and $F^{(2)}$ are isometric to E^2 or H^2 .

Proof

We must eliminate the possibility that the leaves are isometric to S^2 . Assume that the leaves of $F^{(1)}$ are isometric to S^2 . If $x \in M$ denote by F_x the leaf of $F^{(1)}$ through x and by K_x the subgroup $K_x = \{g \in G \mid g(F_x) = F_x\}$. Then it follows from Proposition 5.1.1 that K_x covers $SO(3)$ and is therefore compact. Now $\hat{\zeta}^{(1)}(K_x)$ is the subgroup of $\hat{\zeta}^{(1)}(G)$ fixing $\hat{\zeta}(x)$. Thus there is a $\hat{\zeta}^{(1)}(G)$ invariant metric on $M/F^{(1)}$. So, since $\dim(M/F^{(1)}) = 2$ by Theorem 2.2.2 and $\hat{\zeta}^{(1)}| G_x$ is non-trivial by Proposition 5.1.1, we have $\dim(\hat{\zeta}^{(1)}(G)) = 3$ and $\dim(\mathrm{Ker}(\hat{\zeta})) = 2$. This implies that $\mathrm{Ker}(\hat{\zeta}^{(1)})$ is a proper normal subgroup of K_x contradicting the fact that K_x is simple. We conclude that the leaves of $F^{(1)}, F^{(2)}$ are isometric to E^2 or H^2 .

$\square$

5.2 Invariant Foliation with Hyperbolic Leaves

We consider the case when one of the foliations $F^{(1)}, F^{(2)}$ has hyperbolic leaves. The existence of such a geometry is given by:

Theorem 5.2.1

Let $\alpha : SL(2, \mathbb{R}) \to GL(2, \mathbb{R})$ be a non-trivial homomorphism and let G denote the group $\mathbb{R}^2 \ltimes_\alpha SL(2, \mathbb{R})$. If $K \leq G$ is the subgroup $\{0\} \times SO(2)$ then $(G/K, G)$ is a maximal geometry. Furthermore if $q : G \to G/K$ denotes the quotient map then:

(a) The action of K on $T_{q(e)}G/K$ leaves no vector fixed.

(b) For any G-invariant metric on G/K one of the foliations $F^{(1)}, F^{(2)}$ has leaves isometric to H^2 and the other has leaves isometric to E^2 .

Proof

We first show that there is a discrete subgroup $\Gamma \leq G$ such that $\Gamma \backslash G/K$ is a manifold of finite volume. Up to conjugacy in $GL(2, \mathbb{R})$ there are only two homomorphisms $\alpha_1, \alpha_2 : SL(2, \mathbb{R}) \to SL(2, \mathbb{R})$. These are given by

$$\alpha_1 . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad \alpha_2 . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} .$$

Hence G is isomorphic to $\mathbb{R}^2 \ltimes_{\alpha_i} SL(2, \mathbb{R})$ for $i = 1$ or 2.

Now $\alpha_i(SL(2,\mathbb{Z})) = SL(2,\mathbb{Z})$ for $i = 1,2$ so the group $C = (\mathbb{Z}\times\mathbb{Z}) \ltimes_{\alpha_i} (SL(2,\mathbb{Z}))$ is a discrete subgroup of G. Since $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ has finite volume it follows that $C\backslash G/H$ has finite volume. However $C\backslash G/H$ may not be a manifold. To avoid this problem note that there is a subgroup Γ of finite index in $SL(2,\mathbb{Z})$ which contains no elements of finite order. If we set $C' = (\mathbb{Z}\times\mathbb{Z})\ltimes_{\alpha_i}\Gamma$ then $C'\backslash G/H$ is a manifold of finite volume. Since G/K is diffeomorphic to $\mathbb{R}^4$ the pair $(G/K,G)$ is simply connected and so a geometry.

We now wish to show that $(G/K,G)$ is maximal. Suppose not. Then we can find a G-invariant metric λ on $M = G/K$ such that $\mathrm{Isom}_0(M,\lambda) \neq G$. Then $(M,\mathrm{Isom}_0(M,\lambda))$ is equivalent to one of the symmetric space geometries of Chapter 3. Furthermore, since G/K is diffeomorphic to $\mathbb{R}^4$, this symmetric space must be $E^4, H^4, E\times H^3, E^2\times H^2, H^2\times H^2$ or $\mathbb{C}H^2$. It follows that the solvable group $\mathbb{R}^2\ltimes_{\alpha_i}\mathrm{Aff}(\mathbb{R}) \subseteq \mathbb{R}^2\ltimes_{\alpha_i} SL(2,\mathbb{R})$ has a left invariant metric of non-positive curvature where

$$\mathrm{Aff}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \in GL(2,\mathbb{R}) \;\middle|\; a > 0 \right\}.$$

The Lie algebras $\mathfrak{r}^{(i)}$ of $\mathbb{R}^2\ltimes_{\alpha_i}\mathrm{Aff}(\mathbb{R})$ are given by

$$\begin{aligned}\mathfrak{r}^{(1)} = \{X_1,X_2,Y_1,Y_2 \mid{}& [X_1,X_2] = 0\ ,\ [Y_1,Y_2] = 2Y_2\ ,\ [Y_1,X_1] = X_1\ ,\\ & [Y_1,X_2] = -X_2,\ [Y_2,X_1] = 0\ ,\ [Y_2,X_2] = X_1\}\ .\end{aligned}$$

$$\mathfrak{r}^{(2)} = \{X_1, X_2, Y_1, Y_2 \mid [X_1,X_2] = 0 \ , \ [Y_1,Y_2] = 2Y_2, \ [Y_1,X_1] = X_1 \ ,$$
$$[Y_1,X_2] = -X_2, [Y_2,X_1] = 0 \ , \ [Y_2,X_2] = -X_1\} \ .$$

In both cases the derived algebra $[\mathfrak{r}^{(i)},\mathfrak{r}^{(i)}]$ is spanned by $\{X_1,X_2,Y_2\}$. With respect to this basis $\mathrm{ad}(Y_1)$ is given by the matrix:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and $\mathfrak{r}^{(i)} = [\mathfrak{r}^{(i)},\mathfrak{r}^{(i)}] \ltimes_T \mathbb{R}$ for $i = 1,2$. But from Proposition 5.6 of [|] it follows that if $g = [g,g]\ltimes_\alpha \mathbb{R}$ is the Lie algebra of a Lie group with a left invariant metric of non-positive curvature then the eigenvalues of $\alpha(t)$ have all positive or all negative real parts. Clearly T does not satisfy this condition. We can therefore conclude that $(G/K,G)$ is maximal.

If we denote by L_1 and L_2 the subgroups $L_1 = \mathbb{R}^2 \ltimes_{\alpha_i} SO(2)$, $L_2 = \{0\} \times SL(2,\mathbb{R})$ then L_1,L_2 are closed and contain K. The translates of L_1/K and L_2/K by elements of G form two complementary foliations such that $g(L_1/K)$ and $g(L_2/K)$ are sent into themselves by gKg^{-1} for every $g \in G$. Since K acts non-trivially on L_1/K and L_2/K we must have $F^{(i)} = \{g(L_i/K) | g \in G\}$ for $i = 1,2$. It is now obvious that (a) and (b) are satisfied.

□

The geometries defined in Theorem 5.2.1 are the only ones satisfying the conditions of this section.

Theorem 5.2.2

Let (M,G) be a four dimensional geometry satisfying B1 and B2 and such that for some G invariant metric on M the leaves of $F^{(1)}$ are isometric to H^2. Then (M,G) is equivalent to one of the geometries described in Theorem 5.2.1.

Proof

Since $\{0\} \times SO(2)$ is a maximal compact subgroup of $\mathbb{R}^2 \ltimes_\alpha SL(2,\mathbb{R})$ it is sufficient to prove that G is isomorphic to $\mathbb{R}^2 \ltimes_\alpha SL(2,\mathbb{R})$ for some representation $\alpha: SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$. Let $\mathfrak{r} \ltimes_\alpha \mathfrak{a}$ be the Levi decomposition of $\mathfrak{g}$ with $\mathfrak{r}$ solvable and $\mathfrak{a}$ semi-simple. By Proposition 1.1.3 G is unimodular. It now follows from Proposition 1.1.4 that $\mathfrak{r} \cong \mathbb{R}^2$. Now let $\mathfrak{k}$ denote the Lie algebra corresponding to the subgroup $K_x = \{g \in G \mid g(F_x) = F_x\}$ where F_x is the leaf of $F^{(1)}$ through $x \in M$. From Proposition 5.1.1 we see that $\mathfrak{k} \cong \mathfrak{sl}(2,\mathbb{R})$. It follows immediately that $\mathfrak{a} \cong \mathfrak{sl}(2,\mathbb{R})$ and the universal cover $\tilde{G}$ of G is isomorphic to $\mathbb{R}^2 \ltimes_{\tilde{\alpha}} SL(2,\mathbb{R})$ for some homomorphism $\tilde{\alpha}: \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$. If $q: \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$ denotes the standard covering then either $\tilde{\alpha}$ is trivial or $Ker(\tilde{\alpha}) = Ker(q) \subseteq Z(\widetilde{SL(2,\mathbb{R})})$. If $\tilde{\alpha}$ is trivial then (M,G) must be the geometry described in Theorem 4.4.1 and hence does not satisfy

B2. Thus $\tilde{\alpha}$ is non-trivial and it follows from the condition $\mathrm{Ker}(\tilde{\alpha}) = \mathrm{Ker}(q)$ that there is an automorphism $\phi : SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$ such that $\tilde{\alpha} = \phi \circ q$. It is easy to see that $Z(\tilde{G}) = \{0\} \times \mathrm{Ker}(\tilde{\alpha})$. Since $Z(\widetilde{SL}(2,\mathbb{R})) \cong \mathbb{Z}$ there is an isomorphism $c : \mathbb{Z} \to Z(\widetilde{SL}(2,\mathbb{R}))$ such that $\mathrm{Ker}(\tilde{\alpha}) = c(2\mathbb{Z})$. It follows that if $D \subseteq Z(\tilde{G})$ is such that $G \cong \tilde{G}/D$ then $D = \{0\} \times c(2n\mathbb{Z})$ for some $n \neq 0$. Then $G \cong \mathbb{R} \ltimes_{\tilde{\alpha}_n} SL^{(n)}(2,\mathbb{R})$ where $SL^{(n)}(2,\mathbb{R})$ is an n-fold cover of $SL(2,\mathbb{R})$ with covering map q_n and $\tilde{\alpha}_n = \phi \circ q_n$. Now $Z(G) = \mathrm{Ker}(\tilde{\alpha}_n)$ which is contained in the maximal, connected, compact subgroup $q_n^{-1}(SO(2))$ of $SL^{(n)}(2,\mathbb{R})$. Therefore G does not act faithfully on M unless $n = 1$. Thus $G \cong \mathbb{R} \ltimes_{\phi} SL(2,\mathbb{R})$ as was to be shown.

□

The geometries constructed up until now have all had compact manifold quotients. To show that this is not always the case we have:

Proposition 5.2.3

If $(G/K,G)$ is one of the geometries described in Theorem 5.2.1 and $\Gamma \subseteq G$ is a discrete subgroup with $\Gamma\backslash G/K$ a manifold then $\Gamma\backslash G/K$ is non-compact.

Proof

We have $G = \mathbb{R}^2 \ltimes_{\alpha} SL(2,\mathbb{R})$ and $K = \{0\} \times SO(2)$. Assume that there is a discrete subgroup $\Gamma \subseteq G$ such that $\Gamma\backslash G/K$ is compact. Since K is a compact subgroup this implies that G/Γ is compact. If $q : G \to SL(2,\mathbb{R})$

denotes the quotient map then by Corollary 8.28 of [11] it follows that $\mathbb{R}^2/\mathrm{Ker}(q|\Gamma)$ is compact. Hence $\mathrm{Ker}(q|\Gamma) = \{n_1v_1+n_2v_2 | n_1,n_2\} \in \mathbb{Z}\}$ for some independent vectors $v_1, v_2 \in \mathbb{R}^2$. Since $GL^+(2, \mathbb{R}) \cong \mathbb{R} \times SL(2, \mathbb{R})$ we can extend α to an automorphism $\hat{\alpha}$ of $GL^+(2, \mathbb{R})$. Choose $B \in GL^+(2, \mathbb{R})$ such that $\hat{\alpha}(B)(\mathbb{Z} \times \mathbb{Z}) = \mathrm{Ker}(q|\Gamma)$. The mapping $\phi: G \to G$ defined by setting $\phi(v,T) = (\hat{\alpha}(B)(v), \hat{\alpha}(B)T\hat{\alpha}(B)^{-1})$ is an automorphism of G and maps $\mathrm{Ker}\ (q|\Gamma)$ onto $\mathrm{Ker}\ (q|\phi(\Gamma))$. Hence, without loss of generality, we can assume that $\mathrm{Ker}(q|\Gamma) = \mathbb{Z} \times \mathbb{Z} \subseteq \mathbb{R}^2$. We must now have $q(\Gamma) \subseteq SL(2, \mathbb{Z})$. Since G/Γ is compact there is a compact set $C \subseteq G$ such that for each $g \in G$ there is a $\gamma \in \Gamma$ such that $g\gamma \in C$. If $T \in SL(2, \mathbb{R})$ there is a $g \in G$ such that $q(g) = T$ so if $g\gamma \in C$ we have $T \cdot q(\gamma) \in q(C)$. Hence $SL(2, \mathbb{R})/q(\Gamma)$ is compact contradicting the fact that $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ is non compact. Therefore there are no compact quotients.

□

Remark

Let $G_i = \mathbb{R}^2 \ltimes_{\alpha_i} SL(2, \mathbb{R})$ where α_1, α_2 are the two inequivalent automorphisms described in the proof of Theorem 5.2.1. Then, since $(\alpha_2)^2 = \mathrm{id}$, the mapping $\phi: G_1 \to G_2$ defined by $(v,A) \to (v, \alpha_2(A))$ is an isomorphism sending $\{0\} \times SO(2)$ onto $\{0\} \times SO(2)$. Thus ϕ induces an equivalence between $(G_1/\{0\} \times SO(2), G_1)$ and $(G_2/\{0\} \times SO(2), G_2)$. However, this equivalence is orientation *reversing*.

5.3 Both Invariant Foliations have Flat Leaves

We wish to show that a four dimensional geometry satisfying B1 and B2 is not maximal if the leaves of $F^{(1)}$ and $F^{(2)}$ are isometric to E^2. Let $\mathrm{Isom}_n^+(E^2)$ denote the unique n-fold cover of $\mathrm{Isom}^+(E^2)$. Then $\mathrm{Isom}_n^+(E^2) \cong \mathbb{C} \rtimes_{\alpha_n} S^1$ where $\alpha_n(z)\cdot w = z^n w$ for $z,w \in \mathbb{C}, |z| = 1$.

Lemma 5.3.1

Let $\phi:\mathrm{Isom}_n^+(E^2) \to GL(2, \mathbb{R})$ be a homomorphism which is non-trivial on $\{0\} \times S^1$. Then $\mathbb{C} \times \{1\} \subseteq \mathrm{Ker}(\phi)$ and, up to conjugacy in $GL(2, \mathbb{R})$, $\phi(w,e^{i\theta}) = A_{k\theta}$ for $k \in \mathbb{Z}$, $k \neq 0$, where $A_{k\theta}$ is the matrix:

$$A_{k\theta} = \begin{pmatrix} \mathrm{Cos}\, k\theta & \mathrm{Sin}\, k\theta \\ -\mathrm{Sin}\, k\theta & \mathrm{Cos}\, k\theta \end{pmatrix}.$$

Proof

Let $\det:GL(2, \mathbb{R}) \to \mathbb{R}_*$ denote the determinant map. Since $\{0\} \times S^1$ is compact and connected we must have $(\det\circ\phi)|(\{0\} \times S^1)$ trivial. The only normal subgroup of $\mathrm{Isom}_n^+(E^2)$ containing $\{0\} \times S^1$ is $\mathrm{Isom}_n^+(E^2)$. Therefore the image of ϕ is contained in $SL(2, \mathbb{R})$. Now ϕ cannot be injective, since $SL(2, \mathbb{R})$ has no two dimensional abelian subgroups. Hence $\mathrm{Ker}(\phi) \cap (\mathbb{C} \times \{1\})$ must have dimension greater than 0. But it is easy to see that any normal subgroup of $\mathrm{Isom}_n^+(E^2)$ containing a one dimensional subgroup of $\mathbb{C} \times \{1\}$ must contain all of $\mathbb{C} \times \{1\}$. Therefore $\mathbb{C} \times \{1\} \subseteq \mathrm{Ker}(\phi)$. By hypothesis $\phi(\{0\} \times S^1)$ is non-trivial and hence conjugate to $SO(2)$. The second statement follows immediately.

□

Lemma 5.3.2

If (M,G) is a four dimensional geometry satisfying B1 and B2 and such that for any G invariant metric on M the leaves of $F^{(1)}$ and $F^{(2)}$ are isometric to E^2 then $(M/F^{(i)},\hat{\zeta}^{(i)}(G))$ is a geometry equivalent to $(E^2,\mathrm{Isom}^+(E^2))$.

Proof

We consider only $F^{(1)}$ as the proof for $F^{(2)}$ is the same. Let $F_x^{(1)}$, $F_x^{(2)}$ denote the leaves of $F^{(1)}$ and $F^{(2)}$ through $x \in M$. If $y_1,y_2 \in F_{x_0}^{(1)}$ then we denote by $\sigma(y_1,y_2)$ the holonomy map $\sigma(y_1,y_2):N_{y_1} \to N_{y_2}$ where N_{y_i} is a neighbourhood of y_i in $F_{y_i}^{(2)}$ for $i = 1,2$. To demonstrate the fact that $(M/F^{(1)},\hat{\zeta}^{(1)}(G))$ is a geometry it suffices, by Proposition 2.1.3, to show that $d\sigma(y_1,y_2):P_{y_1}^{(2)} \to P_{y_2}^{(2)}$ is an isometry for any pair of points (y_1,y_2) of M with $\zeta^{(1)}(y_1) = \zeta^{(1)}(y_2)$. Choose $x_0 \in M$ and let $x_1 \in F_{x_0}^{(1)}$. As usual we set $K_x^{(i)} = \{g \in G | g(F_x) = F_x\}$. If $y_0 = \zeta^{(1)}(x_0)$ then $\hat{\zeta}^{(1)}(K_{x_0}^{(1)})$ is the stabilizer of y_0 in $\hat{\zeta}^{(1)}(G)$. From Proposition 5.1.1 $K_{x_0}^{(1)}$ is isomorphic to $\mathrm{Isom}_n^+(E^2)$ for some $n \neq 0$. The subgroup $L_{x_0}^{(1)} \leq K_{x_0}^{(1)}$ corresponding to $\mathbb{C} \times \{1\}$ in $\mathrm{Isom}_n^+(E^2)$ is transitive on $F_{x_0}^{(1)}$. Let $\rho:\hat{\zeta}^{(1)}(K_{x_0}^{(1)}) \to GL(2,\mathbb{R})$ denote the representation coming from the derivative. Then by Lemma 5.3.1 we have $L_{x_0}^{(1)} \subseteq \mathrm{Ker}(\rho\circ\hat{\zeta}^{(1)})$. Let

$g \in L^{(1)}_{x_0}$ be such that $g(x_0) = x_1$. By the definition of $\hat{\zeta}$ $(\zeta^{(1)}|N_{x_0})^{-1} \circ (\hat{\zeta}^{(1)}(g)|\zeta^{(1)}(N_{x_0})) = (\zeta^{(1)}|N_{x_0}) = (\sigma(x_1,x_0)\circ g)|N_{x_0}$ on the appropriate domains of definition. Now $(\rho\circ\hat{\zeta}^{(1)}(g)$ is trivial so $d\zeta^{(1)}(g)_{y_0}$ is trivial. Hence $d(\sigma\circ g)$ is the identity on $P^{(2)}_{x_0}$. It follows that $d\sigma_{x_1}(x_1,x_2)$ is an isometry since $dg_{x_0}:P^{(2)}_{x_0} \to P^{(2)}_{x_1}$ is an isometry. The points x_0,x_1 being an arbitrary pair on the same leaf of $F^{(1)}$ we conclude that $F^{(1)}$ has isometric infinitesmal holonomy as required.

To show that $(M/F^{(1)},\hat{\zeta}^{(1)}(G))$ is equivalent to $(E^2,\mathrm{Isom}^+(E^2))$ we choose a $\hat{\zeta}^{(1)}(G)$ invariant metric λ on $M/F^{(1)}$ such that $d\hat{\zeta}^{(1)}_x : P^{(2)}_x \to T^{(1)}_{\zeta(x)}M/F^{(1)}$ is an isometry for some G invariant metric on M (Proposition 2.1.3). Then if $U \subseteq F^{(2)}_x$ is such that $\zeta^{(1)} U$ is a diffeomorphism then $\zeta^{(1)}:U \to \zeta^{(1)}(U) \subseteq M/F^{(1)}$ is an isometry. But the leaves of $F^{(2)}$ are isometric to E^2 and hence $(M/F^{(1)},\lambda)$ is flat. The conclusion follows since $\hat{\zeta}^{(1)}\ G_x$ is non-trivial by Proposition 5.1.1.

□

We show that there are no *maximal* geometries fulfilling the requirements of this section.

Theorem 5.3.3

If (M,G) is a four dimensional geometry satisfying the conditions B1 and B2 and both foliations $F^{(1)}$, $F^{(2)}$ have leaves isometric to

E^2 for any left invariant metric on M then there is a subgroup $G' \quad \mathrm{Isom}^+(E^4)$ isomorphic to G such that (M,G) is equivalent to (E^4,G').

Proof

We first show that if $F_x^{(1)}$ and $F_x^{(2)}$ denote the leaves of $F^{(1)}$ and $F^{(2)}$ through $x \in M$ then $F_x^{(1)} \cap F_x^{(2)} = \{x\}$. Let $x_0 \in M$ and assume that there is a point $x_1 \neq x_0$ in $F_{x_0}^{(1)} \cap F_{x_0}^{(2)}$. Since $F^{(1)}$ and $F^{(2)}$ are transverse the set $F_{x_0}^{(1)} \cap F_{x_0}^{(2)}$ is countable. If $g \in G_{x_0}$ then $g(F_{x_0}^{(1)} \cap F_{x_0}^{(2)}) = F_{x_0}^{(1)} \cap F_{x_0}^{(2)}$. Therefore, since G_{x_0} is connected, we have $g(x_1) = x_1$ for each $g \in G_{x_0}$. Since $F_{x_0}^{(1)}$ is isometric to E^2 in the induced metric there is a unique geodesic γ in $F_{x_0}^{(1)}$ joining x_0 and x_1. Now if $g \in G_{x_0}$ then $g|F_{x_0}^{(1)}$ is an isometry and therefore fixes γ pointwise. Hence $dg_{x_0}(\gamma'(0) = \gamma'(0)$ for each $g \in G_{x_0}$ contradicting B2. Therefore $F_{x_0}^{(1)} \cap F_{x_0}^{(2)} = \{x_0\}$.

By Lemma 5.3.2 there is an exact sequence $\mathrm{Ker}(\hat{\zeta}) \rightarrowtail G \xrightarrow{\hat{\zeta}} \mathrm{Isom}^+(E^2)$. Thus $\dim(\mathrm{Ker}(\hat{\zeta})) = 2$. Furthermore $\mathrm{Ker}_0(\hat{\zeta})$, the connected component of e in $\mathrm{Ker}(\hat{\zeta})$, is a closed subgroup of G and so, by Proposition 1.1.4, $\mathrm{Ker}_0(\hat{\zeta}) \cong \mathbb{R}^2$. It is easy to see that $\mathrm{Ker}(\hat{\zeta}) = \mathrm{Ker}_0(\hat{\zeta})\cdot(\mathrm{Ker}(\hat{\zeta}) \cap G_x)$ for any $x \in M$. It now follows by Proposition 5.1.1 that the quotient map $q: G/\mathrm{Ker}_0(\hat{\zeta}) \to G/\mathrm{Ker}(\hat{\zeta}) \cong \mathrm{Isom}^+(E^2)$ is a finite covering. We deduce that for some $n \in \mathbb{Z}$ there is an exact sequence $\mathbb{R}^2 = \mathrm{Ker}_0(\hat{\zeta}) \rightarrowtail G \xrightarrow{\hat{\zeta}} \twoheadrightarrow \mathrm{Isom}_n^+(E^2)$. Let $g \in K_{x_0}^{(2)}$ and assume that $\hat{\zeta}_n^{(1)}(g) = e$. Then $g \in K_{x_0}^{(2)} \cap \mathrm{Ker}_0(\hat{\zeta})^{(1)}$ and so $g(F_{x_0}^{(1)} \cap F_{x_0}^{(2)}) = F_{x_0}^{(1)} \cap F_{x_0}^{(2)}$. But $F_{x_0}^{(1)} \cap F_{x_0}^{(2)} = \{x_0\}$ as shown above. Therefore $g \in G_{x_0} \cap \mathrm{Ker}_0(\hat{\zeta})$ and g has finite order. Since $\mathrm{Ker}_0(\hat{\zeta}) \cong \mathbb{R}^2$ we deduce that $g = e$. $\hat{\zeta}_n^{(1)}|K_{x_0}^{(2)}$ is thus injective. Both $K_{x_0}^{(2)}$ and $\mathrm{Isom}_n^+(E^2)$ are connected and of dimension 3 and it follows that $\hat{\zeta}_n^{(1)} : K_{x_0}^{(2)} \to \mathrm{Isom}_n^+(E^2)$ is an isomorphism. The sequence $\mathbb{R}^2 \rightarrowtail G \twoheadrightarrow \mathrm{Isom}_n^+(E^2)$ therefore splits and a short calculation using Lemma 5.3.1 shows that G is isomorphic to $\mathbb{R}^4 \rtimes_\alpha SO(2)$ where $SO(2)$ acts by

$$\alpha(\theta) = \begin{pmatrix} A_{n\theta} & 0 \\ 0 & A_{m\theta} \end{pmatrix} \quad \text{with } n,m \in \mathbb{Z}, \ \theta \in [0,2\pi) .$$

Hence $G \subseteq \mathrm{Isom}^+(E^4)$ and the result follows.

$\square$

CHAPTER 6 : SOLVABLE GROUPS.

6.0 Introduction

In this chapter we consider the last class of geometries; those satisfying

C : The stabilizer of each point is trivial.

It is easy to see that such a geometry is equivalent to a connected, simply connected Lie group acting on itself by left translations. Such a geometry will be denoted (G,G) . In Section 6.1 we show that if (G,G) is maximal then it is solvable. In the case $\dim(G) = 4$ we show further that $G \cong H \rtimes \mathbb{R}$ where H is one of the two nilpotent groups of dimension 3 and $\mathbb{R}$ acts on H by volume preserving automorphisms. The groups of this type with $H \cong \mathbb{R}^3$ are determined in 6.2 and those with H non-abelian are determined in 6.3. In 6.4 we investigate which of those groups constructed in 6.2 and 6.3 have a quotient of finite volume (such a quotient is necessarily compact - see [\\ ] Chapter III). Finally in 6.5 we determine all the maximal geometries with trivial stabilizer.

Throughout this chapter N will denote the non-abelian nil-potent group of dimension 3 presented as:

$$N = \left\{ \begin{pmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \right\} .$$

A basis for the Lie algebra n of N is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

So n is presented as:

$$n = \{X_1, X_2, X_3 | [X_1, X_2] = [X_1, X_3] = 0 \ , \ [X_2, X_3] = X_1\} \quad .$$

6.1 Preliminaries

We first state the obvious equivalence between geometries with trivial stabilizer and Lie groups.

Proposition 6.1.1

If (M,G) is a geometry with $G_x = \{e\}$ then (M,G) is equivalent to G acting on itself by left translations.

Proof

(M,G) is equivalent to standard left action of G on G/G_x . The result follows since $G/G_x = G$.

□

From now on we identify a geometry (M,G) having trivial stabilizer with the group G and abuse notation by speaking of the geometry (G) .

Lemma 6.1.2

If (G) is a maximal geometry then $\mathrm{Aut}(G)$ contains no non-trivial connected compact subgroup.

Proof

Let $K \leq \mathrm{Aut}(G)$ be a connected, compact subgroup. The semi-direct product $G \ltimes K$ acts transitively on G by $(g,k)\cdot x = L_g(k(x))$ where L_g denotes left translation by $g \in G$. This action clearly extends the left action of G on itself and the stabilizer of e is K which is compact. By the assumption that G is a geometry there is a discrete subgroup $\Gamma \leq G$ such that $\Gamma\backslash G$ has finite volume. It follows that $\Gamma\backslash G \ltimes K/K$ is a manifold of finite volume. Hence $(G, G \ltimes K)$ is a geometry. Since (G) is maximal we conclude that $K = \{e\}$.

□

Proposition 6.1.3

If (G) is a maximal geometry then G is a connected, simply connected, unimodular solvable Lie group and $\mathrm{Aut}(G)$ is solvable and simply connected.

Proof

By the definition of a geometry G is connected and simply connected and by Proposition 1.1.3 G is unimodular. Since G is simply connected we can assume that $G = (R \ltimes_\alpha S_1) \times S_2$ where:

(i) R, S_1, S_2 are connected and simply connected.

(ii) R is isomorphic to the solvable radical of G and S_1, S_2 are semi-simple.

(iii) $\mathrm{Ker}(\alpha)$ has finite index in $Z(S_1)$ since a semi-simple matrix group has finite centre (see [5] Chapter 18, Prop. 4.1).

If $S_2 \neq \{e\}$ then there is a non-trivial connected maximal compactly embedded subgroup $K_2 \subseteq S_2$ i.e. $\mathrm{Ad}_{S_2}(K_2)$ is a compact subgroup of $\mathrm{Aut}(S_2)$. Since K_2 commutes with $(R \rtimes_\alpha S_1) \times \{e\}$ it is clear that $\mathrm{Ad}_G(K_2)$ is a compact subgroup of $\mathrm{Aut}(G)$ which contradicts Lemma 6.1.2. Therefore $S_2 = \{e\}$. If $S_1 \neq \{e\}$ then, again, we have a non-trivial connected maximal compactly embedded subgroup $K_1 \subseteq S_1$. It is easy to see that $\mathrm{Ker}(\mathrm{Ad}_G(K_1)) = \mathrm{Ker}(\alpha) \cap K_1$. But the fact that $\mathrm{Ker}(\alpha)$ has finite index in $Z(S_1)$ implies that $\mathrm{Ker}(\alpha) \cap K_1$ has finite index in $Z(S_1) \cap K_1$. It follows that $Z(S_1) \cap K_1/\mathrm{Ker}(\alpha) \cap K_1 \to K_1/\mathrm{Ker}(\alpha) \cap K_1 \to K_1/Z(S_1) \cap K_1 = \mathrm{Ad}_{S_1}(K_1)$ is a finite covering. Hence, since $\mathrm{Ad}_{S_1}(K_1)$ is compact, $\mathrm{Ad}_G(K_1)$ is a compact subgroup of $\mathrm{Aut}(G)$ contradicting Lemma 6.1.2. Therefore $S_1 = \{e\}$ and $G \cong R$ which is solvable.

Since the mapping $\phi:\mathrm{Aut}(G) \to \mathrm{Aut}(g)$ is injective we can regard $\mathrm{Aut}(G)$ as a subgroup of $GL(n, \mathbb{R})$ for some n . It follows that if $S \subseteq \mathrm{Aut}(G)$ is a maximal, non-trivial connected semi-simple analytic subgroup then S has finite centre ([5] Chapter 18 Proposition 4.1). Then any maximal compactly embedded subgroup of S is compact in S and hence a compact subgroup of $\mathrm{Aut}(G)$. This contradicts 6.1.2 so we must have $S = \{e\}$. Hence $\mathrm{Aut}(G)$ is solvable. Finally $\mathrm{Aut}(G)$ must be simply connected since a non-simply connected solvable group contains a non-trivial maximal torus again contradicting 6.1.2.

□

We now prove a number of propositions describing the structure of a maximal four dimensional geometry with trivial stabilizer.

<u>Proposition 6.1.4</u>

If (G) is a maximal four dimensional geometry then $G \cong H \ltimes_\alpha \mathbb{R}$ where H is either $\mathbb{R}^3$ or N and $\alpha(\mathbb{R})$ consists of volume preserving automorphisms.

<u>Proof</u>

By Proposition 1.1.3 G is unimodular. If G is of the form described it follows from Proposition 1.1.4, since $N, \mathbb{R}^3$ and $\mathbb{R}$ are unimodular, that $\mathbb{R}$ acts by volume preserving automorphisms on H.

If G is nilpotent then the Lie algebra $\mathfrak{g}$ of G has a nilpotent ideal $\mathfrak{h}$ with $\dim(\mathfrak{h}) = 3$. Then $\mathfrak{h}$ is isomorphic to $\mathbb{R}^3$ or $\mathfrak{n}$ and $\mathfrak{g} \cong \mathbb{R}^3 \ltimes \mathbb{R}$ or $\mathfrak{n} \ltimes \mathbb{R}$. Since G is simply connected this implies that $G \cong \mathbb{R}^3 \ltimes \mathbb{R}$ or $G \cong N \ltimes \mathbb{R}$.

If G is not nilpotent it is solvable by Proposition 6.1.3. Let h denote the nilradical of g. Since G is simply connected the connected subgroup H corresponding to h is closed, simply connected and normal in G. The result will follow if we can show that h is isomorphic to $\mathbb{R}^3$ or n. Now $[g,g] \subseteq h$ so we have an extension $H \rightarrowtail G \twoheadrightarrow \mathbb{R}^n$ for $1 \leq n \leq 3$ and it follows from Proposition 1.1.4 that $Ad_G|H$ is a group of volume preserving transformations. To demonstrate the proposition we need to show that h is not isomorphic to $\mathbb{R}$ or $\mathbb{R}^2$. If $h \cong \mathbb{R}$ then $g/h \cong \mathbb{R}^3$ and $H \cong \mathbb{R}$. The action of $\mathbb{R}^3$ on $\mathbb{R}$ is trivial since $\mathbb{R}$ has no non-trivial connected group of volume preserving transformations. Hence h is central in g and so g is nilpotent, a contradiction. If $h \cong \mathbb{R}^2$ then $g/h \cong \mathbb{R}^2$ and we have a homomorphism $\alpha: \mathbb{R}^2 \to SL(2,\mathbb{R})$. Since $s\ell(2,\mathbb{R})$ has no two dimensional abelian subalgebras the induced homomorphism of Lie algebras $\alpha_*: \mathbb{R}^2 \to s\ell(2,\mathbb{R})$ has a non-trivial kernel. Let Y be a non-zero vector in $Ker(\alpha_*)$ and choose X in g which projects to Y under the quotient map $g \to g/h \cong \mathbb{R}^2$. It is easy to see that h and X together span an abelian ideal in g. This contradicts the assumption that h is the nilradical of g. $\square$

Let h be any Lie algebra. Then, since the Lie algebra of $Aut(h)$ is isomorphic to the derivation algebra $Der(h)$, $Aut(h)$ acts on $Der(h)$ via the adjoint action of $Aut(h)$ on its Lie algebra. This action preserves the ideal of inner derivations $Inn(h)$. We denote the action of $\phi \in \mathrm{Aut}(h)$ on $\mathrm{Der}(h)/\mathrm{Inn}(h)$ by $\mathrm{Ad}(\phi)\cdot[D]$ where $D \in \mathrm{Der}(h)$.

Proposition 6.1.5

Let h be an arbitrary Lie algebra, $\{Y\}$ a basis for $\mathbb{R}$ as Lie algebra and $\alpha_1,\alpha_2 : \mathbb{R} \to \mathrm{Der}(h)$ homomorphisms defined by $\alpha_i(Y) = D_i$ for $i = 1,2$. If there is $\phi \in \mathrm{Aut}(h)$ such that $\mathrm{Ad}(\phi)[D_1] = k[D_2]$ for $k \in \mathbb{R}$ then $h \ltimes_{\alpha_1} \mathbb{R}$ and $h \ltimes_{\alpha_2} \mathbb{R}$ are isomorphic

Proof

Let g_i denote $h \ltimes_{\alpha_i} \mathbb{R}$ for $i = 1,2$. By hypothesis $\mathrm{Ad}(\phi)(D_1) = k(D_2 + T)$ for some $T \in \mathrm{Inn}(h)$. Define a homomorphism $\beta: \mathbb{R} \to \mathrm{Der}(h)$ by $\beta(Y) = D_2 + T$ and set $g = h \ltimes_\beta \mathbb{R}$. Then the mapping $\Phi: g_1 \to g$ defined by $\Phi(h,t) = (\phi(h),kt)$ is easily seen to be an isomorphism. To show that $g \cong g_2$ it is sufficient to show that there is a vector $V \in g$ such that $V \notin h$ and $[Z,V] = D_2(Z)$ for all $Z \in h$. Now there is an $X \in h$ such that $[Z,X] = T(Z)$ for each $Z \in h$. Also $[Z,Y] = (D_2 + T)(Z)$ in g . Hence the vector $V = Y-X$ has the required properties. Thus $g_1 \cong g \cong g_2$ as was to be shown.

□

Corollary 6.1.6

The Lie algebras of the groups in Proposition 6.1.4 are given by semi-direct products $h \ltimes_D \mathbb{R}$ where $h = \mathbb{R}^3$ or n and D is a trace 0 outer derivation. Two such algebras $h \propto_{D_1} \mathbb{R}$ and $h \propto_{D_2} \mathbb{R}$ are isomorphic iff $\phi D_1 \phi^{-1} = kD_2 + T$ for some $\phi \in \mathrm{Aut}(h)$, $k \in \mathbb{R}$, $T \in \mathrm{Inn}(h)$.

Proof

Combine Propositions 6.1.4 and 6.1.5.

□

In the next two sections we determine, up to isomorphism, all simply connected Lie groups satisfying 6.1.6.

6.2 Extensions $G = \mathbb{R}^3 \times \mathbb{R}$

The Lie algebra of trace 0 outer derivations of $\mathbb{R}^3$ is $\mathfrak{sl}(3, \mathbb{R})$. The orbits of the adjoint action of $GL(3, \mathbb{R})$ or $\mathfrak{sl}(3, \mathbb{R})$ correspond to Jordan canonical forms of matrices in $\mathfrak{sl}(3, \mathbb{R})$. Up to a scalar factor they are:

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

$$D_4 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad D_5(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}, \quad D_6(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -(1+\lambda) \end{pmatrix}$$

The corresponding Lie algebras and groups are numbered $\mathfrak{g}_1, \dots, \mathfrak{g}_6(\lambda), G_1, \dots, G_6(\lambda)$.

6.2.1 G_1

Then $g_1 \cong \mathbb{R}^4$, $G_1 \cong \mathbb{R}^4$.

6.2.2 G_2

$G_2 \cong N \times \mathbb{R}$.

6.2.3 G_3

$$g_3 = \{X_1, X_2, X_3, Y \mid [X_i, X_j] = 0 \text{ for } 1 \le i,j \le 3 \text{ , } [Y,X_1] = 0 \text{ , } [Y,X_2] = X_1 \text{ , } [Y,X_3] = X_2\}$$

$$\mathrm{Exp}(tD_3) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

and G_3 is $\mathbb{R}^4$ with multiplication:

$$(x,y,z,t)\cdot(u,v,w,s) = (x+u+tv+\frac{t^2}{2}w, y+v+tw, z+w, t+s) \text{ .}$$

6.2.4 G_4

$$g_4 = \{X_1, X_2, X_3, Y \mid [X_i, X_j] = 0 \text{ for } 1 \le i,j \le 3 \text{ , } [Y,X_1] = X_1 \text{ , } [Y,X_2] = X_1 + X_2, [Y,X_3] = -2X_3\} \text{ .}$$

$$\mathrm{Exp}(t\,D_4) = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} .$$

So G_4 is $\mathbb{R}^4$ with multiplication

$$(x,y,z,t)\cdot(u,v,w,s) = (x+e^t u+te^t v,\ y+e^t v,\ z+e^{-2t},\ t+s) .$$

6.2.5 $G_5(\lambda)$

$$\mathfrak{g}_5(\lambda) = \{X_1,X_2,X_3,Y \mid [X_i,X_j] = 0 \text{ for } 1 \le i,j \le 3,\ [Y,X_1] = \lambda X_1 - X_2$$
$$[Y,X_2] = X_1+\lambda X_2, [Y,X_3] = -2X_3\}$$

$$\mathrm{Exp}(t\,D_5(\lambda)) = \begin{pmatrix} e^{\lambda t}\,\mathrm{Cos}t & e^{\lambda t}\mathrm{Cos}t & 0 \\ -e^{\lambda t}\,\mathrm{Sin}t & e^{\lambda t}\,\mathrm{Sin}t & 0 \\ 0 & 0 & e^{-2\lambda t} \end{pmatrix}$$

So $G_5(\lambda)$ is $\mathbb{R}^4$ with multiplication

$$(x,y,z,t)\cdot(u,v,w,s) = (x+ue^{\ t}\mathrm{Cos}t + ve^{\ t}\mathrm{Sin}t\ ,$$
$$y-ue^{\lambda t}\mathrm{Sin}t + ve^{\lambda t}\mathrm{Cos}t,\ z+e^{-2\lambda t}w,\ t+s) .$$

It is easy to check that $\mathfrak{g}_5(\lambda) \cong \mathfrak{g}_5(\mu)$ iff $\lambda = \pm\mu$ so we can assume that $\lambda \ge 0$. If $\lambda = 0$ then G (0) is isomorphic to $\widetilde{\mathrm{Isom}}^+(E^2) \times \mathbb{R}$ and hence is not maximal as $\mathrm{Aut}(G_5(0))$ has a compact group of inner automorphisms.

6.2.6 $G_6(\lambda)$

$$\mathfrak{g}_6(\lambda) = \{X_1, X_2, X_3, Y \mid [X_i, X_j] = 0 \text{ for } 1 \le i,j \le 3,\ [Y, X_1] = X_1,\ [Y, X_2] = \lambda X_2,\ [Y, X_3] = -(1+\lambda)X_3\}$$

$$\operatorname{Exp}(tD_6(\lambda)) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{-(1+\lambda)t} \end{pmatrix}$$

So $G_6(\lambda)$ is $\mathbb{R}^4$ with multiplication

$$(x,y,z,t)\cdot(u,v,w,s) = (x+e^t u, y+e^{\lambda t}v, z+e^{-(1+\lambda)t}w, t+s) \quad .$$

The isomorphism classes of the $G_6(\lambda)$ are as follows:

(a) $G_6(\lambda) \cong G_6(0)$ iff $\lambda = 0, -1$.

(b) If $\mu \neq 0$ then $G_6(\lambda) \cong G_6(\mu)$ iff

$$\lambda = \mu,\ 1/\mu,\ -(1+\mu),\ -\frac{1}{1+\mu},\ -\frac{\mu}{1+\mu},\ -\frac{(1+\mu)}{\mu} \quad .$$

There is a better way to express the isomorphism classes of the $G_6(\lambda)$. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are such that $\lambda_1 \ge \lambda_2 \ge \lambda_3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we define $M(\lambda_1, \lambda_2, \lambda_3)$ to be:

$$M(\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad .$$

It is clear that any diagonalizable matrix in $SL(3,\mathbb{R})$ is conjugate to a unique one of these matrices. Hence $g_6(\lambda) \cong \mathbb{R}^3 \ltimes_{M(\lambda)} \mathbb{R}$ for some $\lambda = (\lambda_1,\lambda_2,\lambda_3)$. Furthermore $\mathbb{R}^3 \ltimes_{M(\lambda)} \mathbb{R} \cong \mathbb{R}^3 \ltimes_{M(\mu)} \mathbb{R}$ iff $\mu = \alpha\lambda$ for $\alpha \neq 0$ in $\mathbb{R}$.

6.3 Extensions $G = N \times \mathbb{R}$

We recall the presentations of N and its Lie algebra n :

$$N = \{ \begin{pmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in GL(3,\mathbb{R}) \}$$

$$n = \{X_1,X_2,X_3 \mid [X_1,X_2] = [X_1,X_3] = 0 \ , \ [X_2,X_3] = X_1\}$$

where X_1,X_2,X_3 are the matrices:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

With respect to this basis $\mathrm{Aut}(n)$ is given by

$$\mathrm{Aut}(n) = \{ \begin{pmatrix} a & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{pmatrix} \in GL(3,\mathbb{R}) \mid a = b_2c_3-b_3c_2\} .$$

The corresponding automorphisms of N are given by:

$$\begin{pmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & b_2y+c_2z & ax+b_1y+c_1z+\frac{1}{2}(b_2b_3y^2+2b_3c_2yz+c_2c_3z^2) \\ 0 & 1 & b_3y+c_3z \\ 0 & 0 & 1 \end{pmatrix}$$

The algebra $\mathfrak{sout}(n)$ of trace 0 outer automorphisms of n is the subalgebra of $\mathrm{Der}(n)$ given by:

$$\mathfrak{sout}(n) = \{\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \in \mathrm{Der}(n) \mid M \in \mathfrak{sl}(2,\mathbb{R}) \} .$$

The adjoint action of $\mathrm{Aut}(n)$ on $\mathfrak{sout}(n)$ is:

$$\mathrm{Ad}\begin{pmatrix} a & v \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A^{-1}MA \end{pmatrix}$$

where $v \in \mathbb{R}^2$, $A \in GL(2,\mathbb{R})$, $M \in \mathfrak{sl}(2,\mathbb{R})$, $a = \det(A)$. Hence, up to scaling, the orbits of the adjoint action are represented by the Derivations:

$$D_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad D_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding algebras and groups will be denoted $\mathfrak{g}_7 \ldots \mathfrak{g}_{10}$, $G_7 \ldots G_{10}$.

6.3.1 G_7

$G_7 \cong N \times \mathbb{R}$ which has already been dealt with.

6.3.2 G_8

$$\mathcal{g}_8 = \{X_1,X_2,X_3,Y \mid [X_1,X_2] = [X_1,X_3] = 0 \ , \ [X_2,X_3] = X_1 \ ,$$
$$[Y,X_1] = 0 \ , \ [Y,X_2] = X_2, \ [Y,X_3] = -X_3\} \ .$$

The 1-parameter subgroup of Aut(N) corresponding to D_8 is:

$$\alpha_8(t)\cdot\begin{pmatrix}1 & y & x\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & e^t y & x\\ 0 & 1 & e^{-t}z\\ 0 & 0 & 1\end{pmatrix} \ .$$

So G_8 is $\mathbb{R}^4$ with multiplication

$$(x,y,z,t)\cdot(u,v,w,s) = (x+u+ye^{-t}w,y+e^t v,z+e^{-t}w,t+s) \ .$$

6.3.3 G_9

$$\mathcal{g}_9 = \{X_1,X_2,X_3,Y \mid [X_1,X_2] = [X_1,X_3] = 0 \ , \ [X_2,X_3] = X_1 \ ,$$
$$[Y,X_1] = 0 \ , \ [Y,X_2] = -X_3, \ [Y,X_3] = X_2 \} \ .$$

The 1-parameter subgroup of Aut(N) corresponding to D_9 is:

$$\alpha_9(t)\cdot\begin{pmatrix}1 & y & x\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & y\ \mathrm{Cos}\,t + z\ \mathrm{Sin}\,t & x-yz\ \mathrm{Sin}^2 t+\frac{1}{4}\mathrm{Sin}\,2t(z^2-y^2)\\ 0 & 1 & -y\ \mathrm{Sin}\,t + z\ \mathrm{Cos}\ t\\ 0 & 0 & 1\end{pmatrix} \ .$$

Hence G_9 is $\mathbb{R}^4$ with multiplication:

$$(x,y,z,t)\cdot(u,v,w,s) = (x+u-y(x\mathrm{Sin}t + w\mathrm{Cos}t) + vw\mathrm{Sin}^2t + \tfrac{1}{4}\mathrm{Sin}2t(w^2-v^2), y+v\mathrm{Cos}t + w\mathrm{Sin}t, z-v\mathrm{Sin}t + w\mathrm{Cos}t, t+s) .$$

6.3.4 G_{10}

$$\mathfrak{g}_{10} = \{X_1,X_2,X_3,Y \mid [X_1,X_2] = [X_1,X_3] = 0 , [X_2,X_3] = X_1 , [Y,X_1] = [Y,X_2] = 0 , [Y,X_3] = X_2\} .$$

Now $\{X_1,X_2,Y\}$ spans an abelian ideal in $\mathfrak{g}_{10}$ and so $\mathfrak{g}_{10}$ is isomorphic to one of the Lie algebras $\mathfrak{g}_1,\ldots,\mathfrak{g}_5(\lambda),\mathfrak{g}_6(\lambda)$ of Section 6.2.

6.4 Quotients of finite volume

We wish to determine which of the solvable groups $G_1,\ldots,G_9$ constructed in 6.2 and 6.3 possess a discrete subgroup Γ such that the quotient G_i/Γ has finite volume. We will use the following known results on discrete subgroups of solvable groups. For proofs see [11] Chapters II and III.

Proposition 6.4.1 (Mostow)

Let G be a solvable group and $\Gamma \leq G$ a discrete subgroup such that G/Γ has finite volume. Then G/Γ is compact.

Proposition 6.4.2 (Mostow)

Let G be a simply connected, solvable Lie group and $H \leq G$ its maximal, connected normal nilpotent subgroup (necessarily closed). Then a closed $\Gamma \leq G$ has G/Γ compact iff the following conditions both hold:

(a) $H/H \cap \Gamma$ is compact.

(b) If $p:G \to G/H \cong \mathbb{R}^n$ is the quotient map then $p(\Gamma)$ is a closed subgroup of $\mathbb{R}^n$ with $\mathbb{R}^n/p(\Gamma)$ compact.

Proof

The sufficiency of the conditions is obvious since if (a) and (b) hold then G/Γ is a fibre bundle over a compact base with a compact fibre. The necessity of (a) is demonstrated in Theorem 3.3 of [11] and the necessity of (b) then follows from Theorem 1.13 of [11]. □

Remark

For G nilpotent 6.4.2 is trivial. However it is known that a connected, simply connected nilpotent Lie group G has a subgroup Γ with G/Γ compact iff $\mathfrak{g} \cong \mathfrak{n} \otimes_{\mathbb{Q}} \mathbb{R}$ where $\mathfrak{n}$ is a nilpotent Lie algebra defined over $\mathbb{Q}$ (see [11] Theorem 2.12).

For our purposes we need the following corollary.

Corollary 6.4.3

If $G = \mathbb{R}^n \rtimes_\alpha \mathbb{R}$ is unimodular and not nilpotent then there is a discrete subgroup $\Gamma \leq G$ with G/Γ compact iff there is a non-zero $\lambda \in \mathbb{R}$ such that $\alpha(\lambda)$ has a characteristic polynomial with integral coefficients.

Proof

We recall that $\alpha(t) \in SL(n,\mathbb{R})$ since G is unimodular. If $\lambda \neq 0$ exists with the stated property then there is a basis $v_1,\ldots,v_n$ for $\mathbb{R}^n$ with respect to which $\alpha(\lambda)$ is represented by a matrix in $SL(n,\mathbb{Z})$. The set $\Gamma = \{(r_1v_1 +\ldots+ r_nv_n, s\lambda) \mid r_i, s \in \mathbb{Z}\}$ is a discrete subgroup and 6.4.2 implies that G/Γ is compact.

Conversely assume that $\Gamma \leq G$ is discrete with G/Γ compact. Let $p:G \to \mathbb{R}$ be the obvious quotient map. Then from 6.4.2 we see that for some basis $v_1,\ldots,v_n$ of $\mathbb{R}^n$ we have $\Gamma \cap \mathbb{R}^n \times \{0\} = \{\sum_{i=1}^{n} r_iv_i \mid r_i \in \mathbb{Z}\}$ and $p(\Gamma) = \{k\lambda_0 \mid k \in \mathbb{Z}\}$ for $\lambda_0 \neq 0$ in $\mathbb{R}$. Let $\gamma \in \Gamma$ be such that $p(\gamma) = \lambda_0$. Then, since $\mathbb{R}^n$ is abelian, we have $\gamma x \gamma^{-1} = \alpha(\lambda_0)\cdot x$ for each $x \in \mathbb{R}^n \times \{0\}$. Hence $\alpha(\lambda_0)$ preserves $\Gamma \cap \mathbb{R}^n \times \{0\}$ and is therefore conjugate to an integral matrix in $SL(n,\mathbb{Z})$. It follows that the characteristic polynomial of $\alpha(\lambda_0)$ has integral coefficients.

$\square$

We will now consider each of the groups $G_1 - G_9$ individually to determine whether or not they have a compact quotient.

6.4.4 G_1

Since $G_1 \cong \mathbb{R}^4$ it clearly has a compact quotient.

6.4.5 G_2

It is easy to see that $G_2 \cong N \times \mathbb{R}$ and so clearly has a compact quotient. For example take the direct product of the integral matrices in N and $\mathbb{Z} \in \mathbb{R}$.

6.4.6 G_3

In this case $\alpha(t)$ is given by the matrix:

$$\alpha(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} .$$

So for all $t \in \mathbb{R}$ the characteristic polynomial of $\alpha(t)$ is $(x-1)^3$ and the minimum polynomial for $\alpha(t)$ is x^3-3x^2+3x-1 . Therefore for any $t_0 \neq 0$ we can find $e_1 \in \mathbb{R}^3$ such that $e_1, e_2 = \alpha(t_0) \cdot e_1$, $e_3 = \alpha(t_0) \cdot e_2$ are independent and $\alpha(t_0) \cdot e_3 = 3e_3 - 3e_2 + e_1$. Then $(e_1,0),(e_2,0),(e_3,0)$ and $(0,t_0)$ generate a subgroup of G_3 with compact quotient.

6.4.7 G_4

Here $\alpha(t)$ has the form:

$$\alpha(t) = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

and the characteristic polynomial of $\alpha(t)$ is:

$$p(t,x) = x^3-(2e^t+e^{-2t})x^2 + (e^{2t}+2e^{-t})x-1 \quad .$$

Since G_4 is not nilpotent then, by 6.4.3, if G_4 has a co-compact subgroup we need to find $t_0 \in \mathbb{R}$, $t_0 \neq 0$ such that $2e^{t_0} + e^{-2t_0}$, $e^{2t_0} + 2e^{-t_0} \in \mathbb{Z}$. Setting $\lambda_0 = e^{t_0}$ this means that there are $n,m \in \mathbb{Z}$, $n > 0$, $m > 0$ such that:

$$2\lambda_0 + 1/\lambda_0^2 = n \ , \quad \lambda_0^2 + 2/\lambda_0 = m \ .$$

In other words λ_0 is a root of the polynomials:

$$f(\lambda) \equiv 2\lambda^3 - n\lambda^2 + 1 \ , \quad g(\lambda) = \lambda^3 - m\lambda + 2 \ .$$

If f is irreducible then it is the minimum polynomial for λ_0 and hence must divide g which is clearly impossible. Hence f is reducible. Similarly g is reducible.

<u>Factorings of f</u>

The only possible rational roots of f are ± 1, $\pm 1/2$.

(i) If $f(1) = 0$ then $n = 3$ and $f(\lambda) = (\lambda-1)^2(2\lambda+1)$.

(ii) If $f(-1) = 0$ then $n = -1$ contradicting $n > 0$.

(iii) If $f(\frac{1}{2}) = 0$ then $n = 5$ and $f(\lambda) = (2\lambda-1)(\lambda^2-2\lambda+1)$.

(iv) If $f(-\frac{1}{2}) = 0$ then we are back to (i).

Factorings of g

The only possible rational roots are ± 1 , ± 2 .

(i) If $g(1) = 0$ then $m = 3$ and $g(\lambda) = (\lambda-1)^2(\lambda+2)$.

(ii) If $g(-1) = 0$ then $m = -1$ contradicting $m > 0$.

(iii) If $g(2) = 0$ then $m = 5$ and $g(\lambda) = (\lambda-2)(\lambda^2+2\lambda-1)$.

(iv) $g(-2) = 0$ as in (i).

Then the only way that f and g can share a common roots is when $n = m = 3$ and $\lambda_0 = 1$ i.e. $t_0 = 0$. Hence G_4 does not have a co-compact subgroup.

6.4.8 $G_6(\lambda)$

If $\lambda = 0,-1$ then, as noted in 6.2.6, $G_6(0)$ is isomorphic to $S \times \mathbb{R}$ where S is the group

$$S = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & 1/x & z \\ 0 & 0 & 1 \end{pmatrix} \in GL(2, \mathbb{R}) \right\}.$$

If we define $\Gamma \subseteq S$ to be the discrete group:

$$\Gamma = \{ \begin{pmatrix} (3+\sqrt{5}/2)^{n_1} & 0 & n_2(\sqrt{5}+1) + 2n_3 \\ 0 & (3+\sqrt{5}/2)^{-n_1} & n_2(\sqrt{5}-1) - 2n_3 \\ 0 & 0 & 1 \end{pmatrix} | \; n_1, n_2, n_3 \in \mathbb{Z} \}$$

then $G_6(0)/\Gamma \times \mathbb{Z}$ is compact. If $\lambda = 1 , -\frac{1}{2} , -2$ then the characteristic polynomial of $\alpha(t)$ is the same as for G_4 . The same argument as in 6.4.7 then shows that $G_6(\lambda)$ has no co-compact subgroups if $\lambda = 1$, $-\frac{1}{2}$, -2 .

We now assume that $\lambda \neq -2 , -1 , -\frac{1}{2} , 0 , 1$. Let $H(k_1,k_2)$ denote the group $\mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ where $\mathbb{R}$ acts by

$$\alpha(t) = \begin{pmatrix} e^{k_1 t} & 0 & 0 \\ 0 & e^{k_2 t} & 0 \\ 0 & 0 & e^{-(k_1+k_2)t} \end{pmatrix} , \quad k_1 \geq k_2 \geq -(k_1+k_2)$$

Then, as noted in 6.2.6, $G_6(\lambda) \cong H(k_1,k_2)$ for some $k_1,k_2 \in \mathbb{R}$. The characteristic polynomial of $\alpha(t)$ is $(x-e^{k_1 t})(x-e^{k_2 t})(x-e^{-(k_1+k_2)t})$ so, by 6.4.3, $H(k_1,k_2)$ has a co-compact subgroup iff $e^{k_1 t_0} , e^{k_2 t_0} , e^{-(k_1+k_2)t_0}$ are the roots of a polynomial with integral coefficients for some $t_0 \neq 0$ in $\mathbb{R}$. Conversely let $f(x) = x^3-px^2+qx-1$ be a polynomial with integral coefficients and with three real roots $\mu_1 > \mu_2 \quad \frac{1}{\mu_1\mu_2} > 0$. Then the characteristic polynomial for $\alpha(1)$ in $H(\log \mu_1, \log \mu_2)$ is f and so $H(\log \mu_1, \log \mu_2)$ has a co-compact subgroup. Now given two polynomials $f(x) = x^3-px^2+qx-1$, $f'(x) \equiv x^3-p'x^2+q'x-1$ with roots $\mu_1 \geq \mu_2 \geq (\mu_1\mu_2)^{-1} > 0$, $\mu_1' \geq \mu_2' \geq (\mu_1'\mu_2')^{-1} > 0$ then $H(\log \mu_1, \log \mu_2) \cong H(\log \mu_1', \log \mu_2')$ iff $\log \mu_1/\log \mu_2 = \log \mu_1'/\log \mu_2'$. It is therefore a priori possible that the $H(k_1,k_2)$ obtained in this way fall into a finite number of isomorphism classes. We will show that this is *not* the case. In fact we will show that there is a countably infinite sequence of pairs $\{(\lambda_1^{(n)},\lambda_2^{(n)})\}_{n\in\mathbb{N}}$ such that, setting $H_r = H(\lambda_1^{(r)},\lambda_2^{(r)})$, we have $H_r \cong H_s$ iff $r = s$ and H_r has a co-compact subgroup. The investigation will be in two parts. First we determine the region $\Omega \subseteq \mathbb{R}^2$ formed by pairs (p,q) such that x^3-px^2+qx-1 has three non-zero positive real roots. We show that $\Omega \cap \mathbb{Z} \times \mathbb{Z}$ is infinite. Then we will show that there is a foliation F of Ω whose leaves correspond to the isomorphism classes of the $H(k_1,k_2)$. Infinitely many leaves of F contain points of $\Omega \cap \mathbb{Z} \times \mathbb{Z}$.

(i) Description of Ω

Let D denote the discriminant leaves of polynomials of the form x^3-px^2+qx-1 . Let Ω denote the subset

$$\Omega = \{(p,q) \in \mathbb{R}^2 \mid x^3-px^2+qx-1 \text{ has three distinct, positive, real roots}\}.$$

Then Ω is a connected component of $\mathbb{R}^2-D$. If F is the polynomial $F(X,Y) = X^2Y^2-4(X^3+Y^3) + 18XY-27$ then

$$D = \{(p,q) \in \mathbb{R}^2 | F(p,q) = 0\} \quad .$$

To describe D we make the transformation $\tilde{F} = F \circ T$ where T is the affine map $T(X,Y) = (X+Y+3, X-Y+3)$. Then

$$\tilde{F}(X,Y) = Y^4-2Y^2(X^2+18X+54) + X^4 + 4X^3 \quad .$$

If $\tilde{F}(X,Y) = 0$ then

$$Y^2 = A(X) \pm \sqrt{(A(X))^2 - B(X)} \quad \text{where } A(X) = X^2+18X+54$$
$$B(X) = X^4+4X^3 \quad .$$

Since $(A(X))^2-B(X) = 32(X+9/2)^3$ we must have $X \geq -9/2$ if $F(X,Y) = 0$. There are two components of the set $\tilde{D} = \{(p,q) \in \mathbb{R}^2 \mid \tilde{F}(p,q) = 0\}$.

<u>Branch 1</u> : $Y = \pm\sqrt{A(X) + \sqrt{(A(X))^2-B(X)}}$

We must have $\sqrt{(A(X))^2-B(X)} \geq -A(X)$. If $A(X) \geq 0$ this is always true for $X \geq -9/2$ and if $A(X) < 0$ it implies that $B(X) \leq 0$. Now $X \in [-4,0]$ and so, since the larger root of $A(X)$ is $-9+3\sqrt{3} > -4$ and the smaller is $-9-3\sqrt{3} < -9/2$ this branch is defined only for $X \geq 4$.

Branch 2 : $Y = \pm\sqrt{A(X) - \sqrt{(A(X))^2 - B(X)}}$

If $X \geq 0$ then $A(X), B(X), (A(X))^2 - B(X) \geq 0$ so this branch is defined for $X \in [0,\infty)$. The branch is undefined if $B(X) < 0$, which occurs for $X \in (-4,0)$, or if $A(X) < 0$, which occurs if $X \in [-9/2,-4]$. Hence this branch is defined only for $X \geq 0$.

The locus $\tilde{F}(X,Y) = 0$ looks like Fig. 1 below.

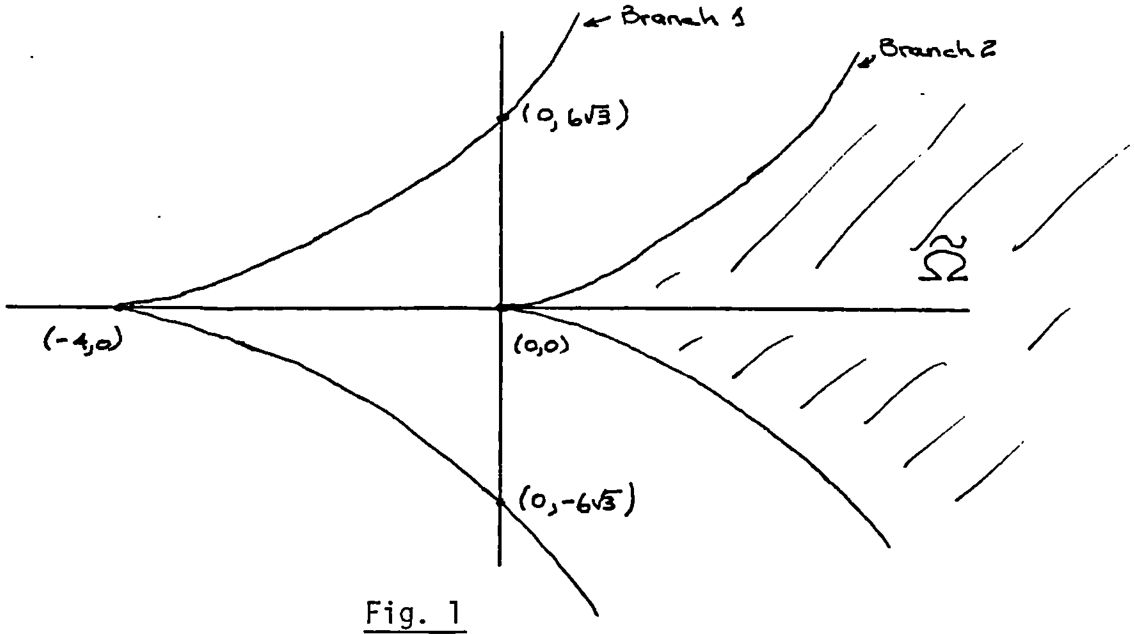


Fig. 1

It is easy to see that the shaded region $\tilde{\Omega}$ is sent on to Ω by T . In fact it suffices to find one point $(x,y) \in \tilde{\Omega}$ such that $T(x,y) \in \Omega$. Now $T(1,0) = (4,4)$ and the polynomial x^3-4x^2+4x-1 has roots $1, \frac{1}{2}(3\pm\sqrt{3})$ which are all real and positive. Now we have

$$T^{-1}(\mathbb{Z} \times \mathbb{Z}) = \{(x,y) \in \mathbb{R}^2 \mid x \text{ and } y \text{ are both integral or both half integral}\} .$$

Therefore, since $\pm\sqrt{A(X) - \sqrt{(A(X))^2 - B(X)}} \to \pm\infty$ as $X \to \infty$, the set $\tilde{\Omega} \cap T^{-1}(\mathbb{Z} \times \mathbb{Z})$ is infinite and so $\Omega \cap \mathbb{Z} \times \mathbb{Z}$ is infinite. The discriminant locus looks like Fig. 2.

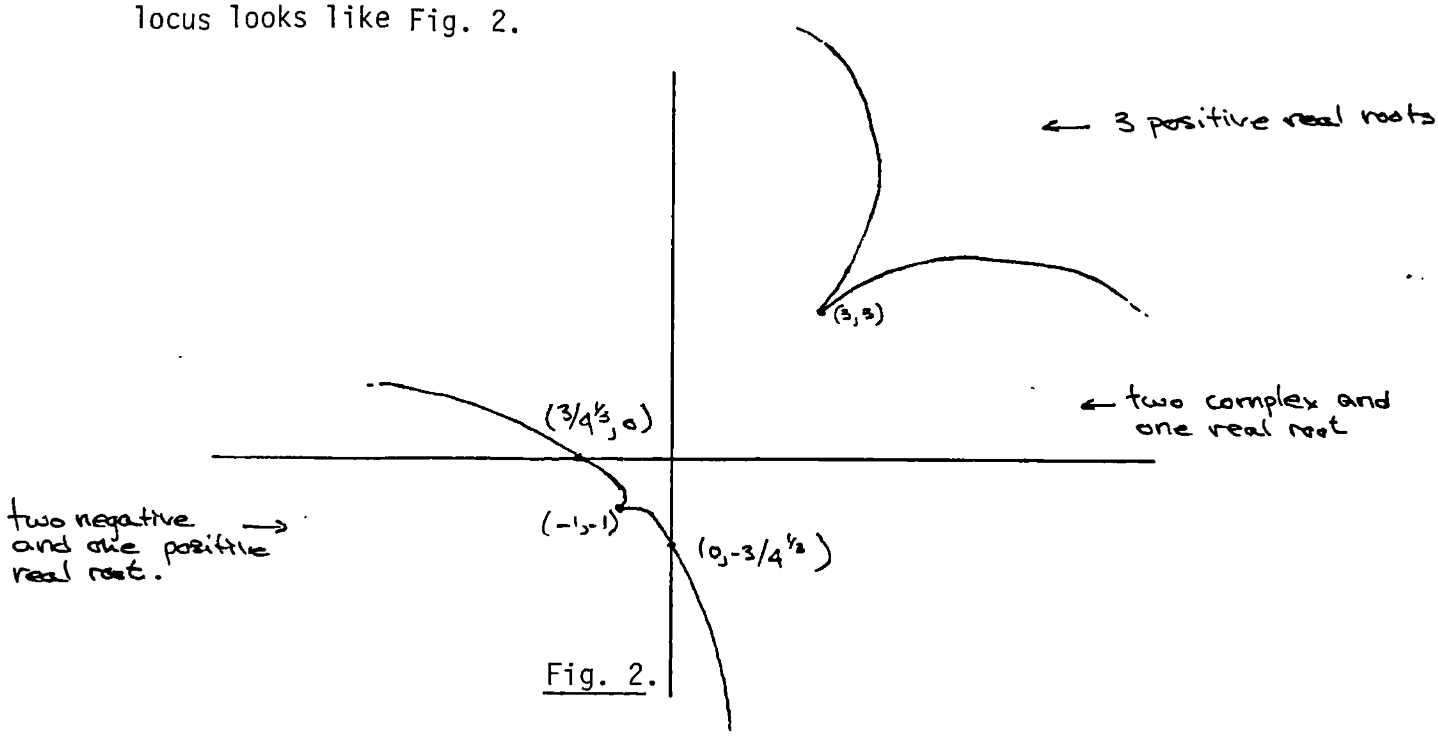


Fig. 2.

Remark

The region between the two branches corresponds to polynomials with one real root and two complex roots. It clearly contains integral points e.g. $(0,0),(1,1),(2,2),(0,n),(n,0)$ for $n \geq -1$.

(ii) Let Ω be the region described in (i). We define an equivalence relation on Ω as follows:

If $(p_0,q_0),(p_1,q_1) \in \Omega$ and the roots of
$x^3 - p_0x^2 + q_0x - 1$ are $\lambda_3 > \lambda_2 > \lambda_1 > 0$
$x^3 - p_1x^2 + q_1x - 1$ are $\mu_3 > \mu_2 > \mu_1 > 0$

$$\text{then} \quad (p_0,q_0) \sim (p_1,q_1) \quad \text{iff} \quad \frac{\log \lambda_3}{\log \lambda_1} = \frac{\log \mu_3}{\log \mu_1} \quad .$$

We will show that the equivalence classes foliate Ω in such a way that infinitely many leaves contain points of $\Omega \cap \mathbb{Z} \times \mathbb{Z}$. As noted above this implies that there are infinitely many non-isomorphic groups of the form $G_6(\lambda)$ containing a discrete co-compact subgroup.

Let $\lambda_i : \Omega \to \mathbb{R}$ for $1 \le i \le 3$ denote the functions defined by taking the roots of x^3-px^2+qx-1 in increasing order for $(p,q) \in \Omega$. i.e.

$$\lambda_3(p,q) > \lambda_2(p,q) > \lambda_1(p,q) > 0$$

$$\lambda_i^3 - p\lambda_i^2 + q\lambda_i - 1 \equiv 0 \quad \text{on} \quad \Omega \quad \text{for} \quad 1 \le i \le 3 \ .$$

Applying the implicit function theorem to the function $f(p,q,x) = x^3-px^2+qx-1$ we see that the λ_i are smooth. Differentiating with respect to p we obtain

$$(3\lambda_i^2 - 2p\lambda_i + q)\,\frac{\partial \lambda_i}{\partial p} = \lambda_i^2 \qquad \text{for} \quad i = 1,2,3 \quad .$$

Now $3\lambda_i^2(p,q) - 2p\lambda_i(p,q) + q$ is non-zero for $(p,q) \in \Omega$ since $(p,q) \notin D$. So, setting $h(p,q,t) = 3t^2 - 2pt + q$ we have

$$\frac{\partial \lambda_i}{\partial p} = \frac{\lambda_i^2}{h(\lambda_i)} \quad \text{for} \quad i = 1,2,3 \ .$$

Similarly, differentiating with respect to q , we obtain

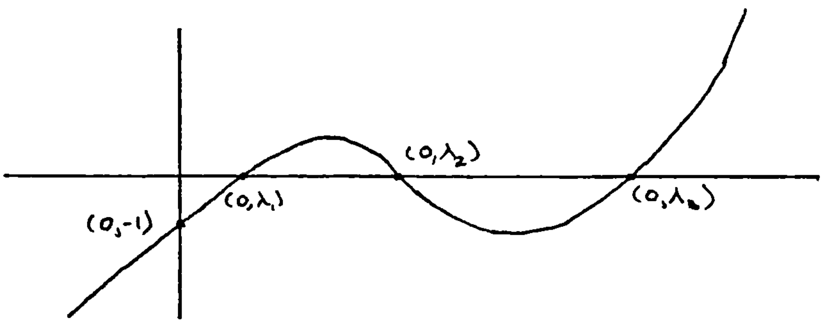
$$\frac{\partial \lambda_i}{\partial q} = \frac{-\lambda_i}{h(\lambda_i)} \quad \text{for} \quad i = 1,2,3 \ .$$

Define $\Phi : \Omega \to \mathbb{R}$ by $\Phi(p,q) = \dfrac{\log \lambda_3(p,q)}{\log \lambda_1(p,q)}$. The sets $\Phi(p,q) = \text{const}$ are the equivalence classes for the relation above. Differentiating we obtain

$$\frac{\partial \Phi}{\partial p} = \frac{1}{(\log \lambda_1)^2} \left[\frac{\lambda_3 \log \lambda_1}{h(\lambda_3)} - \frac{\lambda_1 \log \lambda_3}{h(\lambda_1)} \right]$$

$$\frac{\partial \Phi}{\partial q} = \frac{1}{(\log \lambda_1)^2} \left[\frac{\log \lambda_3}{h(\lambda_1)} - \frac{\log \lambda_1}{h(\lambda_3)} \right] .$$

Now $\lambda_1 \lambda_2 \lambda_3 = 1$ so $\lambda_3 > 1$, $\lambda_1 < 1$ and therefore $\log(\lambda_1) < 0$ and $\log(\lambda_3) > 0$. For $(p,q) \in \Omega$ the graph of $x^3 - px^2 + qx - 1$ looks like:



So $h(p,q,\lambda_1(p,q)) > 0$ and $h(p,q,\lambda_3(p,q)) > 0$. It now follows that $\partial\Phi/\partial p < 0$ and $\partial\Phi/\partial q > 0$. Hence Φ is a submersion. The sets $\Phi = \text{const.}$ form a foliation F whose leaves are never horizontal or vertical. We can write the leaf through (p_0, q_0) as $(p(q),q)$ where $p(q_0) = p_0$ and $\frac{dp}{dq} = - \partial\Phi/\partial q \ \partial\Phi/\partial p \ > 0$. The foliation looks like Fig. 3:

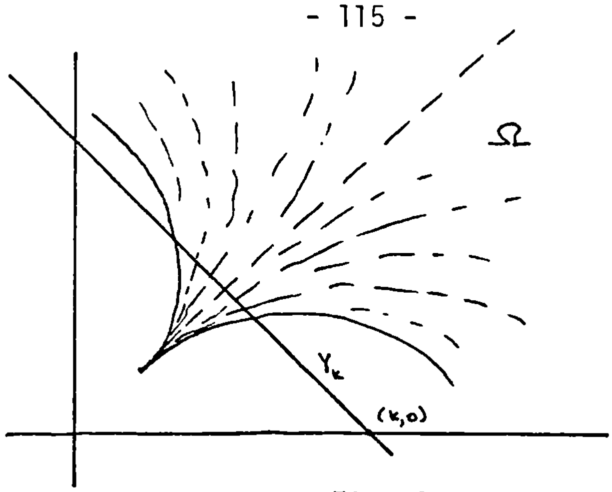


Fig. 3.

If γ_k is the curve $(x,-x+k)$ for $k \in \mathbb{Z}$ it is clear that the family $\{\gamma_k\}_{k \in \mathbb{N}}$ has the following properties:

P1: Each leaf of the foliation intersects γ_k exactly once if $k > 6$.

P2: The length of $(\gamma_k \cap \Omega)$ goes to ∞ as $k \to \infty$.

P3: $\mathbb{Z} \times \mathbb{Z} \subseteq \bigcup_{k \in \mathbb{N}} \gamma_k$.

P4: If $A_n = \gamma_n \cap (\mathbb{Z} \times \mathbb{Z} \cap \Omega)$ then $|A_n| < \infty$ and $|A_n| \to \infty$ as $n \to \infty$.

Define a sequence $\{(p_r,q_r)\}_{r=1}^{\infty}$ as follows:

(a) Let n_0 be the smallest integer such that $|A_{n_0}| \neq 0$. If the points of A_{n_0} are $(v_1,w_1) - (v_{k_0},w_{k_0})$ set $(p_i,q_i) = (v_i,w_i)$ for $i = 1,..,k_0$.

(b) Assume that $(p_1,q_1) - (p_r,q_r)$ have been defined and denote by L_i the leaf of F through (p_i,q_i). By P1 and P4 we can choose a smallest n_j such that $|A_{n_j} - (L_1 \ \ldots \ L_r)| \geq 1$. If the points of $A_{n_j} - (L_1 \ \ldots \ L_r)$ are $(v_1,w_1) - (v_{k_j},w_{k_j})$ we set $(p_{r+i},q_{r+i}) = (v_i,w_i)$ for $i = 1,\ldots,k_j$.

From P3 above $(\mathbb{Z} \times \mathbb{Z} \cap \Omega) \subseteq \bigcup_{r=1}^{\infty} L_r$ so any group $G_6(\lambda)$ with a co-compact discrete subgroup is represented by one of the leaves L_r . If the roots of $x^3-p_r x^2+q_r x-1$ are $\lambda_1^{(r)},\lambda_2^{(r)},\lambda_3^{(r)}$ then the groups $G_6(\log \lambda_3^{(r)}/\log \lambda_1^{(r)})$ are all non-isomorphic and have a co-compact discrete subgroup.

6.4.9 $G_5(\lambda)$

In this case we have

$$\alpha(t) = \begin{pmatrix} e^{\lambda t}\mathrm{Cos}\, t & e^{\lambda t}\mathrm{Sin}\, t & 0 \\ -e^{\lambda t}\mathrm{Sin}\, t & e^{\lambda t}\mathrm{Cos}\, t & 0 \\ 0 & 0 & e^{-2\lambda t} \end{pmatrix} .$$

By the remark following the discussion of the discriminant in 6.4.8 we know that there are $n,m \in \mathbb{Z}$ such that $x^3-nx^2+mx - 1$ has one positive real root, say e^{-2s} , and two complex roots $e^{s\pm i\theta}$. If we set $t_0 = \theta$ and $\lambda_0=s/\theta$ then for the group $G_5(\lambda_0)$ $\alpha(t_0)$ has a characteristic polynomial with integral coefficients. Hence $G_5(\lambda_0)$ has a lattice.

6.4.10 G_7

Since $G_7 \cong N \times \mathbb{R}$ this group has been dealt with in 6.4.5.

6.4.11 : G_8

We will show that G_8 has a discrete co-compact subgroup. Recall that $G_8 = N \rtimes_\alpha \mathbb{R}$ where:

$$\alpha(t)\cdot\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & e^t x & z\\ 0 & 1 & e^{-t}y\\ 0 & 0 & 1\end{pmatrix}.$$

It is clearly enough to show that for some non-zero $t_0 \in \mathbb{R}$ there is a lattice $\Gamma \subseteq N$ preserved by $\alpha(t_0)$. It is easy to see that if we set

$$\gamma(a,b,c) = \begin{pmatrix}1 & \frac{1}{2}(a\sqrt{5}+a+2b) & \frac{1}{10}(5c+(b^2-a^2)\sqrt{5}+ab(\sqrt{5}-5))\\ 0 & 1 & \frac{1}{10}((a+2b)\sqrt{5}-5a)\\ 0 & 0 & 1\end{pmatrix}$$

then $\Gamma = \{\gamma(n,m,p)\,|\,n,m,p \in \mathbb{Z}\}$ is a lattice in N. If $e^{t_0} = (3+\sqrt{5})/2$ then it is easy to see that $\alpha(t_0)\cdot\gamma(n,m,p) = \gamma(2n+m,n+m,p+2n^2+2nm+m^2)$. Hence $\Gamma\times\{0\}$ and $(e, \log(\frac{3+\sqrt{5}}{2}))$ generate a lattice in G_8.

6.4.12 : G_9

$G_9 = N \rtimes_\alpha \mathbb{R}$ where $\alpha(2k\pi) = \mathrm{id}$ for $k \in \mathbb{Z}$. If $\Gamma \subseteq N$ is co-compact then $\Gamma \times \{2\pi\mathbb{Z}\}$ is a discrete co-compact subgroup of G_9 since $(e,2k\pi)$ is contained in the centre of G_9.

6.5 Maximal Geometries

In this, last section we draw together the results of 6.1-6.4 and determine the maximal four dimensional geometries with $G_x = \{e\}$.

Theorem 6.5.1

The group G_3 defined in 6.2.3 acting on itself by left translations is a maximal geometry.

Proof

We have already shown in 6.4.6 that G_3 has a co-compact discrete subgroup. It remains to show that G_3 is maximal. The Lie algebra $\mathcal{g}_3$ is given by $\mathbb{R}^3 \ltimes_\alpha \mathbb{R}$ where

$$\alpha(t) = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} .$$

Hence G_3 is nilpotent since $\alpha(t)$ is a nilpotent endomorphism for each $t \in \mathbb{R}$. It now follows from Theorem 2 of [12] that any maximal geometry extending (G_3) is of the form $(G_3, G_3 \ltimes K)$ where K is a connected compact group of automorphisms of G_3 . But if $\mathcal{g}_3$ is given the presentation

$$\mathcal{g}_3 = \{X_1,X_2,X_3,X_4 \mid [X_i,X_j] = 0 \text{ for } 1 \le i,j \le 3 , [X_4,X_1] = 0$$
$$[X_4,X_2] = X_1 , [X_4,X_3] = X_2\}$$

then, with respect to the basis $\{X_1,..,X_4\}$ we have

$$\mathrm{Aut}_0(\mathfrak{g}_3) = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & a/d_4 & b/d_4 & d_2 \\ 0 & 0 & a_1/d_4^2 & d_3 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \in GL(3,\mathbb{R}) \mid d_4 > 0 \right\} .$$

This group is solvable and simply connected and so has no non-trivial connected compact subgroups. Hence $\mathrm{Aut}_0(G_3)$ has no non-trivial connected compact subgroups. We conclude that (G_3) is a maximal geometry.

□

Theorem 6.5.2

The group $G_6(\lambda)$ acting on itself by left translations is a maximal geometry iff $\lambda = \dfrac{\log \mu_1}{\log \mu_2}$ where $\mu_1, \mu_2 > 0$ are roots of a polynomial of degree 3 with integral coefficients. There are countably many non-isomorphic such geometries.

Proof

The existence of a discrete co-compact subgroup in $G_6(\lambda)$ iff has the stated form was established in 6.4.8. The fact that there are infinitely many distinct such geometries was also shown in 6.4.8. It remains to show that if $(G_6(\lambda))$ is a geometry then it is maximal. If $(G_6(\lambda))$ is a geometry but is not maximal then there is an injective homomorphism $\phi: G_6(\lambda) \to L$ where L denotes the transformation group associated to one of the maximal geometries described in Chapters 3,4,5.

If L_e denotes the stabilizer of e for the action of L on $G_6(\lambda)$ then either $L_e \cong SO(2)$ or the geometry $(L/L_e, L)$ is equivalent to one of the symmetric space geometries of Chapter 3. We will show that either assumption leads to a contradiction.

<u>Case (1)</u> : $L_x \cong SO(2)$

By the results of Chapters 4 and 5 L is isomorphic to one of the following groups

$L_1 = \mathbb{R}^3 \ltimes_\alpha (\mathbb{R} \times SO(2))$ (see Theorem 4.2.3)

$L_2 = \mathbb{R} \times (N \ltimes SO(2))$ (see Theorem 4.2.6)

$L_3 = \mathbb{R} \times (\widetilde{SL}(2, \mathbb{R}) \ltimes SO(2))$ (see Theorem 4.4.2)

$L_4 = \mathbb{R}^2 \ltimes_\alpha SL(2, \mathbb{R})$ (see Theorem 5.2.2).

At the Lie algebra level we have an injection $\phi_*: \mathfrak{g}_6(\lambda) \to \ell_i$. Now $\mathfrak{g}_6(\lambda) = \mathbb{R}^3 \ltimes_M \mathbb{R}$ where:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -(1+\lambda) \end{pmatrix} .$$

Therefore ℓ_i must have a three dimensional abelian subalgebra $\mathfrak{r}$ and a vector $V \notin \mathfrak{r}$ normalizing $\mathfrak{r}$ such that

$$\mathrm{ad}_{\ell_i}(V)|_{n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -(1+\lambda) \end{pmatrix}$$

(i) L_1

We have

$$\ell_1 = \{X_1,X_2,X_3,X_4,X_5 \mid [X_i,X_j] = 0 \text{ for } 1 \le i,j \le 3,\ [X_4,X_1] = X_1\ ,$$
$$[X_4,X_2] = X_2, [X_4,X_3] = -2X_3, [X_5,X_1] = -X_2, [X_5,X_2] = X_1\ ,$$
$$[X_5,X_3] = 0\}\ .$$

It is easy to see that the only three dimensional abelian subalgebra is the ideal n spanned by $\{X_1,X_2,X_3\}$. If $V = \sum_{i=1}^{5} a_i X_i$ then

$$\mathrm{ad}_{\ell_1}(V)|_{n} = \begin{pmatrix} b_1 & b_2 & 0 \\ -b_2 & b_1 & 0 \\ 0 & 0 & -2b_1 \end{pmatrix}\ .$$

Hence we must have $b_1 = 1$, $b_2 = 0$ and so $\lambda = 1$. This contradicts the fact proved in 6.4.8 that $G_6(1)$ has no discrete co-compact subgroup.

(ii) L_2

L_2 is isomorphic to the group of isometries of a nilpotent group $\mathbb{R} \times N$ equipped with a left invariant metric. It follows from [12]

Theorem 2(4) that $G_6(\lambda) \cong \mathbb{R} \times N$. This contradicts the fact that $G_6(\lambda)$ is not nilpotent for any λ .

(iii) L_3

We have

$$\ell_3 = \{X_1, X_2, X_3, X_4, X_5 \mid [X_1, X_i] = 0 \text{ for } 1 \le i \le 5,\ [X_2, X_3] = 2X_4 ,$$
$$[X_3, X_4] = 2X_1, [X_2, X_4] = 2X_3, [X_5, X_2] = -2X_4 ,$$
$$[X_5, X_3] = 0 ,\ [X_5, X_4] = 2X_2\} .$$

The only three dimensional abelian subalgebra of ℓ_3 is spanned by $\{X_1, X_3, X_5\}$ which is easily seen to be its own normalizer.

(iv) L_4

Since $\ell_4 = \mathbb{R}^2 \ltimes_\alpha \mathfrak{sl}(2, \mathbb{R})$ with α non-trivial ℓ_4 has no three dimensional abelian subalgebras.

Case (2) : $(L/L_e, L)$ symmetric.

Since $G_6(\lambda)$ is diffeomorphic to $\mathbb{R}^4$ the symmetric space $(L/L_e, L)$ must be one of $H^4, E^4, E\times H^3, E^2\times H^2, H^2\times H^2$ or $\mathbb{C}H^2$. It follows that $G_6(\lambda)$ has a left invariant metric of non-positive curvature. We recall that $\mathfrak{g}_6(\lambda) \cong \mathbb{R}^3 \rtimes_{M_\lambda} \mathbb{R}$ where

$$M_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -(1+\lambda) \end{pmatrix} .$$

(i) $\underline{\lambda \neq 0,-1}$

Since $\mathfrak{g}_6(\lambda)$ is unimodular and $[\mathfrak{g}_6(\lambda),\mathfrak{g}_6(\lambda)] = \mathbb{R}^3 \times \{0\}$ it follows from [8] Theorems 1.5, 1.6 that M_λ acts as a skew adjoint transformation with respect to some inner product ρ on $\mathbb{R}^3 \times \{0\}$. If $< , >$ denotes the standard inner product on $\mathbb{R}^3$ then there is a positive definite symmetric matrix A such that $\rho(x,y) = <Ax,y>$. The adjoint of M_λ with respect to ρ is given by $M^* = A^{-1}M^tA$. Since $M^* = -M$ and $M^t = M$ it follows that M and $-M$ have the same eigenvalues. Hence $\{1,\lambda,-(1+\lambda)\} = \{-1,-\lambda,(1+\lambda)\}$. But this implies that $\lambda = 0$ or -1 contradicting our assumption

(ii) $\underline{\lambda = 0 \text{ or } \lambda = -1}$

Since $G_6(0) \cong G_6(-1)$ we consider $\lambda = 0$. Then

$$\mathfrak{g}_6(0) = \{X_1,X_2,Y_1,Y_2 \mid [X_1,X_2] = [Y_1,Y_2] = 0 .$$
$$[Y_2,X_1] = [Y_2,X_2] = 0 , [Y_1,X_1] = X_1, [Y_1,X_2] = -X_2\} .$$

Letting $\mathfrak{r}$ and $\mathfrak{s}$ denote the subalgebras spanned by $\{X_1,X_2\}$ and $\{Y_1,Y_2\}$ respectively we have $\mathfrak{r} = [\mathfrak{g}_6(0), {}_6(0)]$. The roots of $\mathfrak{s}$ in $\mathfrak{r}$ are ξ_1,ξ_2 where

$$\xi_1(a_1Y_1+a_2Y_2) = a_1 \ , \quad \xi_2(a_1Y_1+a_2Y_2) = -a_1 \ .$$

But there is no vector V in $\mathfrak{s}$ with $\xi_1(V)$, $\xi_2(V) > 0$ contradicting Proposition 5.6 of [1].

We conclude that $(G_6(\lambda))$ is a maximal geometry if it is a geometry.

□

Theorem 6.5.3

The group G_8 defined in 6.3.2 acting on itself by left translations is a maximal geometry.

Proof

The existence of a discrete co-compact subgroup was established in 6.4.11. It remains to show that (G_8) is maximal. As in the proof of 6.5.2 there is an injective homomorphism $\phi : G_8 \to L$ where L is the transformation group associated to one of the maximal geometries described in Chapters 3, 4, 5. If L_e denotes the stabilizer of e for the action of L on G_8 we have $L_e \cong SO(2)$ or $(L/L_e, L)$ is one of the symmetric space geometries of Chapter 3. We will show that either assumption leads to a contradiction.

<u>Case (1)</u> : $L_e \cong SO(2)$

Then L is isomorphic to one of the following

$L_1 = \mathbb{R}^3 \ltimes_\alpha (\mathbb{R} \times SO(2))$ (see Theorem 4.2.3)

$L_2 = \mathbb{R} \times (N \ltimes SO(2))$ (see Theorem 4.2.6)

$L_3 = \mathbb{R} \times (\widetilde{SL(2,\mathbb{R})} \ltimes SO(2))$ (see Theorem 4.4.2)

$L_4 = \mathbb{R}^2 \ltimes_\alpha SL(2,\mathbb{R})$ (see Theorem 5.2.2)

<u>(i) L_1</u>

Let $\pi : L_1 \to SO(2)$ denote the obvious projection and let k denote the kernel of the associated map of Lie algebras $\pi_* \phi_* : g_8 \to \mathbb{R}$. Since g_8 is not isomorphic to $\mathbb{R}^3 \ltimes_\alpha \mathbb{R}$ it follows that k is a 3 dimensional ideal in g_8. Now $g_8 \cong n \ltimes \mathbb{R}$ and $k \supseteq [g_8, g_8] = n \times \{0\}$. Hence $k = n \times \{0\}$ and $\phi_*(n \times \{0\}) \subseteq \mathbb{R}^3 \ltimes_\alpha \mathbb{R}$. But $\mathbb{R}^3 \ltimes_\alpha \mathbb{R}$ has no subalgebra isomorphic to n.

<u>(ii) L_2</u>

Since L_2 is the group of isometries of the nilpotent group $\mathbb{R} \times N$ equipped with a left invariant metric it follows from [12] Theorem 2(4) that $G_8 \cong \mathbb{R} \times N$. This contradicts the fact that G_8 is not nilpotent.

<u>(iii) L_3</u>

This possibility is eliminated in the same way as L_1.

<u>(iv) L_4</u>

The group L_4 cannot contain G_8 since by Proposition 5.2.3 the geometry associated to L_4 has no compact quotients.

Case (2) : $(L/L_e, L)$ symmetric

As in the case (2) of 6.5.2 this implies that G_8 has a left invariant metric of non-positive curvature. Since g_8 is unimodular it follows from Theorems 1.5, 1.6 of [8] that $g_8 \cong r \ltimes s$ where r is an abelian ideal of g_8 and s is a complementary abelian subalgebra. Since r is contained in the nilradical of g_8 and $g_8 \cong n \ltimes \mathbb{R}$ we must have $r = h \times \{0\}$ for some abelian ideal $h \subseteq n$. Now

$$g_8 = \{X_1, X_2, X_3, Y \mid [X, X_2] = [X_1, X_3] = 0 \ , \ [X_2, X_3] = X_1 \ ,$$
$$[Y, X_1] = 0 \ , \ [Y, X_2] = X_2 \ , \ [Y, X_3] = -X_3\} \ .$$

It is easy to see that r is spanned by X_1 and $aX_2 + bX_3$ for some $a, b \in \mathbb{R}$. Clearly for any choice of a, b the algebra g_8/r is not abelian.

□

Theorem 6.5.4

If (G) is a maximal four dimensional geometry then G is isomorphic to G_3 , $G_6(\lambda)$ for some $\lambda \in \mathbb{R}$, or G_8 .

Proof.

By Corollary 6.1.6 G must be isomorphic to one of the groups $G_1 - G_9$ constructed in sections 6.2 and 6.3. Now $G_1 \cong \mathbb{R}^4$ and G_2 , $G_7 \cong \mathbb{R} \times N$ are not maximal by 6.1.2 since they have compact groups of automorphisms (see Lemma 4.2.4). The groups $G_5(\lambda)$ and G_9 are of the form $H \ltimes_\alpha \mathbb{R}$ where the image of α in Aut(H) is compact. Therefore by Proposition 6.1.3 $G_5(\lambda)$ and G_9 are not maximal geometries. Finally G_4 is not a geometry since it was shown in 6.4.7 that it has no co-compact subgroup. The only possibilities are $G \cong G_3$, $G \cong G_8$ or $G \cong G_6(\lambda)$ for the λ defined in Theorem 6.5.2.

$\square$

SUMMARY OF MAXIMAL GEOMETRIES.

We list the maximal 4-dimensional geometries determined in Chapters 3, 4, 5, 6.

1. Riemannian Globally Symmetric Spaces (Chapter 3)

   (i) The simply connected spaces of constant curvature E^4, H^4, S^4 with their maximal connected group of isometries.

   (ii) The Hermitian Symmetric spaces:

   Complex Projective space $\mathbb{C}P^2 = SU(3)/S(U(2)\times U(1))$.

   Complex Hyperbolic space $\mathbb{C}H^2 = SU(2,1)/S(U(2)\times U(1))$.

   (iii) The reducible spaces $E\times S^3$, $E\times H^3$, $E^2\times S^2$, $E^2\times H^2$, $S^2\times S^2$, $S^2\times H^2$, $H^2\times H^2$ with their connected groups of isometries.

2. Geometries with stabilizer SO(2) (Chapters 4,5)

   (i) (Theorem 4.2.2) Let G be the group $\mathbb{R}^3 \rtimes_\alpha (\mathbb{R}\times SO(2))$ where

$$\alpha(t,\theta) = \begin{pmatrix} e^t\mathrm{Cos}\theta & e^t\mathrm{Sin}\theta & 0 \\ -e^t\mathrm{Sin}\theta & e^t\mathrm{Cos}\theta & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} :$$

If K is the subgroup $\{0\} \times SO(2)$ then $(G/K,G)$ is a maximal geometry.

(ii) (Theorem 4.2.5) Let G be the group $\mathbb{R}\times(N \ltimes_{\alpha} SO(2))$ where

$$N = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R})\}$$

$$\alpha(\theta)\cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\mathrm{Cos}\theta+z\mathrm{Sin}\theta & y-xz\mathrm{Sin}^2\theta+\frac{1}{4}\mathrm{Sin}2\theta(z^2-x^2) \\ 0 & 1 & -x\mathrm{Sin}\theta + z\mathrm{Cos}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

If K is the subgroup $\{0\} \times \{I\} \times SO(2)$ then $(G/K,G)$ is a maximal geometry.

(iii) (Theorem 4.4.1) Let G be the group $(\mathbb{R}\times \widetilde{SL(2, \mathbb{R})}) \ltimes_{\alpha} SO(2)$ where $\alpha(SO(2))$ is a maximal compact group of automorphisms of $\widetilde{SL(2, \mathbb{R})}$. If $K = \{0\} \times \{e\} \times SO(2)$ then $(G/K,G)$ is a maximal geometry.

(iv) (Theorem 5.2.1) Let $G = \mathbb{R}^2 \ltimes_{\alpha} \widetilde{SL(2, \mathbb{R})}$ where $\alpha: SL(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is a non-trivial automorphism. If $K = \{0\} \times SO(2) \subseteq \{0\} \times SL(2, \mathbb{R})$ then $(G/K,G)$ is a maximal geometry with quotients of finite volume but no compact quotients (Proposition 5.2.3).

3. Solvable Groups

The following simply connected solvable groups acting on themselves by left translations are maximal geometries:

(i) (Theorem 6.5.1) $G = \mathbb{R}^3 \ltimes_\alpha \mathbb{R}$ where α is the homomorphism:

$$\alpha(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) (Theorem 6.5.2) $G = \mathbb{R}^3 \ltimes_\alpha \mathbb{R}$ where α is the homomorphism:

$$\alpha(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{-(1+\lambda)t} \end{pmatrix}.$$

and $\lambda = \log \mu_1/\log \mu_2$ with μ_1, μ_2 the roots of a polynomial of degree 3 with integral coefficients. There is a countable infinity of non-isomorphic such groups.

(iii) $G = N \ltimes_\alpha \mathbb{R}$ where (Theorem 6.5.3)

$$\alpha(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}$$

REFERENCES

1. Azencott, R. and Wilson, E.N. : Homogeneous Manifolds with Negative Curvature - Transactions of the American Mathematical Society 215 (1976) pg. 323-362.

2. Borel, A. : Compact Clifford-Klein Forms of Symmetric Spaces - Topology 2 (1963) pg. 111-122.

3. Gordon, C. : Riemannian Isometry Groups Containing Transitive Reductive Subgroups - Mathematische Annalen 248 (1980) pg. 185-192.

4. Halmos, P.R. : Measure Theory - Van Nostrand, 1950.

5. Hochschild, G. : The Structure of Lie Groups, Holden-Day, 1965.

6. Helgason, S. : Differential Geometry and Symmetric Spaces - Academic Press, 1962.

7. Lawson, H. Blaire Jr. : The Quantitative Theory of Foliations - CBMS Regional Conference Series 27, American Mathematical Society 1977.

8. Milnor, J. : Curvatures of Left Invariant Metrics on Lie Groups - Advances in Mathematics 21 (1976).

9. D'Atri, J.E. and Ziller, W. : Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups. Memoirs of the American Mathematical Society 215 (1979).

10. Palais, R. : Topological Transformation Groups - Memoirs of the American Mathematical Society, 22 (1957).

11. Ragunathan, M.S. : Discrete Subgroups of Lie Groups - Springer Verlag, 1972.

12. Wilson, E.N. : Isometry Groups on Homogeneous Nilmanifolds - Geometriae Dedicata 12 (1982) pg. 337-346.

13. Scott, P. : The Geometries of 3-manifolds - Bulletin of the London Mathematical Society 15 (1983) pg. 401-487.

14. Thurston, W.P. : Hyperbolic Geometry and 3-manifolds - in Low Dimensional Topology, London Mathematical Society Lecture Notes 48 (ed. Brown and Thickstun, Cambridge University Press 1982) pg. 9-25.